HW#1 solutions

Problem 13 from Section 1.2.

No, V is not a vector space. Indeed, suppose it is, and (z_1, z_2) is zero element in V, i.e. $(a_1, a_2) + (z_1, z_2) = (a_1, a_2)$ for any $(a_1, a_2) \in V$. Take $(a_1, a_2) = (1, 1)$. We get $z_1 = 0, z_2 = 1$. At the same time, if one takes $(a_1, a_2) = (0, 0)$, then $(a_1, a_2) + (b_1, b_2) \neq (0, 1)$ for any $(b_1, b_2) \in V$, and therefore the element $(0, 0)$ does not have an inverse one.

Problem 17 from Section 1.2.

No, V is not a vector space. Otherwise we would have $(a, 0) = 1 \cdot (a, 0) = (1 + 0) \cdot (a, 0) = 1 \cdot (a, 0) + 0 \cdot (a, 0) = (a, 0) + (a, 0) = (2a, 0)$, and therefore $a = 2a$, or $a = 0$ for any $a \in \mathbb{F}$. But for any field $1 \neq 0$.

Problem 18 from Section 1.2.

No, V is not a vector space. Notice that the addition is not commutative.

Problem 21 from Section 1.2.

The solution is straightforward.

Problem 10 from Section 1.3.

The fact that W_1 is a subspace of $\mathbb{F}^n$ is straightforward. The set W_2 is not a subspace since it does not contain zero vector.

Problem 19 from Section 1.3.

If $W_1 \subseteq W_2$ (or $W_2 \subseteq W_1$) then $W_1 \cup W_2 = W_2$ (or $W_1 \cup W_2 = W_1$) and therefore must be a subspace of V. Suppose now that $W_1 \cup W_2$ is a subspace of V, and at the same time there are vectors $v_1 \in W_1 \backslash W_2$ and $v_2 \in W_2 \backslash W_1$. Set $v_1 + v_2 = v_3$. Since $W_1 \cup W_2$ is a subspace of V, $v_3 \in W_1 \cup W_2$, hence either $v_3 \in W_1$, or $v_3 \in W_2$. In the first case we have that $v_2 \in W_1$, in the second case we have that $v_1 \in W_2$. In either case we get a contradiction.

Problem 6 from Section 1.4.

The statement is incorrect. In the case of the field $\mathbb{F}$ of characteristic 2 those vectors do not generate $\mathbb{F}^3$. Let us prove that in the case of the field $\mathbb{F}$ of characteristic different from 2 (i.e. $1 + 1 \neq 0$) the statement holds. Denote $v_1 = (1, 1, 0), v_2 = (1, 0, 1), v_3 = (0, 1, 1)$. It is enough to show that some linear combinations of v_1, v_2, v_3 give the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. But we can explicitly check that $(1, 0, 0) = \frac{1}{2}(v_1 + v_2 - v_3)$, $(0, 1, 0) = \frac{1}{2}(v_1 + v_3 - v_2)$, and $(0, 0, 1) = \frac{1}{2}(v_2 + v_3 - v_1)$.

Problem 10 from Section 1.4.

Since the matrices M_1, M_2, M_3 are symmetric, any linear combination is also symmetric. Therefore

$span\{M_1, M_2, M_3\}$ is a subset of the space of symmetric matrices. At the same time if A is a symmetric matrix of the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, then $A = aM_1 + cM_2 + bM_3$, and therefore the set of symmetric matrices is contained in $span\{M_1, M_2, M_3\}$.

Problem 12 from Section 1.4.

It was proven already that $span\{W\}$ is a subspace, so if $W = span\{W\}$, then W is a subspace. On the other hand, if $W \subseteq V$ is a subspace, then any linear combination of vectors from W is in W, and hence $span\{W\} \subseteq W$. Since the inclusion $W \subseteq span\{W\}$ always holds, we must have $W = span\{W\}$.

Problem 2 from Section 1.5.

a) linearly dependent;
b) linearly independent;
c) linearly independent;
d) linearly dependent;
e) linearly dependent;
f) linearly independent;
g) linearly dependent;
h) linearly independent;
i) linearly independent;
j) linearly dependent.

Problem 10 from Section 1.5.

For example, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 0)$. Certainly, there are many other examples.

Problem 2 from Section 1.6.

a) Basis;
b) Not a basis;
c) Basis;
d) Basis;
e) Not a basis.

Problem 3 from Section 1.6.

a) Not a basis;
b) Basis;
c) Basis;
d) Basis;
e) Not a basis.

Problem 4 from Section 1.6.

No, three vectors cannot generate a four dimensional space.

Problem 12 from Section 1.6.

It is clear that $v + w \in span\{u, v, w\}$ and $u + v + w \in span\{u, v, w\}$, hence $span\{u + v + w, v + w, w\} \subseteq span\{u, v, w\}$. Also, since $v = (v + w) - w$, and $u = (u + v + w) - (v + w)$, we have that $u, v, w \in span\{u + v + w, v + w, w\}$, and hence $span\{u, v, w\} \subseteq span\{u + v + w, v + w, w\}$. This proves that $span\{u + v + w, v + w, w\} = span\{u, v, w\}$. If $\{u, v, w\}$ is a basis in V, then $V = span\{u, v, w\}$ is a three dimensional vector space. Since $\{u + v + w, v + w, w\}$ is a generating set of three vectors in three dimensional space, it must be a basis.