121A HW 3 Solutions

2.4.2:

(a) Not invertible because $\dim(\mathbb{R}^2) = 2 \neq 3 = \dim(\mathbb{R}^3)$

(b) Not invertible, same as (a)

(c) Invertible. A quick argument shows $N(T) = \{0\}$, and since the domain space and target space have the same dimension, this implies T is bijective, so invertible.

(d) Not invertible. As in (a), (b) the dimensions don't match

(e) Not invertible. Same reason

(f) Invertible. Not hard to show $N(T) = \{0\}$, so T is injective. As in (c) this implies T is bijective, so invertible.

2.4.3:

(a) Not isomorphic. They have different dimension.

(b) Isomorphic. They have the same dimension.

(c) Isomorphic. Same dimension

(d) Not isomorphic. $V = N(tr)$, where $tr: M_2(\mathbb{R}) \longrightarrow \mathbb{R}$ is the trace map. We saw previously that this has dimension 3, so the spaces have different dimension.

2.4.14: Let $T: V \longrightarrow \mathbb{F}^3$ be the map

$$T\left(\begin{bmatrix} a & a+b \\ 0 & c \end{bmatrix}\right) = (a, a+b, c)$$

It is straightforward to check T is linear, so we show T is bijective.

Injectivity: Easy to check $N(T) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ so T is injective.

Surjectivity: Let $(a_1, a_2, a_3) \in F^3$. Set $a = a_1$, $b = a_2 - a_1$, $c = a_3$. Then,

$$T\left(\begin{bmatrix} a & a+b \\ 0 & c \end{bmatrix}\right) = (a, a+b, c) = (a_1, a_1 + a_2 - a_1, a_3) = (a_1, a_2, a_3).$$

So, T is bijective and hence invertible.

2.4.16: First, notice Φ is linear: if $A_1, A_2 \in M_n(F)$ and $c \in F$ then

$$\begin{aligned}\Phi(cA_1 + A_2) &= B^{-1}(cA_1 + A_2)B \\ &= (B^{-1}cA_1 + B^{-1}A_2)B \\ &= cB^{-1}A_1B + B^{-1}A_2B = c\Phi(A_1) + \Phi(A_2)\end{aligned}$$

Since the domain space and co-domain space have the same dimension, we only need to check Φ is injective to show it is bijective, and hence invertible.

<u>Φ is injective</u>: If $\Phi(A) = O$, then

$$B^{-1}AB = O$$

$$\overset{I}{\cancel{B}B^{-1}}A\overset{I}{\cancel{B}B^{-1}} = BOB^{-1}$$

$$A = O$$

So $N(\Phi) = \{0\}$

<u>Alternative Solution</u>: Let $T: M_n(F) \longrightarrow M_n(F)$ be defined by

$$T(A) = BAB^{-1}$$

As above, T is linear, then just check that $T = \Phi^{-1}$.

<u>2.5.2</u>:

(a) $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -1 \\ 5 & -4 \end{bmatrix}$

<u>2.5.3</u>:

(a) $\begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ (e) $\begin{bmatrix} 5 & -6 & 3 \\ 0 & 4 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{bmatrix}$

2.5.10: Note, by a previous HW, we know $tr(CD) = tr(DC)$. So, if A is similar to B then there is Q such that $A = QBQ^{-1}$. Then,

$$\begin{aligned} tr(A) &= tr(QBQ^{-1}) \\ \text{HW} \rightarrow \quad &= tr(Q^{-1}QB) \\ &= tr(IB) \\ &= tr(B) \end{aligned}$$

2.5.11: Let $Q = [I]_\alpha^\beta$ and $R = [I]_\beta^\gamma$

(a) Notice,

$$RQ = [I]_\beta^\gamma [I]_\alpha^\beta \overset{\text{Thm 2.11}}{=} [II]_\alpha^\gamma = [I]_\alpha^\gamma$$

(b) Notice,

$$Q^{-1} = \left([I]_\alpha^\beta\right)^{-1} \overset{\text{Thm 2.18}}{=} [I^{-1}]_\beta^\alpha = [I]_\beta^\alpha$$

3.1.3:

(a) swap row 1 and row 3, so inverse is itself

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) 3 times row 2, so inverse is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) -2 times row 1 added to row 3, so inverse is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

3.1.8: This was done in detail in discussion. Handle each row operation separately. If Q is obtained by type 1, then just do the same operation to get back to P.

If Q is obtained by type 2, where the row is scaled by c, then scale the same row by $\frac{1}{c}$ to get back to P.
If Q is obtained by type 3, where we add $c \cdot$ row i to row j, then add $-c \cdot$ row i to row j to get back to P.

3.2.2:

(a) 2
(b) 3
(c) 2
(d) 1
(e) 3
(f) 3
(g) 1

3.2.5:

(a) rank = 2

inverse: $\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

(b) rank = 1

(c) rank = 2

(d) rank = 3

inverse: $\begin{bmatrix} -1/2 & 3 & -1 \\ 3/2 & -4 & 2 \\ 1 & -2 & 1 \end{bmatrix}$

(e) rank = 3

inverse: $\begin{bmatrix} 1/6 & -1/3 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/6 & 1/3 & 1/2 \end{bmatrix}$

(f) rank = 2

(g) rank = 4

inverse: $\begin{bmatrix} -51 & 15 & 7 & 12 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{bmatrix}$

(h) rank = 3

3.2.6:

(a) T is invertible,

$$T^{-1}(a_0 + a_1x + a_2x^2) = -(a_0 + 2a_1 + 10a_2) + -(a_1 + 4a_2)x - a_2x^2$$

(b) T is not invertible, $N(T) \neq \{0\}$, for example, $1 \in N(T)$.

(c) T is invertible,

$$T^{-1}(a_1, a_2, a_3) = (\tfrac{1}{6}a_1 - \tfrac{1}{3}a_2 + \tfrac{1}{2}a_3,\ \tfrac{1}{2}a_1 - \tfrac{1}{2}a_3,\ -\tfrac{1}{6}a_1 + \tfrac{1}{3}a_2 + \tfrac{1}{2}a_3)$$

(d) T is invertible,

$$T^{-1}(a_0 + a_1x + a_2x^2) = \left(a_2,\ \tfrac{1}{2}a_0 - \tfrac{1}{2}a_1,\ \tfrac{1}{2}a_0 + \tfrac{1}{2}a_1 - a_2\right)$$

(e) T is invertible,

$$T^{-1}(a_1, a_2, a_3) = a_2 + \left(\tfrac{1}{2}a_3 - \tfrac{1}{2}a_1\right)x + \left(\tfrac{1}{2}a_1 + \tfrac{1}{2}a_3 - a_2\right)x^2$$

(f) T is not invertible, it is not injective since $N(T) \neq \{0\}$. For example, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in N(T)$

<u>4.1.2</u>:

(a) 30

(b) -17

(c) -8

<u>4.1.3</u>:

(a) $-10 + 15i$

(b) $36 + 41i$

(c) -24