Math 121A HW4 Solutions

<u>4.1.11</u>: This was done in detail in discussion.

Let $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in M_2(\mathbb{R})$. Then, using properties (i), (ii), (iii) of δ we get

$$\delta\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) \overset{(i)}{=} a_1\,\delta\left(\begin{bmatrix} 1 & 0 \\ a_3 & a_4 \end{bmatrix}\right) + a_2\,\delta\left(\begin{bmatrix} 0 & 1 \\ a_3 & a_4 \end{bmatrix}\right)$$

$$(i) \longrightarrow = a_1a_3\,\delta\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) + a_1a_4\,\delta\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + a_2a_3\,\delta\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) + a_2a_4\,\delta\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right)$$

$$(ii)+(iii) \longrightarrow = 0 + a_1a_4 + a_2a_3\,\delta\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) + 0$$

$$(*) \qquad = a_1a_4 + a_2a_3\,\delta\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

<u>Claim</u>: $\delta\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1$

Given the claim, $(*)$ simplifies to $a_1a_4 - a_2a_3$ which is the same as the determinant so we will be done.

<u>Pf of Claim</u>: Note: $\delta\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 0$ by (ii), so we have

$$0 = \delta\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \delta\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) + \delta\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + \delta\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) + \delta\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right)$$

where we use (i) to split as above. Using (ii) and (iii) we get,

$$-1 = -\delta\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \delta\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \qquad \square \text{ Claim}$$

<u>4.2.3</u>: Use row operation & interactions with determinant to find $k = 42$.

<u>4.2.4</u>: Same process as #3 to find $k = 2$.

<u>4.2.28</u>: $\det(E_1) = -1$

$\det(E_2) = c$ where c is the scalar used to get to E_2 from I

$\det(E_3) = 1$

<u>4.2.30</u>: Let $j \in \mathbb{N}$ be such that $n = 2j$ or $n = 2j+1$ (based on whether n is even or odd). It is not hard to see that it takes j row swaps to get from A to B. Hence,

$$\det(B) = (-1)^j \det(A)$$

4.3.6: Use the formula $x_k = \frac{\det(M_k)}{\det(A)}$ to get

$x_1 = -43$

$x_2 = -109$

$x_3 = -17$

4.3.7: Same as #6.

$x_1 = 0$

$x_2 = -12$

$x_3 = 16$

4.3.10:

Proof: Let M be nilpotent and let k be such that $M^k = O$. Then, by properties of determinant, we have

$$(\det(M))^k = \det(M^k) = \det(O) = 0$$

So $\det(M)$ is a solution to the equation

$$x^k = 0$$

and hence, it must be that $\det(M) = 0$. □

4.3.12:

Proof: Let Q be orthogonal. Since $\det(A^t) = \det(A)$ for any matrix A, we have

$$1 = \det(I) \underset{\substack{\uparrow \\ Q \text{ orthogonal}}}{=} \det(QQ^t) = \det(Q)\det(Q^t) = (\det(Q))^2$$

Hence, $\det(Q)$ is a solution to the equation

$$x^2 = 1$$

So, $\det(Q) = \pm 1$. □

4.4.4:

(a) 0 (c) -49 (e) $-28 - i$ (g) 95

(b) 36 (d) 10 (f) $17 - 3i$ (h) -100

5.1.2:

(a) $[T]_\beta = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$

Not diagonal, so not a basis of eigenvectors

(b) $[T]_\beta = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

Is diagonal, so is a basis of eigenvectors

(c)

$$[T]_\beta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Is diagonal, so is a basis of eigenvectors

(d)

$$[T]_\beta = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -2 & 0 \\ -4 & 0 & 0 \end{bmatrix}$$

Not diagonal, so not a basis of eigenvectors

(e)

$$[T]_\beta = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Not diagonal, so not a basis of eigenvectors

(f)

$$[T]_\beta = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Is diagonal, so is a basis of eigenvectors

5.1.3:

(a) (i) $\lambda_1 = -1$, $\lambda_2 = 4$

(ii) $E_{\lambda_1} = \left\{ \begin{bmatrix} -a \\ a \end{bmatrix} \middle| \, a \in F \right\}$

$E_{\lambda_2} = \left\{ \begin{bmatrix} \frac{2}{3}a \\ a \end{bmatrix} \middle| \, a \in F \right\}$

(iii) One basis is $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \right\}$

(iv) One Q is $\begin{bmatrix} -1 & 2/3 \\ 1 & 1 \end{bmatrix}$

(b) (i) $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

(ii) $E_{\lambda_1} = \left\{ \begin{bmatrix} -a \\ -a \\ a \end{bmatrix} \middle| \, a \in F \right\}$

$E_{\lambda_2} = \left\{ \begin{bmatrix} -a \\ a \\ 0 \end{bmatrix} \middle| \, a \in F \right\}$

$E_{\lambda_3} = \left\{ \begin{bmatrix} -a \\ 0 \\ a \end{bmatrix} \middle| \, a \in F \right\}$

(iii) One basis is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(iv) One Q is $\begin{bmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(c) (i) $\lambda_1 = 1$, $\lambda_2 = -1$

(ii) $E_{\lambda_1} = \left\{ \begin{bmatrix} \frac{i+1}{2}a \\ a \end{bmatrix} \middle| \, a \in F \right\}$

$E_{\lambda_2} = \left\{ \begin{bmatrix} \frac{i-1}{2}a \\ a \end{bmatrix} \middle| \, a \in F \right\}$

(iii) One basis is $\left\{ \begin{bmatrix} \frac{i+1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{i-1}{2} \\ 1 \end{bmatrix} \right\}$

(iv) One Q is $\begin{bmatrix} \frac{i+1}{2} & \frac{i-1}{2} \\ 1 & 1 \end{bmatrix}$

(d) (i) $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$

(ii) $E_{\lambda_1} = \left\{ \begin{bmatrix} 1/2\,a \\ 2a \\ a \end{bmatrix} \middle|\, a \in F \right\}$

$E_{\lambda_2} = \left\{ \begin{bmatrix} a \\ b \\ a \end{bmatrix} \middle|\, a, b \in F \right\}$

(iii) One basis is $\left\{ \begin{bmatrix} 1/2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(iv) One Q is $\begin{bmatrix} 1/2 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

<u>5.1.4</u>:

(a) $\lambda_1 = 3$, $\lambda_2 = 4$

One basis is $\left\{ \begin{bmatrix} 3/5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$

(b) $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$

One basis is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) $\lambda_1 = -1$, $\lambda_2 = 2$

One basis is $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) $\lambda_1 = -2$, $\lambda_2 = -3$

One basis is $\{2+x,\ 3/2 + x\}$

(e) $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 4$

One basis is $\{-3 + x,\ -3/4 - 13/4\,x + x^2,\ 1 + x\}$

(f) $\lambda_1 = 1$, $\lambda_2 = 3$

One basis is $\{-2 + x,\ -4 + x^2,\ -8 + x^3,\ x\}$

(g) $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$, $\lambda_4 = 3$

One basis is $\{-1 + x,\ 1,\ -\frac{2}{3} + x^2,\ -\frac{7}{2} + 3x + x^3\}$

(h) $\lambda_1 = 1$, $\lambda_4 = -1$

One basis is $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

(i) $\lambda_1 = 1$, $\lambda_2 = -1$

One basis is $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \right\}$

(j) $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 5$ One basis is $\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

5.1.17 This was done in detail in discussion.

(a) If $A=\lambda A^t$, then since transpose only swaps elements of a matrix without affecting values, we must have $\lambda=\pm 1$.

(b) For $\lambda=1$, $A=A^t$ so A is symmetric

For $\lambda=-1$, $A=-A^t$ so A is antisymmetric

(c) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

(d) For $k,\ell \in \{1,\dots,n\}$ with $k<\ell$ define matrices A_k, $B(k,\ell)$, $C(k,\ell)$ in $M_n(F)$ as follows

$$(A_k)_{ij} = \begin{cases} 1 & \text{if } i=j=k \\ 0 & \text{o.w.} \end{cases}$$

$$(B(k,\ell))_{ij} = \begin{cases} 1 & \text{if } i=k,\ j=\ell \\ 1 & \text{if } i=\ell,\ j=k \\ 0 & \text{o.w.} \end{cases}$$

$$(C(k,\ell))_{ij} = \begin{cases} 1 & \text{if } i=k,\ j=\ell \\ -1 & \text{if } i=\ell,\ j=k \\ 0 & \text{o.w.} \end{cases}$$

Then, the set

$$\{A_k \mid k=1,\dots,n\} \cup \left\{ B(k,\ell), C(k,\ell) \,\middle|\, \begin{array}{l} k,\ell \in \{1,\dots,n\} \\ k<\ell \end{array} \right\}$$

is a basis of eigenvectors.

5.1.22: This was done in detail in discussion.

(a) Let $g(t)=a_n t^n + \cdots + a_1 t + a_0$. Then,

$$\begin{aligned} g(T)(x) &= a_n T^n(x) + \cdots + a_1 T(x) + a_0 I(x) \\ &= a_n \lambda^n x + \cdots + a_1 \lambda x + a_0 x \\ &= (a_n \lambda^n + \cdots + a_1 \lambda + a_0)x \\ &= g(\lambda)x \end{aligned}$$

(b) Just replace every "T" in the statement/proof by $M \in M_n(F)$

(c) We have

$$\begin{aligned} g(A) = 2A^2 - A + I &= 2\begin{bmatrix} 7 & 6 \\ 9 & 10 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 12 \\ 18 & 20 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 10 \\ 15 & 19 \end{bmatrix} \end{aligned}$$

Then,

$$g(A)\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 58 \\ 87 \end{bmatrix} = 29\begin{bmatrix} 2 \\ 3 \end{bmatrix} = g(4)\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$