Math 121A, Homework 2

§**1.7**: 5, 7; §**2.1**: 1, 5, 6, 19, 20; §**2.2**: 2, 5, 14, 16; §**2.3**: 2, 4, 12, 13

1.7.5. This was done in detail in discussion, so some details may need to be filled in by the student. First assume β is a basis. Since a basis spans,we only need to see that each nonzero $x \in V$ has a unique representation as

$$x = c_1 u_1 + \cdots + c_n u_n$$

where the $c_i \in F$ are nonzero scalars and each $u_i \in \beta$. If not, x has two representations, say

$$c_1 u_1 + \cdots + c_n u_n = x = d_1 v_1 + \cdots + d_m v_m$$

where are the scalars are nonzero and the vectors are from β. Since $c_1 \neq 0$ we can divide by it and get

$$u_1 = \frac{1}{c_1} (d_1 v_1 + \cdots + d_m v_m - c_2 u_2 - \cdots - c_n u_n)$$

So u_1 is a linear combination of vectors from β. This contradicts that β is linearly independent.

Now assume β has the property in the problem statement. We need to see that β is a basis. It is not hard to see that β spans, so we need to show linear independence. First note, that by the property on β, each vector $u \in \beta$ has a unique representation as

$$u = 1 \cdot u$$

If β is not linearly independent, then there is are vectors $u, u_1, \ldots, u_n \in \beta$ and nonzero scalars $c_1, \ldots, c_n$ such that

$$u = c_1 u_1 + \cdots + c_n u_n$$

This contradicts u having the unique representation stated above.

1.7.7. This was done in detail in discussion, so some details may need to be filled in by the student. Note that $S \cup \beta$ is a spanning set, since β is spanning, so by Theorem 1.12, it suffices to find a maximal linearly independent subset of $S \cup \beta$. Let

$$\mathcal{F} = \{S \cup B \subseteq S \cup \beta \mid S \cup B \text{ is linearly independent } \}$$

It is not hard to verify that $\mathcal{F}$ has the maximal principle: if $\mathcal{C} \subseteq \mathcal{F}$ is a chain, then an upperbound is the union of $\mathcal{C}$, one just needs to check that this union is an element of $\mathcal{F}$.

Since $\mathcal{F}$ satisfies the maximal principle, it has a maximal element, $S \cup B^*$. By definiton of $\mathcal{F}$, $S \cup B^*$ is linearly independent. To see it is a maximal linearly independent subset of $S \cup \beta$, suppose not. Then, there is $v \in (S \cup \beta) - (S \cup B^*)$ such that $S \cup B^* \cup \{v\}$ is linearly independent. But this contradicts the maximality of $S \cup B^*$ in $\mathcal{F}$.

2.1.1.

a. True

b. False

c. False (need T to be linear)

d. True

e. False

f. False

g. True

h. False

2.1.5. T is linear: let $p, q \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$.

$$\begin{aligned} T(cp(x) + q(x)) &= x(cp(x) + q(x)) + (cp(x) + q(x))' \\ &= cxp(x) + xq(x) + cp'(x) + q'(x) \\ &= c(xp(x) + p'(x)) + xq(x) + q'(x) \\ &= cT(p(x)) + T(q(x)) \end{aligned}$$

If $p(x) \in N(T)$, then it is easy to see that $p(x) = 0$, so $N(T) = \{0\}$ and the basis is $\emptyset$. Thus, T is injective.

Since T is injective, a basis for $R(T)$ is $\{T(1), T(x), T(x^2)\} = \{x, x^2 + 1, x^3 + 2x\}$. By dimension considerations, T is not onto. And we also have

$$\text{rank}(T) + nullity(T) = 3 + 0 = 3 = \dim(P_2(\mathbb{R}))$$

2.1.6. T is linear: Let $c \in F$ and $A, B \in M_n(F)$.

$$\begin{aligned} T(cA + cB) &= (cA_{11} + B_{11}) + (cA_{22} + B_{22}) + \ldots + (cA_{nn} + B_{nn}) \\ &= c(A_{11} + A_{22} + \ldots + A_{nn}) + (B_{11} + B_{22} + \ldots + B_{nn}) \\ &= cT(A) + T(B) \end{aligned}$$

A basis for $N(T)$ is the set of matrices A such that A has a 1 off its main diagonal, and 0 elsewhere, or A has a 1 in the top left and a -1 in some other diagonal entry, and 0 everywhere else. More formally, a basis is

$$B = \{A_1^{j_*,k_*} \mid j_*, k_* \in \{1, \ldots, n\} \text{ and } j_* \neq k_*\} \cup \{A_2^{j_*} \mid j_* = 2, \ldots, n\}$$

$$(A_1^{j_*,k_*})_{jk} = \begin{cases} 1 & \text{if } (j,k) = (j_*, k_*) \\ 0 & \text{otherwise} \end{cases} \qquad (A_2^{j_*})_{jk} = \begin{cases} 1 & \text{if } j = k = 1 \\ -1 & \text{if } j = k = j_* \\ 0 & \text{otherwise} \end{cases}$$

A basis for $R(T) = \{1\}$, so

$$nullity(T) + \text{rank}(T) = n^2 - 1 + 1 = n^2 = \dim(M_n(F))$$

Moreover, T is not injective, but it is surjective.

2.1.19. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by:

$$\begin{aligned} T(1,0,0) &= (0,0) \\ T(0,1,0) &= (1,0) \\ T(0,0,1) &= (0,1) \end{aligned}$$

Define $U : \mathbb{R}^3 \to \mathbb{R}^2$ by:

$$\begin{aligned} U(1,0,0) &= (0,0) \\ U(0,1,0) &= (0,1) \\ U(0,0,1) &= (1,0) \end{aligned}$$

Then $N(T) = \{(a,0,0) \mid a \in \mathbb{R}\} = N(U)$ and $R(T) = \mathbb{R}^2 = R(U)$. There are other examples of course.

2.1.20. Suppose V_1 is a subspace of V. We have$T(V_1)$ is a subspace of W by the subspace criterion:

(a) $0 \in V_1$ since V_1 is a subspace, so $0 = T(0) \in T(V_1)$.

(b) If $T(x), T(y) \in T(V_1)$ then $x, y \in V_1$ and since V_1 is a subspace, $x + y \in V_1$. Hence, $T(x) + T(y) = T(x + y) \in T(V_1)$ where the equality comes from linearity of T.

(c) If $T(x) \in T(V_1)$ and $c \in F$ then $cx \in V_1$ since its a subspace, so $cT(x) = T(cx) \in T(V_1)$ where again, the equality is from linearity.

Let $T^{-1}(W_1) = \{x \in V \mid T(x) \in W_1\}$. We have $T^{-1}(W_1)$ is a subspace of V by the subspace criterion:

(a) $0 \in T^{-1}(W_1)$ since $T(0) = 0 \in W_1$ because W_1 is a subspace and T is linear. $\implies 0 \in T_(W_1)$

(b) If $x, y \in T^{-1}(W_1)$ then $T(x), T(y) \in W_1$, so $T(x + y) = T(x) + T(y) \in W_1$ since W_1 is a subspace and T is linear. This gives $x + y \in T^{-1}(W_1)$.

(c) If $x \in T^{-1}(W_1)$ and $c \in F$, then $T(cx) = cT(x) \in W_1$ since W_1 is a subspace and T is linear. Hence, $cx \in T^{-1}(W_1)$.

2.2.2. a. $\begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 1 & -3 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 2 & 1 \\ -1 & 4 & 5 \\ 1 & 0 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & & \\ 1 & 0 & & \\ 1 & 0 & \text{\Large 0} & \\ \vdots & & & \\ 1 & 0 & & \end{bmatrix}$

f. $\begin{bmatrix} & & & & & 1 \\ & \text{\Large 0} & & & 1 & \\ & & & 1 & & \\ & & \cdots & & & \\ & 1 & & & & \\ 1 & & & \text{\Large 0} & & \\ 1 & & & & & \end{bmatrix}$

g. $\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & 1 \end{bmatrix}$

2.2.5. a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$

e. $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \end{bmatrix}$

f. $\begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix}$

g. $[a]$

2.2.14. Fix $n \in \mathbb{N}$. Suppose

$$a_1T_1 + a_2T_2 + \ldots + a_nT_n = 0 \tag{1}$$

By induction on $j \leq n$ we show $a_j = 0$.
Note that (1) means

$$a_1T_1(p(x)) + a_2T_2(p(x)) + \ldots + a_nT_n(p(x)) = 0 \tag{2}$$

for every polynomial $p(x) \in P(x)$.
Base case ($j = 1$):
Apply (2) to $p(x) = x$.
$a_1T_1(x) + a_2T_2(x) + \ldots + a_nT_n(x) = 0 \implies a_1 + 0 + \ldots + 0 = 0 \implies a_1 = 0$
Inductive step
Assume $a_1, a_2, \ldots, a_j = 0$; we want to show $a_{j+1} = 0$. Apply (2) to $p(x) = x^{j+1}$.
$a_1T_1(x^{j+1}) + a_2T_2(x^{j+1}) + \ldots + a_nT_n(x^{j+1}) = 0$
By assumption $a_1, a_2, \ldots, a_j = 0$, so $0 + \cdots + 0 + a_{j+1}T_{j+1}(x^{j+1}) + \ldots + a_nT_n(x^{j+1}) = 0 \implies (j+1)!a_{j+1} + 0 + \cdots + 0 = 0 \implies a_{j+1} = 0$
Hence, by induction, $a_1, \ldots, a_n = 0$, so $T_1, \ldots T_n$ are linearly independent.

2.2.16. This was done in detail in discussion, so some details may need to be filled in by the student. Assume $\dim(V) = \dim(W) = n$. Start with a basis $\{v_1, \ldots, v_m\}$ for $N(T)$. Inductively, for $i = m+1, \ldots, n$ find vectors

$$v_i \in V - span(v_1, \ldots v_m, \ldots, v_{i-1})$$

Then, $\beta = \{v_1, \ldots, v_n\}$ is linearly independent, so is a basis for V. For $i = m+1, \ldots, n$, let

$$w_i = T(v_i)$$

Show inductively that $\{w_{m+1}, \ldots, w_n\}$ is linearly independent. (If not, $w_i = c_{m+1}w_{m+1} + \cdots c_{i-1}wi - 1$ use this to show $v_i \in span(v_1 \ldots, v_{i-1})$ which is a contradiction.) So $\{w_{m+1}, \ldots, w_n\}$ is a linearly independent set in W, so we can extend it to a basis $\gamma = \{w_1, \ldots, w_n\}$. Then,

$$[T]_\beta^\gamma = \begin{bmatrix} \vec{0} & \cdots & \vec{0} & e_{m+1} & \cdots & e_n \end{bmatrix}$$

where e_j is the j^{th} standard basis vector in F^n i.e. the vector with a 1 in slot j and 0 elsewhere, and $\vec{0}$ denotes the zero the vector in F^n.

2.3.2. a. $A(2B+3C) = \begin{bmatrix} 20 & -9 & 18 \\ 5 & 10 & 8 \end{bmatrix}$

$(AB)P = A(BD) = \begin{bmatrix} 29 \\ -26 \end{bmatrix}$

b. $A^t = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{bmatrix}$

$A^tB = \begin{bmatrix} 23 & 19 & 40 \\ 26 & -1 & 10 \end{bmatrix}$

$BC^t = \begin{bmatrix} 12 \\ 16 \\ 29 \end{bmatrix}$

$CB = \begin{bmatrix} 27 & 7 & 9 \end{bmatrix}$

$CA = \begin{bmatrix} 20 & 26 \end{bmatrix}$

2.3.4. a. $\begin{bmatrix} 1 \\ -1 \\ 4 \\ 6 \end{bmatrix}$

b. $\begin{bmatrix} -6 \\ 2 \\ 0 \\ 6 \end{bmatrix}$

c. [5]

d. [12]

2.3.12. a. Suppose $x_1, x_2 \in V$ and $x_1 \neq x_2$. If $T(x_1) = T(x_2)$, then $UT(x_1) = UT(x_2)$. So UT is not injective. Contradiction.
No, U need not be injective. For example, let $T : \mathbb{R} \to \mathbb{R}^2 : 1 \mapsto (1, 0)$ and let $U : \mathbb{R}^2 \to \mathbb{R} : (1, 0) \mapsto 1, (0, 1) \mapsto 0$. Then $UT : \mathbb{R} \to \mathbb{R}$ is injective but U is not.

b. Let $z \in Z$. Then there is $v \in V$ such that $UT(v) = z$. So if $T(v) = w \in W$, then $U(w) = z$. So U is surjective.
No, T need not be surjective. The previous example works here too.

c. UT is injective: if $UT(v) = 0$, then $T(v) \in N(U)$. $N(U) = \{0\}$ by injectivity of U, so $T(v) = 0$. Similarly, injectivity of T gives that $v = 0$. So $N(UT) = \{0\}$.
UT is surjective: If $x \in Z$, then there is $w \in W$ such that $U(w) = z$. Also, there is $v \in V$ such that $T(v) = w$. Hence, $UT(v) = U(T(v)) = U(w) = z$.

2.3.13. (i) Using properties of summations and matrix multiplication:

$$\begin{aligned}
tr(AB) &= \sum_{l=1}^{n} (AB)_{ll} \\
&= \sum_{l=1}^{n} \left(\sum_{k=1}^{n} A_{lk} B_{kl} \right) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} A_{lk} B_{kl} \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} B_{kl} A_{lk} \\
&= \sum_{k=1}^{n} \left(\sum_{l=1}^{n} B_{kl} A_{lk} \right) \\
&= \sum_{k=1}^{n} (BA)_{kk} \\
&= tr(BA)
\end{aligned}$$

(ii) Note that, $(A^t)_{ii} = A_i i$ for $i = 1, \ldots, n$. So,

$$tr(A^t) = \sum_{i=1}^{n} A^t_{ii} = \sum_{i=1}^{n} A_{ii} = tr(A)$$