Math 141, Introduction to Topology
Sample Midterm

Problem 1.

a) What is a metric space?

b) Does the function $d(x, y) = \sqrt{|x - y|}$ define a metric space on $\mathbb{R}$?

Problem 2.

Let X, Y be metric spaces and $f : X \to Y$ be continuous. If X is compact, show that $f(X)$ is compact.

Problem 3.

Give an explicit example

a) of a countable union of closed subsets of $\mathbb{R}$ which is a proper subset of $\mathbb{R}$ and closed;

b) of a countable union of closed subsets of $\mathbb{R}$ which is a proper subset of $\mathbb{R}$ and open;

c) of a countable union of closed subsets of $\mathbb{R}$ which is not closed and not open.

Problem 4.

Let (X, d) be a complete metric space with $\{x_n\}$ a sequence in X. Suppose that $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ converges. Show that $\{x_n\}$ converges.

Problem 5.

Let $(X, \|\cdot\|)$ be a normed linear space, and $\mathcal{B}(X, X)$ be the space of bounded linear operators from X to itself. Prove that for any $T, S \in \mathcal{B}(X, X)$ the composition $T \circ S$ is also a bounded linear operator, and $\|T \circ S\| \leq \|T\| \cdot \|S\|$.