Final Sample

Problem 1.

Let ϕ be defined by $\phi(x) = \frac{15}{16}(x^2-1)^2$ for $|x|<1$ and $\phi(x)=0$ otherwise. Let f be a function with continuous derivative. Find the limit

$$\lim_{n\to\infty}\int_{-1}^{1} n^2\phi'(nx)f(x)dx.$$

Problem 2.

Let $\{a_n\}_{n\in\mathbb{N}}$ be sequence of nonzero real numbers. Prove that the sequence of functions

$$f_n(x) = \frac{1}{a_n}\sin(a_n x) + \cos(x+a_n)$$

has a subsequence converging to a continuous function.

Problem 3.

Let $C[0,1]$ be the space of continuous functions on $[0,1]$. Define

$$d(f,g) = \int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}dx.$$

Show that d is a metric on $C[0,1]$. Is the metric space $(C[0,1],d)$ complete?

Problem 4.

Let $f:[0,\infty)\to\mathbb{R}$ be a uniformly continuous function such that the improper integral $\int_0^\infty f(x)dx$ converges (and is finite). Show that

$$\lim_{x\to\infty} f(x) = 0.$$

Problem 5.

Let f be a 2π-periodic C^2 function on $\mathbb{R}$ with zero average over the period, that is $\int_{-\pi}^{\pi} f(x)dx = 0$. Show that

$$\int_{-\pi}^{\pi}(f(x))^2dx \le \int_{-\pi}^{\pi}(f'(x))^2dx.$$