Real Analysis
Math 205B/H140B, Winter 2016

Homework 7, due February 26, 2016 in class

Problem 1.

For the linear functional $L : C[0,1] \to R$,

$$L(f) = f(0) + f(1/2) + f(1),$$

find the function $\alpha \in BV[0,1]$ such that $L(f) = \int_0^1 f d\alpha$ for all $f \in C[0,1]$.

Problem 2.

For the linear functional $L : C[0,1] \to R$,

$$L(f) = f(0) - \frac{1}{2}f(1/2) + \frac{1}{5}f(1),$$

find the function $\alpha \in BV[0,1]$ such that $L(f) = \int_0^1 f d\alpha$ for all $f \in C[0,1]$.

Problem 3.

For the linear functional $L : C[0,1] \to R$,

$$L(f) = \int_0^{0.3} f(x)dx - \int_{0.6}^1 f(x)dx,$$

find the function $\alpha \in BV[0,1]$ such that $L(f) = \int_0^1 f d\alpha$ for all $f \in C[0,1]$.

Problem 4.

For the linear functional $L : C[0,1] \to R$,

$$L(f) = \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{f(x)}{2^n} dx,$$

find the function $\alpha \in BV[0,1]$ such that $L(f) = \int_0^1 f d\alpha$ for all $f \in C[0,1]$.

Problem 5.

For the linear functional $L : C[0,1] \to R$,

$$L(f) = \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} (-1)^n \frac{f(x)}{2^n} dx,$$

find the function $\alpha \in BV[0,1]$ such that $L(f) = \int_0^1 f d\alpha$ for all $f \in C[0,1]$.

Problem 6.

For the linear functional $L : C[0,1] \to R$,

$$L(f) = \int_0^{1/2} \frac{f(x) + 2f(1-x)}{3} dx,$$

find the function $\alpha \in BV[0,1]$ such that $L(f) = \int_0^1 f d\alpha$ for all $f \in C[0,1]$.

Problem 7.

Suppose that $f \in C[0,1]$ and for any $n \in \mathbb{N}$ we have $\int_0^1 f(x)e^{nx}dx = 0$. Does it imply that $f \equiv 0$?

Problem 8.

Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that

$$\int_0^1 |f'(x)|^2 dx = A^2, \quad A \geq 0.$$

Prove that

$$|f(x) - f(y)| \leq A|x-y|^{1/2}$$

for all $x, y \in [0,1]$.

Problem 9.

Find

$$\lim_{n\to\infty} \int_0^1 \frac{e^x}{1+x^2} \cos(nx) \sin(n^2 x) dx$$

Problem 10.

Prove that if $f : [0,1] \to \mathbb{R}$ is Riemann integrable, then

$$\lim_{n\to\infty} \int_0^1 f(x) \cos^2(nx) dx = \frac{1}{2} \int_0^1 f(x) dx$$