Midterm Sample

Problem 1.

For which values of real parameters α, β does the function $f : [0,1] \to \mathbb{R}$,

$$\begin{cases} 0, & x = 0; \\ x^\alpha \sin x^\beta, & x > 0, \end{cases}$$

have bounded variation?

Problem 2.

Suppose that $\{f_\alpha\}_{\alpha \in A}$, $f_\alpha : [0,1] \to \mathbb{R}$, is an equicontinuous family. Suppose also that $f_\alpha(0) = 0$ for all $\alpha \in A$. Prove that $\{F_\alpha\}_{\alpha \in A}$, $F_\alpha : [0,1] \to \mathbb{R}$, $F_\alpha(x) = \int_0^x f_\alpha(t)dt$, is also an equiucontinuous family.

Problem 3.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f_n(x) = f(nx)$, $n \in \mathbb{N}$, and the family of functions $\{f_n\}_{n \in \mathbb{N}}$ is equicontinuous on $[-1,1]$. What conclusion can you draw about f?

Problem 4.

Let f_n, $n = 1, 2, 3, \ldots$, and f be Riemann integrable real-valued functions defined on $[0,1]$. For each of the following statements, determine whether the statement is true or not:

(a) If $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|\, dx = 0$ then $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|^2\, dx = 0$;

(b) If $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|^2\, dx = 0$ then $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|\, dx = 0$.

Problem 5.

Prove that the space $C^1[0,1]$ (the space of continuously differentiable functions on $[0,1]$ with the metric $\|f - g\|_{C^1} = \max(\|f - g\|_\infty, \|f' - g'\|_\infty)$ is connected.