Final Sample

Problem 1.

Let $M_{n\times n}$ denote the space of $n \times n$ matrices over $\mathbb{R}$, identified in the usual way with $\mathbb{R}^{n\times n}$. Let the function F from $M_{n\times n}$ to $M_{n\times n}$ be defined by

$$F(X) = X^3 + X^2 + X.$$

Find the derivative of the function F.

Problem 2.

a) TRUE or FALSE: For any continuously differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$ with $grad\ f \neq 0$ at $0 \in \mathbb{R}^3$, there exist two other continuously differentiable functions $g, h : \mathbb{R}^3 \to \mathbb{R}$ such that the map

$$(x, y, z) \mapsto (f(x, y, z), g(x, y, z), h(x, y, z))$$

from $\mathbb{R}^3$ to $\mathbb{R}^3$ is one-to-one in some neighborhood of the origin.

b) TRUE or FALSE: For any continuously differentiable functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ with $grad\ f \neq 0$, $grad\ g \neq 0$ at $0 \in \mathbb{R}^3$, there exists a continuously differentiable function $h : \mathbb{R}^3 \to \mathbb{R}$ such that the map

$$(x, y, z) \mapsto (f(x, y, z), g(x, y, z), h(x, y, z))$$

from $\mathbb{R}^3$ to $\mathbb{R}^3$ is one-to-one in some neighborhood of the origin.

Problem 3.

Consider the forms $\omega = e^{xy+z^2}dx$, $\theta = xdx + ydy + zdz$, and $\eta = xdy + x^2dy$. Find $d\omega$, $\theta \wedge \eta$, $d(\omega \wedge \eta)$, and $\int_\gamma \theta$, where the curve γ is given by $\gamma(t) = (t(t-1), t^2\sin(2\pi t), 1 - \cos(2\pi t))$.

Problem 4.

Suppose $f, g : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions, and γ is a smooth closed curve without self intersections, positively oriented. Denote by $D \subset \mathbb{R}^2$

the domain enclosed by γ. Prove that

$$\int_\gamma < f\mathrm{grad}\, g, \bar{n} > ds = \int_D (f\Delta g + < \mathrm{grad}\, f, \mathrm{grad}\, g >) dA,$$

where Δg is the Laplacian of the function g, and $\bar{n}$ is a normal to the curve γ pointing outward of the domain D.

Problem 5.

Let S be the surface obtained by rotating the curve

$$x = \cos t, y = \sin 2t, \; t \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right],$$

around y-axis. Find the volume of the region inside of S.