Real Analysis
Math 205C/H140C, Spring 2016

Homework 2, due April 15, 2016 in class

Problem 1.

Prove that a differentiable function $f : \mathbb{R} \to f(\mathbb{R})$, $f(\mathbb{R}) \subseteq \mathbb{R}$, with $f'(x) \neq 0$ for all $x \in \mathbb{R}$ must be a bijection.

Problem 2.

TRUE or FALSE: A differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ with $\det Df(x) \neq 0$ for all $x \in \mathbb{R}^n$ must be a bijection.

Problem 3.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at every point and such that $f(0) = 0$ and $f'(0) \neq 0$. Is it true that f is locally invertible in a neighborhood of zero? In other words, can one replace the condition $f \in C^1$ in the Inverse Function Theorem by merely differentiability of f?

Problem 4.

Let $U \subset \mathbb{R}^n$ be an open set, and $f : U \to \mathbb{R}^n$ be a C^1 function such that $\det Df(x) \neq 0$ for all $x \in U$. Prove that $f(U)$ is also open.

Problem 5.

Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(r, \theta) = (x(r, \theta), y(r, \theta)) = (r\cos\theta, r\sin\theta)$. Find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$. At what points the map f is locally invertible?

Problem 6.

Consider the system of equations

$$\begin{cases} x^2 - y^2 - u^3 + v^2 + 4 = 0 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0 \end{cases}$$

Prove that there are neighborhoods U of $(2, -1)$ and W of $(2, 1)$ such that for each $(x, y) \in U$ there is a unique (u, v) in W such that (x, y, u, v) is a solution of the system.

Problem 7.

Do the conditions

$$\begin{cases} 2x + y + 2z + u - v = 1 \\ xy + z - u + 2v = 1 \\ yz + xz + u^2 + v = 0 \end{cases}$$

define the first three variables (x, y, z) as a function $\phi(u, v)$ near the point $(x, y, z, u, v) = (1, 1, -1, 1, 1)$? If so, find the derivative matrix $D\phi(1, 1)$.

Problem 8.

Consider the polynomial $x^{2016} - 1$. Notice that it has a simple root 1. Let $c(\lambda)$ be a continuous family of roots of polynomials $x^{2016} + \lambda x^{10} - 1$, where λ is close to zero, and $c(0) = 1$. Prove that $c(\lambda)$ is a function that is well defined and differentiable in some neighborhood of zero, and find $\frac{dc(\lambda)}{d\lambda}(0)$.

Problem 9.

Let $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ be of class C^1. Suppose that $f(x_0) = 0$ and $Df(x_0)$ has rank n. Show that for any point $\xi \in \mathbb{R}^n$ sufficiently close to the origin, the equation $f(x) = \xi$ has a solution.

Problem 10.

Suppose $f(x, y) \in C^1$ and $f(0, 0) = 0$. What conditions on f will guarantee that the equation $f(f(x, y), y) = 0$ can be solved for y as a C^1 function of x near $(0, 0)$?