Real Analysis
Math 205C/H140C, Spring 2016

Homework 3, due April 22, 2016 in class

Problem 1.

Let $S \subset \mathbb{R}^n$ be a set, and a collection $\{U_\alpha\}_{\alpha \in A}$ of open rectangles be such that $S \subseteq \cup_{\alpha \in A} U_\alpha$. Prove that one can find a countable sequence $\{U_{\alpha_i}\}_{i \in \mathbb{N}}$ such that $S \subseteq \cup_{i \in \mathbb{N}} U_{\alpha_i}$.

Problem 2.

Let $f : [0,1] \to \mathbb{R}$ be a positive continuous function. Use Fubini's theorem to derive an expression for the volume of the set $\{(x,y,z) \mid z \in [0,1],\ x^2 + y^2 \leq f(z)\} \subset \mathbb{R}^3$.

Problem 3.

Let $Q = [0,1] \times [0,1] \times [0,1] \subset \mathbb{R}^3$, $f : Q \to \mathbb{R}$ be a continuous function, and

$$F(x_1, x_2, x_3) = \int_{[0,x_1] \times [0,x_2] \times [0,x_3]} f.$$

Find DF.

Problem 4.

Let $Q = [0,1] \times [0,1]$, $f : Q \to \mathbb{R}$ be a C^1 function, and define

$$F(x) = \int_0^1 f(x,y)dy.$$

Prove that $F'(x) = \int_0^1 \frac{\partial f(x,y)}{\partial x} dy$.

Problem 5.

TRUE or FALSE: Finite union of rectifiable sets in $\mathbb{R}^n$ is rectifiable.

Problem 6.

TRUE or FALSE: Countable union of rectifiable sets in $\mathbb{R}^n$ is rectifiable.

Problem 7.

TRUE or FALSE: Every compact subset of $\mathbb{R}^2$ is rectifiable.

Problem 8.

TRUE or FALSE: Every set $E \subset \mathbb{R}^n$ that has zero measure is rectifiable.

Problem 9.

Give an example of a bounded open set $U \subset \mathbb{R}^2$ and a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\int_{\bar{U}} f$ exists but $\int_U f$ does not.

Problem 10.

Suppose that $f : [0,1] \times [0,1] \to \mathbb{R}$ is bounded, and

- for each $x \in [0,1]$ the function $g_x : [0,1] \to \mathbb{R}$, $g_x(y) = f(x,y)$, is integrable;
- for each $y \in [0,1]$ the function $h_y : [0,1] \to \mathbb{R}$, $h_y(x) = f(x,y)$, is integrable;
- $\int_0^1 \left(\int_0^1 g_x(y)dy \right) dx = \int_0^1 \left(\int_0^1 h_y(x)dx \right) dy$.

Does it imply that the function $f : [0,1] \times [0,1] \to \mathbb{R}$ is integrable, and

$$\int_{[0,1]\times[0,1]} f = \int_0^1 \left(\int_0^1 g_x(y)dy \right) dx = \int_0^1 \left(\int_0^1 h_y(x)dx \right) dy \quad ?$$