REAL ANALYSIS
MATH 205C/H140C, SPRING 2016

Homework 4, due May 2, 2016 in class

Problem 1.

Denote by $\mathbb{D}$ the unit disc in $\mathbb{R}^2$, $\mathbb{D} = \{(x, y) \mid x^2 + y^2 < 1\}$, and by $\mathbb{D}_{1-\varepsilon}$ - the disc centered at $(0, 0)$ of radius $1 - \varepsilon$. Suppose that $f : \mathbb{D} \to \mathbb{R}$ is a non-negative continuous function. Prove that $\int_{\mathbb{D}} f$ exists if and only if the limit $\lim_{\varepsilon \to 0+} \int_{\mathbb{D}_\varepsilon} f$ exists.

Problem 2.

TRUE or FALSE: Suppose $f : \mathbb{D} \to \mathbb{R}$ is a continuous function (not necessarily non-negative). Then $\int_{\mathbb{D}} f$ exists if and only if the limit $\lim_{\varepsilon \to 0+} \int_{\mathbb{D}_\varepsilon} f$ exists.

Problem 3.

Set $A = \{(x, y) \in \mathbb{R}^2 \mid x > 1, y > 1\} \subset \mathbb{R}^2$, and let $f : A \to \mathbb{R}$ be given by $f(x, y) = \frac{1}{x^3 y^2}$. Does the integral $\int_A f$ exist? Explain. If yes, find it.

Problem 4.

Set $A = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1,\ 0 < y < 1\} \subset \mathbb{R}^2$, and let $f : A \to \mathbb{R}$ be given by $f(x, y) = \frac{1}{x^3 y^2}$. Does the integral $\int_A f$ exist? Explain. If yes, find it.

Problem 5.

Give an example of a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the integral $\int_{\mathbb{R}^2} f$ exists.

Problem 6.

Suppose that for some continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ and a sequence of compact rectifiable subsets $C_n \subset \mathbb{R}^2$ we have $\cup_{n=1}^{\infty} C_n = \mathbb{R}^2$, $C_n \subset \operatorname{int} C_{n+1}$, and the limit $\lim_{n\to\infty} \int_{C_n} f$ exists (and is finite). Does it imply that the integral $\int_{\mathbb{R}^2} f$ exists?

Problem 7.

Let $f(x, y) = \frac{1}{(x+y)^2}$, and the sets $A, B, C, D \subset \mathbb{R}^2$ be given by

$$A = \left\{ (x, y) \in \mathbb{R}^2 \mid x > 0,\ \ y \in \left(\frac{1}{2}x, 2x \right) \right\},$$

$$B = \left\{ (x, y) \in \mathbb{R}^2 \mid x > 0,\ \ y \in \left(\frac{1}{2}x^2, 2x^2 \right) \right\},$$

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid x > 0,\ y > 0,\ \ y < x + x^2,\ \ x < y + y^2 \right\},$$

$$D = \left\{ (x, y) \in \mathbb{R}^2 \mid x > 0,\quad x < y < x + x^2 \right\}.$$

Which of the integrals $\int_A f, \int_B f, \int_C f, \int_D f$ do exist? Explain.

Problem 8.

TRUE or FALSE: For any compact subset $C \subset \mathbb{R}^n$ there exists a C^∞ function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $supp\, \varphi = C$.

Problem 9.

TRUE or FALSE: For any bounded open subset $U \subset \mathbb{R}^1$ there exists a C^∞ function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $supp\, \varphi = \overline{U}$.

Problem 10.

TRUE or FALSE: For any bounded open subset $U \subset \mathbb{R}^n$ there exists a C^∞ function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $supp\, \varphi = \overline{U}$.