REAL ANALYSIS
MATH 205C/H140C, SPRING 2016

Homework 6, due May 19, 2016 in class

Problem 1.

Prove that a bilinear form $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is alternating if and only if $\phi(v, v) = 0$ for all $v \in \mathbb{R}^3$.

Problem 2.

Is it true that $\omega \wedge \omega = 0$ for any k-form (with $k \geq 1$)?

Problem 3.

Let ω be an exterior k-form, where k is an odd integer. Show that $\omega \wedge \omega = 0$.

Problem 4.

Let ϕ, ψ, and θ be the following forms in $\mathbb{R}^3$:

$$\begin{aligned} \phi &= xdx - ydy \\ \psi &= zdx \wedge dy + xdy \wedge dz \\ \theta &= zdy \end{aligned}$$

Compute $\phi \wedge \psi$, $\theta \wedge \phi \wedge \psi$, $d\phi$, $d\psi$, $d\theta$.

Problem 5.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map given by

$$f(x_1, \ldots, x_n) = (y_1, \ldots, y_n),$$

and let $\omega = dy_1 \wedge dy_2 \wedge \ldots dy_n$. Show that

$$f^*\omega = det(Df)dx_1 \wedge \ldots dx_n.$$

Problem 6.

Given a k-form ω in $\mathbb{R}^n$, define an $(n-k)$-form $\star\omega$ by setting $\star(dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = sgn(\sigma)(dx_{j_1} \wedge \ldots \wedge dx_{j_{n-k}})$, and extending it by linearity, where $i_1 < i_2 < \ldots i_k$, $j_1 < \ldots < j_{n-k}$, and σ is a permutation of the set $(1, 2, \ldots, n)$ given by $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$. This is *Hodge star operator*.
If $\omega = a_{12}dx_1 \wedge dx_2 + a_{13}dx_1 \wedge dx_3 + a_{23}dx_2 \wedge dx_3$ is a 2-form in $\mathbb{R}^3$, find $\star\omega$.

Problem 7.

Prove that $\star \star \omega = (-1)^{k(n-k)}\omega$.

Problem 8.

Show that the form $\omega = 2xy^3dx + 3x^2y^2dy$ is closed and compute $\int_c \omega$, where c is the arc of the parabola $y = x^2$ from $(0, 0)$ to (x, y).

Problem 9.

Let ω be a 1-form in an open connected set $U \subset \mathbb{R}^n$. Assume that for each closed differentiable curve c in U, the integral $\int_c \omega$ is a rational number. Prove that ω is closed. Does it have to be exact in U?

Problem 10.

Let ω be a closed 1-form in $\mathbb{R}^2 \setminus \{(0,0)\}$. Assume that ω is bounded (i.e. its coefficients are bounded) in a disc centered at $(0,0)$. Show that ω is exact in $\mathbb{R}^2 \setminus \{(0,0)\}$.