Midterm Sample

Problem 1.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} x^{2016}\left(1-\cos\left(\frac{y}{x}\right)\right), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Determine all the points at which f is differentiable.

Problem 2.

Denote $U = \{(x,y) \mid x+y>0\} \subset \mathbb{R}^2$. Let the transformation $f : U \to \mathbb{R}^2$ be given by $f(x,y) = (x-y, x^2+y^2)$. Prove that f is locally one-to-one. Is it globally one-to-one?

Problem 3.

Suppose $q(x)$ is a continuous function on $[0,1]$. Set $Q_1(x) = \int_0^x q(t)dt$, and

$$Q_{k+1}(x) = \int_0^x q(t)Q_k(t)dt$$

for $k \geq 1$. Prove that $Q_k(1) = \frac{1}{k!}\left(Q_1(1)\right)^k$ for each $k \in \mathbb{N}$.

Problem 4.

Let $U = \{(x,y) \in \mathbb{R}^2 \mid x>0, y>0, 1<x^2+y^2<4\}$. Find $\int_U f(x,y)$, where $f(x,y) = \frac{xy}{x^2+y^2}$.

Problem 5.

Consider the set $S = \{(x,y,z) \mid x^2+y^2+z^2-2xyz=1\} \subset \mathbb{R}^3$. Find all the points of the set S that do not have a neighborhood where the set can be represented as a graph of a smooth function.