Complex Analysis, HW # 5

Problem 1.

TRUE or FALSE: Suppose $v : \mathbb{C} \to \mathbb{R}$ is a continuous function such that for any point $z \in \mathbb{C}$ and any $n \in \mathbb{N}$ we have $\frac{1}{2\pi}\int_0^{2\pi} v(z + \frac{e^{i\theta}}{n})d\theta = v(z)$. Then v is harmonic.

Problem 2.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Prove that for any $r \geq 0$ one has

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} r^{2n}|a_n|^2$$

Problem 3.

If f is an entire function satisfying the estimate

$$|f(z)| \leq 1 + |z|^{2015+\sqrt{2}\sin|z|}, \quad z \in \mathbb{C},$$

show that f is a polynomial and determine the best upper bound for the degree of f.

Problem 4.

Find a conformal transformation which maps G onto the unit disk $\mathbb{D}$, where

$$G = \left\{ z \in \mathbb{C} : |z| > 1 \text{ and } \left| z - \frac{1+i}{2} \right| < \frac{1}{\sqrt{2}} \right\}.$$

Problem 5.

a) Does there exist a function f holomorphic in $\mathbb{C}\backslash\{0\}$, such that $|f(z)| \geq \frac{1}{\sqrt{|z|}}$ for all $z \in \mathbb{C}\backslash\{0\}$?

b) Does there exist a function f holomorphic in $\mathbb{C}\backslash\{0\}$, such that $|f(z)| \leq \frac{1}{\sqrt{|z|}}$ for all $z \in \mathbb{C}\backslash\{0\}$?

Problem 6.

Let f be a meromorphic function in $\mathbb{C}$ and such that $|f(z)| = 1$ when $|z| = 1$. Prove that f is a rational function.

Problem 7.

Find explicitly all the conformal automorphisms of the first quadrant.