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Nonremovable Zero Lyapunov Exponents*

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ABSTRACT. Skew products over a Bernoulli shift with a circle fiber are studied. We prove that in the space of such products there exists a nonempty open set of mappings each of which possesses an invariant ergodic measure with one of the Lyapunov exponents equal to zero. The conjecture that the space of C^2-diffeomorphisms of the 3-dimensional torus into itself has a similar property is discussed.

KEY WORDS: Lyapunov exponent, partially hyperbolic system, nonuniform hyperbolicity, dynamical system, skew product, Bernoulli diffeomorphism

1. Introduction

To what extent is the behavior of a generic dynamical system hyperbolic?

A number of problems in modern theory of smooth dynamical systems can be viewed as some forms of this question. It was shown in the 1960s that uniformly hyperbolic systems (Anosov diffeomorphisms, Axiom A) are not dense in the space of dynamical systems [1]. This necessitated weakening the notion of hyperbolicity. The notions of partial and (important here) nonuniform hyperbolicity appeared (Pesin's theory [2]). In Pesin's theory, hyperbolic behavior is characterized by nonzero Lyapunov exponents for some invariant measure. The most natural case is that of a system with a smooth invariant measure. This case was studied in various aspects, for example, in [3–7]. However, the question about Lyapunov exponents can also be considered for maps that do not *a priori* have a natural invariant measure.

An invariant measure is said to be *good* if it can be obtained from Lebesgue measure by the Krylov–Bogolyubov procedure.

Problem 1. Does a generic smooth dynamical system on a compact Riemannian manifold have nonzero Lyapunov exponents for each good measure?

This problem (in a slightly different form) was posed by M. Shub and A. Wilkinson [6] in connection with the question about existence of SRB-measures for a generic dynamical system and is still open. Even if one removes any conditions on the invariant measures, the problem remains open and meaningful.

Conjecture 1. In the space of diffeomorphisms of the three-dimensional torus, there exists an open set of mapping having an ergodic invariant measure with one of the Lyapunov exponents equal to zero.

This conjecture suggests that the answer in Problem 1 may be negative, although the conjecture and the problem are very far from each other.

We intend to present the proof of the conjecture in a series of papers, where M. Nalsky is the main author. The present paper is the first paper in the series.

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2. Main Results

The following theorem is the main result of the paper.

Consider the Bernoulli shift $\sigma\colon \Sigma^2 \to \Sigma^2$ on the space of two-way sequences of zeros and ones. Let

$$f_j\colon S^1 \to S^1, \qquad j = 0, 1, \tag{1}$$

be two diffeomorphisms of a circle. We set $M = \Sigma^2 \times S^1$. Consider the step skew product

$$F\colon M \to M, \qquad (\omega, x) \to (\sigma\omega, f_{\omega_0}(x)). \tag{2}$$

We use the term "*step* skew product" since the map on a fiber depends only on the zeroth element of the corresponding sequence in the base and hence resembles a step function on Σ^2.

Theorem 1. *In the space* $(\mathrm{Diff}^1(S^1))^2$ *of pairs of diffeomorphisms of the circle equipped with the* C^1*-topology, there exists an open set* U *such that for each pair in* U *the corresponding step skew product* (2) *has an invariant ergodic measure with zero Lyapunov exponent along the fiber.*

The definition of Lyapunov exponent along the fiber for the map (2) is given at the beginning of Sec. 4.

Theorem 1 follows from Theorems 2 and 3.

Theorem 2. *Consider the set of pairs of mappings* (1) *such that*

(i) *the action of the semigroup* $G^+(f_0, f_1)$ *generated by the diffeomorphisms* f_0 *and* f_1 *is minimal*; *that is, each orbit is dense in* S^1;

(ii) *for each point* $x \in S^1$, *there exists a* $j \in \{0, 1\}$ *such that* $|f_j'(x)| > 1$;

(iii) *there exists a diffeomorphism* $f \in G^+(f_0, f_1)$ *with a hyperbolic attracting periodic point.*

Then the corresponding step skew product (2) *has an ergodic invariant measure with zero Lyapunov exponent along the fiber.*

Remark 1. It follows from an analysis of the proofs and some considerations like the compactness of the circle that one can weaken condition (ii) in the following way:

(ii′) *for each point* $x \in S^1$, *there exists an element* $g \in G^+(f_0, f_1)$ *such that*

$$|g'(x)| > 1.$$

Supplement to Theorem 2. *The measure in Theorem* 2 *can be taken to be nonatomic.*

Theorem 3. *The space* $(\mathrm{Diff}^1(S^1))^2$ *contains a nonempty open subset of pairs possessing properties* (i), (ii), *and* (iii).

We intend to derive the conjecture stated above from Theorem 1 in the following way.

Each system F of the form (2) in Theorem 1 admits a "smooth realization" in the sense of [10]. Namely, there exists a smooth self-map $\mathscr{F}$ of the three-dimensional torus with an invariant partially hyperbolic set Λ foliated by circles such that the following holds. The restriction of $\mathscr{F}$ to Λ is topologically conjugated to F, and the restriction of the conjugacy to each central fiber is a smooth map. A small perturbation of $\mathscr{F}$ has an invariant set homeomorphic to Λ (see [8]). In a subsequent paper, which is now in preparation, we intend to prove that the conjecture holds for all maps in a small neighborhood of $\mathscr{F}$. Note that for the complete realization of this program one should take an initial map $\mathscr{F}$ of class C^2 and consider a C^2-small neighborhood of $\mathscr{F}$. In the present paper, we consider a C^1-map F and a C^1-neighborhood in the space of step skew products.

This plan essentially uses the approach in [9–11] and is a continuation of these studies.

3. The Structure of the Paper

To prove Theorem 2, we construct the desired measure as a limit of invariant measures that are uniformly distributed on periodic orbits and whose Lyapunov exponents tend to zero. Moreover, these periodic orbits will be in some sense "similar" to each other.

The idea is realized as follows. In Sec. 4, we show that if the limit of a sequence of ergodic measures is ergodic, then the Lyapunov exponent along the fiber for the limit measure is the limit of Lyapunov exponents for the terms of the sequence. In Sec. 5, sufficient conditions for the ergodicity of the limit measure are given. (One of the authors has been inspired by the idea, due to Katok and Stepin, of approximating ergodic systems by periodic systems; the paper [17] has influenced the content of Sec. 5 most substantially.) Section 6 describes how, given a periodic orbit, to construct another periodic orbit sufficiently similar to the original orbit but with larger period and smaller absolute value of the Lyapunov exponent. After that, in Sec. 7, we construct the desired sequence of periodic orbits and verify that it satisfies the sufficient conditions for the ergodicity of the limit measure. This proves Theorem 2.

Moreover, in Sec. 8 we prove the Supplement to Theorem 2, which claims that the measure found in the theorem is nonatomic.

Finally, in Sec. 9 we give an example of a C^1-open set of pairs of diffeomorphisms satisfying Theorem 2. This proves Theorem 3.

4. Ergodicity and Lyapunov exponents

Definition 1. Let F be a skew product of the form (2).

The *Lyapunov exponent along the fiber at a point* (w, x) is the following number (defined at the points where the limit exists):

$$\lambda^c(w, x) := \lim_{n\to\infty} \frac{1}{n} \ln |(f_{w_n} \circ f_{w_{n-1}} \circ \cdots \circ f_{w_0})'(x)|.$$

If the map F has an ergodic probability measure ν, then for ν-almost all points the Lyapunov exponent along the fiber is well defined and independent of the point. In this case, the function $\lambda^c(w, x)$ is a constant on a set of ν-full measure and one can speak of the Lyapunov exponent along the fiber with respect to the measure ν. (We denote it by $\lambda^c(\nu)$.)

When speaking of convergence of measures, we always mean $*$-weak convergence: μ_n converges to μ if

$$\int \varphi \, d\mu_n \longrightarrow \int \varphi \, d\mu \quad \text{as } n \to \infty$$

for each continuous function φ.

Lemma 1. *Let μ_n and μ be ergodic probability measures for the skew product F, and let $\mu_n \to \mu$ as $n \to \infty$. Then $\lambda^c(\mu_n) \to \lambda^c(\mu)$.*

Proof. By definition, the Lyapunov exponent along the fiber at a point x is the time average of the function $\varphi = \ln(\partial F/\partial x)$ at x. It follows from the ergodicity of the measures μ_n (respectively, μ) that this time average is equal to the space average of φ with respect to the corresponding measure for μ_n- (respectively, μ-)almost every point. The function φ is continuous. Since $\mu_n \to \mu$ $*$-weakly, we have

$$\lambda^c(\mu_n) = \int \varphi \, d\mu_n \longrightarrow \int \varphi \, d\mu = \lambda^c(\mu).$$

5. Sufficient Conditions for Ergodicity

Let G be an arbitrary continuous self-map of a metric compact space Q. Let X_n be periodic orbits of G, P_n their periods, and μ_n atomic measures uniformly distributed on the respective X_n.

Definition 2. The *n-measure of a point* x_0 is the atomic measure uniformly distributed on n successive iterations of x_0 under G:

$$\nu_n(x_0) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{G^i(x_0)},$$

where δ_x is the δ-measure supported at a point x.

Proposition 1. *The ergodicity of a measure μ is equivalent to the $*$-weak convergence of the n-measures of μ-almost all points to μ.*

Proof. By definition of the n-measure, we have

$$\forall \varphi \in C(Q),\ x_0 \in Q \quad \int_Q \varphi\, d\nu_n(x_0) = \overline{\varphi}_n(x_0) := \frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ G^i(x_0).$$

It follows that the convergence of the n-measures to μ is equivalent to the convergence of the partial time averages $\overline{\varphi}_n(x_0)$ to the space average $\overline{\varphi} := \int \varphi\, d\mu$. The latter convergence on a set of full μ-measure is just the ergodicity of μ. □

The following lemma is a key point in the proof of the ergodicity of the limit measure.

Lemma 2. *Let G be a continuous self-map of a compact metric set Q, let $\{X_n\}$ be a sequence of periodic orbits with increasing periods P_n, let μ_n be atomic probability measures uniformly distributed on the respective orbits, and let μ be a limit point of the sequence $\{\mu_n\}$.*

Suppose that for each continuous function φ on Q and each $\varepsilon > 0$ there exists an $N = N(\varepsilon, \varphi) \in \mathbb{N}$ such that for each $m > N$ there exists a subset $\widetilde{X}_{m,\varepsilon} \subset X_m$ satisfying the following conditions:

(1) $\mu_m(\widetilde{X}_{m,\varepsilon}) > 1 - \varepsilon$;

(2) *for each n such that $m > n \geqslant N$ and for all $x \in \widetilde{X}_{m,\varepsilon}$, one has*

$$\left| \int \varphi\, d\nu_{P_n}(x) - \int \varphi\, d\mu_n \right| < \varepsilon.$$

Then the measure μ is ergodic.

Remark 2. In Secs. 6 and 7, for a skew product F satisfying the assumptions of Theorem 2 we construct periodic orbits that satisfy the assumptions of Lemma 2 and whose Lyapunov exponents along the fiber tend to zero. Lemma 2 implies the ergodicity of a limit measure for the sequence of measures uniformly distributed on these periodic orbits. Lemma 1 implies that the Lyapunov exponent along the fiber for the map F with respect to the limit measure is zero. This will prove Theorem 2.

Proof of Lemma 2. By passing to a subsequence, we can assume without loss of generality that μ is the limit of μ_n.

By Proposition 1, the ergodicity of μ is equivalent to the $*$-weak convergence of the measures $\nu_n(x)$ to μ. Therefore, we need to prove the existence of a set $\widetilde{X}$ of full μ-measure such that

$$\forall \varphi \in C(Q),\, x \in \widetilde{X} \quad \nu_n(x) \to \mu \quad \text{as } n \to \infty.$$

The following auxiliary statement holds [13].

Proposition 2. *Let $\mu_m \to \mu$, and let Y_m be μ_m-measurable sets. Consider the upper topological limit*

$$Y := \overline{\text{lim top}}_{m\to\infty} Y_m = \{y \mid \exists m_i \to \infty,\, y_i \in Y_{m_i} \colon y_i \to y\}.$$

Then $\mu(Y) \geqslant \overline{\lim}_{m\to\infty} \mu_m(Y_m)$.

We take arbitrary $\varphi \in C(Q)$ and $\varepsilon > 0$ and choose an $n > N(\varepsilon, \varphi)$. Consider the set

$$\widetilde{X}_\varepsilon = \overline{\text{lim top}}_{m\to\infty} \widetilde{X}_{m,\varepsilon},$$

By Proposition 2, $\mu(\widetilde{X}_\varepsilon) \geqslant 1 - \varepsilon$.

By the definition of $\widetilde{X}_\varepsilon$, for each $x \in \widetilde{X}_\varepsilon$ there exists a sequence $x_{m_i} \in \widetilde{X}_{m_i,\varepsilon}$ that converges to x. Hence for each $n > N$ we have

$$\left| \int \varphi\, d\nu_{P_n}(x) - \int \varphi\, d\nu_{P_n}(x_{m_i}) \right| < \varepsilon \tag{3}$$

provided that m_i is sufficiently large, since G is continuous and $\widetilde{X}_\varepsilon$ is compact.

The assumptions of the lemma imply that

$$\left|\int \varphi\, d\nu_{P_n}(x_{m_i}) - \int \varphi\, d\mu_n\right| < \varepsilon \tag{4}$$

for $m_i > n$.

Since $\mu_n \to \mu$, we have, for sufficiently large n,

$$\left|\int \varphi\, d\mu_n - \int \varphi\, d\mu\right| < \varepsilon. \tag{5}$$

By combining (3)–(5), we obtain

$$\left|\int \varphi\, d\nu_{P_n}(x) - \int \varphi\, d\mu\right| < 3\varepsilon. \tag{6}$$

We proved that for arbitrary φ and ε and for all $x \in \widetilde{X}_\varepsilon$ the sequence of partial time averages $\overline{\varphi}_k(x)$ has countably many elements $\overline{\varphi}_{P_n}(x)$ in the (3ε)-neighborhood of the space average $\overline{\varphi} := \int \varphi\, d\mu$.

By the Birkhoff–Khinchin theorem, there exists a set Y of full μ-measure such that the time averages for φ converge for all points in Y.

Therefore, the time averages are (3ε)-close to the space average on the set $Y_\varepsilon := Y \cap \widetilde{X}_\varepsilon$, $\mu(Y_\varepsilon) > 1 - \varepsilon$, for arbitrary ε. Let

$$\widetilde{X} := \bigcap_{k=1}^{\infty} \bigcup_{\varepsilon < 1/k} Y_\varepsilon.$$

The set $\widetilde{X}$ is of full measure. By (6), the time averages on $\widetilde{X}$ are arbitrarily close to the space averages and hence coincide with the latter. This proves the ergodicity of μ. □

6. Main Lemma

Let f_0 and f_1 be diffeomorphisms of a circle. We denote the semigroup generated by these diffeomorphisms by $G^+(f_0, f_1)$.

The following two propositions readily follow from the compactness of a circle.

Proposition 3. *If the semigroup $G^+(f_0, f_1)$ acts minimally (that is, each orbit is dense in S^1), then for each interval J on the circle there exist $K = K(J)$ and $\delta_0 = \delta_0(J) > 0$ such that every interval I of length less than δ_0 can be taken into J by some composition of at most K maps f_0 and f_1.*

Proposition 4. *Suppose that for each $x \in S^1$ there exists an i such that $f_i'(x) > 1$. Then there exist $\nu > 1$ and $\delta_1 > 0$ such that for each interval I of length less than δ_1 on the circle we have*

$$\forall x \in I: \quad f_j'(x) > \nu$$

for one of the maps f_j.

We set

$$L = \max_j \max_{x \in S^1} |f_j'(x)|$$

Definition 3. Let X be a P-periodic orbit of a map F, and let $\varepsilon > 0$. A point y is said to be (ε, P)*-good* for X if there exists an $x \in X$ such that

$$\forall k = 0, 1, \ldots, P-1 \quad \operatorname{dist}(F^k(x), F^k(y)) < \varepsilon.$$

Lemma 3. *Suppose that a skew product F of the form (2) satisfies the assumptions of Theorem 2. Let X be an arbitrary P-periodic orbit of F with multiplier α along the fiber, $0 < \alpha < 1$. Suppose that $\lambda := (\ln \alpha)/P < 0$.*

Then for each $\varepsilon > 0$ there exists a periodic orbit Y of F of period $P' > 2P$ and with Lyapunov exponent $\lambda' < 0$ along the fiber such that the following conditions hold:

(1) $|\lambda'| < |\lambda|(1 - (\ln \nu)/(2 \ln L))$;

(2) *there exist $\widetilde{Y} \subset Y$ and a projection $\pi\colon \widetilde{Y} \to X$ such that*

(a) *all points of* $\widetilde{Y}$ *are* (ε, P)*-good for* X*, and one can take* $x = \pi(y)$ *in Definition* 3;

(b) *the fraction* $\varkappa := \#\widetilde{Y}/\#Y$ *of points where* π *is defined can be estimated as*

$$\varkappa \geqslant 1 - \frac{2|\lambda|}{\ln L};$$

(c) *the cardinality of* $\pi^{-1}(x)$ *is the same for all* $x \in X$.

Proof. A periodic orbit of a skew product is determined by its initial point (ω, x), where $x \in S^1$ and $\omega \in \Sigma^2$ is a periodic sequence: $\omega = \dots www \dots = (w)$, where $w = (w_0 \dots w_{n-1})$ is a finite zero-one sequence.

Given an orbit X with initial point $(\omega, x) = ((w), x)$, we seek Y as an orbit with initial point $(\omega', x') = ((w'), x')$, where

$$w' = w^k R_1 R_2 = \underbrace{w \dots w}_{k \text{ times}} R_1 R_2.$$

Here k is a large positive integer to be chosen later, w, w', R_1 and R_2 are finite words, and $x, x' \in S^1$.

We set

$$T_w := f_{w_{n-1}} \circ f_{w_{n-2}} \circ \dots \circ f_{w_0},$$

where w is a word of length n.

Let $\varepsilon > 0$. We take constants α_- and α_+ sufficiently close to α such that $0 < \alpha_- < \alpha < \alpha_+ < 1$. The choice will be specified below (see inequality (8)). There exists an interval J that contains x, does not contain any other fixed points of T_w, and satisfies the condition

$$\forall x \in J \quad \alpha_- \leqslant T'_w(x) \leqslant \alpha_+ \quad \text{and} \quad L^P |J| < \varepsilon. \tag{7}$$

Then

$$\forall k \quad |T_{w^k}(J)| \leqslant \alpha_+^k |J|.$$

We take $K(J)$ and $\delta_0(J)$ as in Proposition 3 and ν and δ_1 as in Proposition 4. Note that ν and δ_1 are independent of X.

Set $\delta = \min(\delta_0(J), \delta_1, |J| L^{-K(J)})$.

Let $r = r(k)$ be a positive integer such that

$$\frac{\delta}{L} \leqslant \alpha_+^k L^{r(k)} |J| < \delta.$$

(Such a number necessarily exists provided that k is sufficiently large.)

By Proposition 4, there exists a composition of $r(k)$ maps f_i (we consider the first map on the interval $T_{w^k}(J)$ and each successive map on the range of its predecessor) such that each map expands at least by a factor of ν at each point of the respective interval. We denote the word corresponding to this composition by R_1. Let $I = T_{R_1} \circ T_{w^k}(J)$. Then

$$|I| < \delta \leqslant \delta_0(J).$$

By the definition of $\delta_0(J)$, there exists a word R_2, $|R_2| < K(J)$, such that $T_{R_2}(I) \subset J$.

The derivative $T'_{w'}$ satisfies the estimate

$$|T'_{w'}| < \alpha_+^k L^{r(k)} L^K < \frac{\delta}{|J|} L^K \leqslant 1$$

on the interval J.

The map $T_{w'}$ is a contraction on J and takes J into itself; hence it has a unique fixed point $x' \in J$. The orbit $Y = Y(k)$ with initial point $(\omega', x') = ((w'), x')$ is completely determined for each sufficiently large k. Let us estimate the Lyapunov exponent along the fiber of F at x' from below.

There exists a constant $C_1 = C_1(w, J)$ such that

$$|T'_{w'}| > C_1(w, J) \alpha_-^k \nu^{r(k)}$$

on J. By the definition of $r(k)$, there exists a constant $C_2 = C_2(w, J)$ such that

$$r(k) > \frac{1}{\ln L}(-k \ln \alpha_+ + C_2(w, J));$$

therefore,

$$\ln T'_{w'}(x') \geqslant k \ln \alpha_- + r(k) \ln \nu + C_3(w, J) \geqslant k\left(\ln \alpha_- - \frac{\ln \nu}{\ln L} \ln \alpha_+\right) + C_4(w, J).$$

Now let us specify the choice of α_- and α_+. They are taken to be so close to α that the following inequality holds:

$$\ln \alpha_- - \frac{\ln \nu}{\ln L}\alpha_+ \geqslant \ln \alpha\left(1 - \frac{\ln \nu}{1.5 \ln L}\right). \tag{8}$$

By substituting (8) into the preceding estimate, we obtain

$$\ln T'_{w'} \geqslant k \ln \alpha\left(1 - \frac{\ln \nu}{1.5 \ln L}\right) + C_4(w, J).$$

The Lyapunov exponent of the orbit $Y(k)$ can be estimated as follows:

$$\lambda(Y(k)) = \frac{\ln T'_{w'}(x')}{kP + r(k) + K} \geqslant \frac{k \ln \alpha(1 - \frac{\ln \nu}{1.5 \ln L}) + C_4(w, J)}{kP} = \lambda(X)\left(1 - \frac{\ln \nu}{1.5 \ln L}\right) + O\left(\frac{1}{k}\right).$$

Set $\lambda := \lambda(X)$ and $\lambda' := \lambda(Y(k))$. Then

$$\lambda' \geqslant \lambda\left(1 - \frac{\ln \nu}{2 \ln L}\right)$$

for sufficiently large k. This proves the first assertion of Lemma 3. Let us prove the second assertion.

Let us define the set $\widetilde{Y}$ and the projection π. Let $M = M(\varepsilon, w)$ be the minimum positive integer such that $2^{-MP} \leqslant \varepsilon$. For $k > M$, we set

$$\widetilde{Y} = \{F^j(\omega', x') \mid MP \leqslant j \leqslant (k - M - 1)P\}.$$

Note that $\widetilde{Y}$ is contained in the set of first kP iterations of (ω', x').

We define the projection $\pi : \widetilde{Y} \to X$ as follows:

$$\pi(F^j(\omega', x')) = F^\rho(\omega, x),$$

where ρ is the residue of division of j by P. Obviously, the number of points in $\pi^{-1}(\tilde{\omega}, \tilde{x})$ is independent of $(\tilde{\omega}, \tilde{x})$ and is equal to $(k - 2M - 1)$. Thus assertion 2(c) of the lemma holds.

Proposition 5. *All the points in $\widetilde{Y}$ are $(2\varepsilon, P)$-good for X.*

Proof. Let us estimate the distance in the base first. By the choice of M, the distance between Σ^2-coordinates of the points $\tilde{y} \in \widetilde{Y}$ and $\pi(\tilde{y}) \in X$ is not greater than ε.

Now let us estimate the distance along the fiber. The image of x' after Pl iterations, $0 \leqslant l \leqslant k$, is contained in J. We denote this image by x'_l. The interval J was chosen in such a way (see (7)) that after t iterations, $0 \leqslant t < P$, the points x and x'_l cannot diverge by a distance greater that ε.

Therefore, the orbits of x and x' for the first kP iterations diverge along the fiber by a distance less than ε.

Hence the first P iterations of $\tilde{y} \in \widetilde{Y}$ and $\pi(\tilde{y})$ diverge in $\Sigma^2 \times S^1$ by a distance less than 2ε. This proves assertion 2(a) of Lemma 3. □

Let us estimate the fraction of points of Y that are not good for X.

$$1 - \frac{\#\widetilde{Y}}{\#Y} = \frac{(2M-1)P + r + K}{kP + r + K} \leqslant \frac{C_6(w, J, \varepsilon) + r}{kP} \leqslant \frac{C_7(w, J, \varepsilon)}{kP} + \frac{-k \ln \alpha}{kP \ln L} = -\frac{\lambda}{\ln L} + O\left(\frac{1}{k}\right).$$

This implies assertion (2b) of Lemma 3 for sufficiently large k. The other assertions of the lemma have already been proved. □

7. Construction of the Sequence of Periodic Orbits

Lemma 4. *Suppose that the diffeomorphisms f_0 and f_1 satisfy the assumptions of Theorem 2. Then the skew product F has a set of periodic orbits X_n such that*

(1) *the periods P_n increase and tend to infinity*;

(2) *the Lyapunov exponents λ_n^c along the fiber are negative and tend to zero*;

(3) *the assumptions of Lemma 2 hold.*

Proof. We take an arbitrary sequence ε_n such that $\sum \varepsilon_n < \infty$. In Sec. 8, the choice of this sequence will be subjected to an additional restriction that does not affect the argument in the present section. As an attracting orbit X_1 along the fiber, we take the periodic orbit corresponding to the hyperbolic attracting periodic point of the diffeomorphism f in condition (iii).

Using Lemma 3, we inductively construct a sequence of periodic orbits X_n taking $\varepsilon = \varepsilon_n$ at each step. For this sequence, the periods tend to infinity and the Lyapunov exponents along the fiber tend to zero exponentially.

Let us verify that the assumptions of Lemma 2 are satisfied. We take arbitrary $\varepsilon > 0$ and $\varphi \in C(M)$. By Lemma 3, a sequence of projections $\pi_n\colon \widetilde{X}_{n+1} \to X_n$ is defined for our orbits such that

$$\prod_{n=1}^{\infty} \varkappa_n = \prod_{n=1}^{\infty} \frac{\#\widetilde{X}_{n+1}}{\#X_n} > 0.$$

(Indeed, the product is convergent, since $1 - \varkappa_n$ is not greater than $2|\lambda_n|/\ln L$ and hence is dominated by a decreasing geometric progression.)

We take a $\delta = \delta(\varepsilon, \varphi)$ such that

$$\omega_\delta(\varphi) := \sup_{\operatorname{dist}(x,y)<\delta} |\varphi(x) - \varphi(y)| < \varepsilon.$$

We take an $N = N(\varepsilon, \varphi)$ such that

$$\sum_N^{\infty} \varepsilon_k < \delta(\varepsilon, \varphi) \quad \text{and} \quad \prod_N^{\infty} \varkappa_k > 1 - \varepsilon.$$

Since the cardinality of the preimage of a point under the projections π_n is independent of the point, it follows that the set $\widetilde{X}_{m,\varepsilon} \subset X_m$ where the total projection $\pi_{m,N} = \pi_{m-1} \circ \dots \circ \pi_N$ is defined contains most of the orbit X_m:

$$\frac{\#\widetilde{X}_{m,\varepsilon}}{\#X_m} = \prod_{k=N}^{m-1} \varkappa_k \geqslant \prod_N^{\infty} \varkappa_k > 1 - \varepsilon.$$

We take arbitrary m and n, $m > n > N(\varepsilon, \varphi)$. On the set $\widetilde{X}_{m,\varepsilon}$, the total projection $\pi_{m,n} = \pi_{m-1} \circ \ldots \circ \pi_n$ is defined. All the points of the set $\widetilde{X}_{m,\varepsilon} \subset X_m$ are $(\delta(\varepsilon, \varphi), P_n)$-good for the orbit X_n. Hence for $x \in \widetilde{X}_{m,\varepsilon}$ we have

$$\left| \int \varphi \, d\nu_{P_n}(x) - \int \varphi \, d\mu_n \right| < \omega_\delta(\varphi) < \varepsilon.$$

All assumptions of Lemma 2 are satisfied. □

According to the remark in Sec. 5, this completely proves Theorem 2.

8. The Limit Measure is Nonatomic

In Sec. 7, we finished the proof of Theorem 2 by constructing an ergodic invariant measure with zero central Lyapunov exponent. The Supplement to Theorem 2 claims that there exists a nontrivial limit measure, that is, a measure not supported on a periodic orbit. Let us prove this.

Proof of the Supplement to Theorem 2. We retain the notation of the preceding section. We take any point $x \in X_n$. Note that in the ε_{n+1}-neighborhood of x there are $\varkappa_{n+1} \frac{P_{n+1}}{P_n}$ points of the orbit X_{n+1}. The μ_{n+1}-measure of these points is $\varkappa_{n+1}$ times less than the μ_n-measure of the initial point; that is, after one step of the induction procedure the point "splits" into ε_{n+1} points and the measure is multiplied by $\varkappa_{n+1}$. We obtain the following sequence of estimates:

$$\mu_{n+1}(U_{\varepsilon_{n+1}}(x)) \geqslant \varkappa_{n+1}\mu_n(\{x\}),$$
$$\mu_{n+2}(U_{\varepsilon_{n+2}+\varepsilon_{n+1}}(x)) \geqslant \varkappa_{n+2}\mu_{n+1}(U_{\varepsilon_{n+1}}(x)) \geqslant \varkappa_{n+2}\varkappa_{n+1}\mu_n(\{x\}),$$

and so on. By passing to the limit, we obtain

$$\mu(\overline{U_{r_n}(x)}) \geqslant \prod_{n+1}^{\infty} \varkappa_k \cdot \mu_n(x) > 0,$$

where by r_n we denote the sum $r_n = \sum_{k=n+1}^{\infty} \varepsilon_k$. Therefore, the μ-measure of a closed ball of radius r_n centered at an arbitrary point of X_n is positive.

Note that we can define the sequence ε_n in Sec. 7 simultaneously with rather than before the construction of the sequence of orbits X_n by choosing X_n depending on ε_n and ε_{n+1} depending on X_n. Namely, let d_n be the minimum distance between two distinct points of X_n, and set $\varepsilon_{n+1} = (\min_{j\leqslant n} d_j)/(3\cdot 2^n)$. Then the series $\sum \varepsilon_k$ is convergent, and moreover, $r_n = \sum_{k=n+1}^{\infty} \varepsilon_k \leqslant d_n/3$. Therefore, any two r_n-balls with centers at distinct points of X_n are disjoint. But the μ-measure of each of these balls is positive, and hence μ cannot be supported at less than P_n points. On the other hand, n is arbitrary, and the periods P_n tend to infinity. Therefore, μ cannot be supported on a finite set. □

9. Proof of Theorem 3

The following result was established in [9, 10].

Theorem 4. *Let f_0 be a map with exactly two fixed points a and b such that $f_0'(a) \in (1,2)$ and $f_0'(b) \in (1/2,1)$, and let f_1 be an irrational rotation of a circle. Then there exists a neighborhood $U \subset (\mathrm{Diff}(S^1))^2$ of the pair (f_0, f_1) such that the semigroup $G^+(\widetilde{f}_0, \widetilde{f}_1)$ acts minimally for each pair $(\widetilde{f}_0, \widetilde{f}_1) \in U$.*

We take arbitrary f_0 and f_1 satisfying the assumptions of Theorem 4. Let U be a neighborhood of (f_0, f_1) in $(\mathrm{Diff}(S^1))^2$ where the action of every pair is minimal.

Let f_2 be a map close to f_1 such that $f_2'(x) > 1$ on the set $I = \{x \in S^1 \mid f_0'(x) \leqslant 1\}$. By taking f_2 sufficiently close to f_1, we can guarantee that $(f_0, f_2) \in U$.

For the pair (f_0, f_2), properties (ii) and (iii) in Theorem 2 hold. But properties (ii) and (iii) are C^1-open, and therefore, there exists a neighborhood V of the pair (f_0, f_2) such that every pair of maps in V satisfies these conditions.

For each pair (g_0, g_1) in the set $U \cap V$, conditions (i), (ii) and (iii) hold. The proof of Theorem 3 is complete.

References

1. R. Abraham and S. Smale, Nongenericity of Ω-stability. In: Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 5–8.
2. Ya. Pesin, "Characteristic Lyapunov exponents and smooth ergodic theory," Usp. Mat. Nauk, **32**, No. 4 (196), 55–112 (1977).
3. J. Bochi, "Genericity of zero Lyapunov exponents," Ergodic Theory Dynam. Systems, **22**, No. 6, 1667–1696 (2002).
4. J. Bochi and M. Viana, "Uniform (projective) hyperbolicity or no hyperbolicity: a dichotomy for generic conservative maps," Ann. Inst. H. Poincare Anal. Non Liniaire, **19**, No. 1, 113–123 (2002).
5. A. Baraviera and C. Bonatti, "Removing zero Lyapunov exponents," Ergodic Theory Dynam. Systems, **23**, 1655–1670 (2003).

6. M. Shub and A. Wilkinson, "Pathological foliations and removable zero exponents," Invent. Math., **139**, 495–508 (2000).
7. D. Dolgopyat and Y. Pesin, "Every compact manifold carries a completely hyperbolic diffeomorphism," Ergodic Theory Dynam. Systems, **22**, No. 2, 409–435 (2002).
8. M. W. Hirsch, C. C. Pugh, and M. Shub, Invariant manifolds, Lect. Notes in Math., Vol. 583, Springer-Verlag, Berlin–New York, 1977.
9. A. Gorodetskii and Yu. Ilyashenko, "Some new structurally stable properties of invariant sets and attractors of dynamical systems," Funkts. Anal. Prilozhen., **33**, No. 2, 16–30 (1999).
10. A. Gorodetskii and Yu. Ilyashenko, "Some properties of skew products over the horseshoe and solenoid," Trudy MIRAN, **231**, 6–118 (2000); English transl. Proc. Steklov Inst. Math., **231**, 96–118 (2000).
11. A. Gorodetskii, Minimal Attractors and Partially Hyperbolic Invariant Sets of Dynamical Systems, PhD thesis, Moscow State University, 2001.
12. V. Kleptsyn and M. Nalsky, "Contraction of Orbits in Random Dynamical Systems on the Circle," Funkts. Anal. Prilozhen., **38**, No. 4, 36–54 (2004).
13. A. N. Kolmogorov and S. V. Fomin, Elements of the Theory of Functions and Functional Analysis, Graylock Press, Rochester, N.Y., 1957.
14. Yu. Ilyashenko and W. Li, Nonlocal bifurcations. Amer. Math. Soc., Providence, R.I., 1999.
15. A. Katok and B. Hasselblat, Introduction to the Modern Theory of Dynamical System, Cambridge Univ. Press, 1995.
16. A. Katok and A. Stepin, "Approximation of ergodic dynamical systems by periodic transformations," Dokl. Akad. Nauk SSSR, **171**, 1268–1271 (1966); English transl. Soviet Math. Dokl, **7**, 1638–1641 (1966).
17. A. Katok and A. Stepin, "Approximations in ergodic theory," Usp. Mat. Nauk., **22**, No. 5 (137), 81–106 (1967).

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