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# How often surface diffeomorphisms have infinitely many sinks and hyperbolicity of periodic points near a homoclinic tangency

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**Abstract**

Here we study an amazing phenomenon discovered by Newhouse [S. Newhouse, Non-density of Axiom A(a) on $S^2$, in: Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., 1970, pp. 191–202; S. Newhouse, Diffeomorphisms with infinitely many sinks, Topology 13 (1974) 9–18; S. Newhouse, The abundance of wild hyperbolic sets and nonsmooth stable sets of diffeomorphisms, Publ. Math. Inst. Hautes Études Sci. 50 (1979) 101–151]. It turns out that in the space of $C^r$ smooth diffeomorphisms $\mathrm{Diff}^r(M)$ of a compact surface $M$ there is an open set $U$ such that a Baire generic diffeomorphism $f \in U$ has infinitely many coexisting sinks. In this paper we make a step towards understanding "how often does a surface diffeomorphism have infinitely many sinks." Our main result roughly says that *with probability one* for any positive $D$ a surface diffeomorphism has only finitely many *localized* sinks either of *cyclicity* bounded by $D$ or those whose period is relatively large compared to its *cyclicity*. It verifies a particular case of Palis' Conjecture saying that even though diffeomorphisms with infinitely many coexisting sinks are Baire generic, they have probability zero.

One of the key points of the proof is an application of Newton Interpolation Polynomials to study the dynamics initiated in [V. Kaloshin, B. Hunt, A stretched exponential bound on the rate of growth of the number of periodic points for prevalent diffeomorphisms I, Ann. of Math., in press, 92 pp.; V. Kaloshin, A stretched exponential bound on the rate of growth of the number of periodic points for prevalent diffeomorphisms II, preprint, 85 pp.].
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## 1. Introduction

Let $M$ be a smooth 2-dimensional manifold and $\mathrm{Diff}^r(M)$ be the space of $C^r$ smooth diffeomorphisms with the uniform $C^r$ topology. According to the standard terminology *Baire generic* (*residual*) set of diffeomorphisms is a subset of $\mathrm{Diff}^r(M)$ which contains a countable intersection of open dense sets. During the time Thom [39] had been developing singularity theory he conjectured that Baire generically a diffeomorphism or a flow has only finitely many hyperbolic periodic attractors. It turns out that this conjecture has a negative answer.

We say that a diffeomorphism $f \in \mathrm{Diff}^r(M)$ exhibits a *homoclinic tangency* (HT) if it has a saddle periodic point $p = f^n(p)$ for some $n \in \mathbb{N}$ such that stable and unstable manifolds $W^s(p)$ and $W^u(p)$ of $p$ respectively have a point $q$ of tangency (see Fig. 1). Denote by $\mathcal{HT} \subset \mathrm{Diff}^r(M)$ the set of diffeomorphisms exhibiting homoclinic tangency. The picture of homoclinic tangency seems fragile and easily destroyable by a small perturbation for a single saddle periodic point. Surprisingly however, Newhouse [25] proved that the $C^r$-closure of $\mathcal{HT}$ contains an open set $U \subset \mathrm{Diff}^r(M)$. These open sets are called *Newhouse domains*. Later Newhouse [26,27] proved that in such a domain there is a Baire generic set of diffeomorphisms having *infinitely many coexisting sinks*. Examples of coexistence of infinitely many sinks have been found in various situations:

*Henon family.* In the 1970s, M. Henon [16] made an extensive numerical study of the behavior under iteration of maps $P_{a,b}: R^2 \to R^2$ of the form $(x, y) \mapsto (1 - ax^2 + by, x)$, where $a, b \in R$.

In particular, Henon found numerical evidence supporting the existence of a strange attractor for $P_{a,b}$ when $a = 1.4$ and $b = 0.3$. In the parameter plane $(a, b)$ it was shown [40] that arbitrarily near $(a, b) = (2, 0)$ there is an open set $U$ such that for a Baire generic parameter in it, the corresponding $P_{a,b}$ has infinitely many coexisting sinks.

*Polynomial automorphisms of* $\mathbb{C}^2$. Buzzard [4], using results of Forness–Gavosto [7], showed that for a large enough $d$ in the space of holomorphic self-maps $H_d(\mathbb{C}^2)$ of $\mathbb{C}^2$ of degree $d$, there exists an open set $\mathcal{N} \subseteq H_d(\mathbb{C}^2)$ such that for a Baire generic parameter in it, the corresponding self-map has infinitely many coexisting sinks.

Newhouse's discovery of existence of infinite number of attractors for topologically generic dynamical system leads to the following natural question: *What is the probability of this phenomena in some measure theoretical sense?* In the case of finite parameter families as above it corresponds to the question: What is the measure of the set of Baire generic parameters with infinitely many coexisting sinks?

Another result which shows importance of investigation of perturbations of HT is the following

**Conjecture 1.** (Palis [31]) *For any $r \geqslant 1$ any surface diffeomorphism $f \in \mathrm{Diff}^r(M)$ can be approximated by one that is either essentially hyperbolic or exhibiting HT.*

Essential hyperbolicity refers here to a diffeomorphism that has a finite number of hyperbolic attractors whose basins of attraction cover a set of full Lebesgue measure. For $r = 1$ this conjecture has been proven by Pujals–Sambarino [36].

The primary goal of this paper is to analyze trajectories *localized in a neighborhood of a fixed HT*. A loose statement of the main result is in the abstract. A sink periodic orbit is the simplest attractor. We now define notions of an unfolding of an HT and localized trajectories of finite complexity associated to that HT.

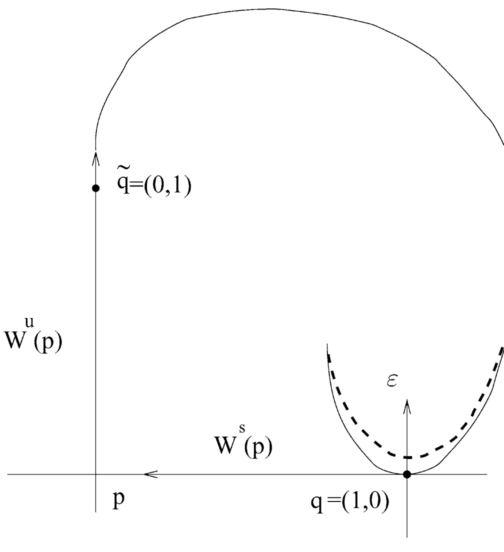


Fig. 1. Homoclinic tangency.

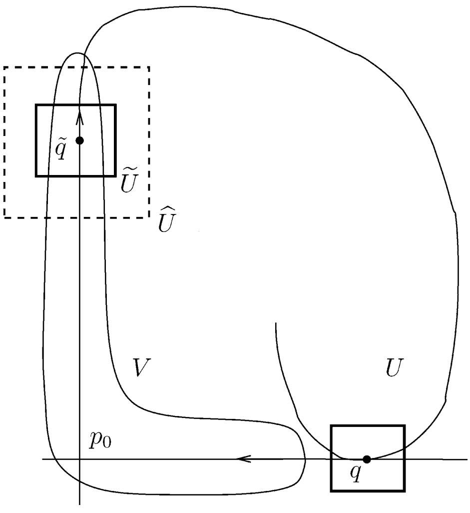


Fig. 2. Localization for homoclinic tangency.

Consider a 1-parameter family of perturbations $\{f_\varepsilon\}_{\varepsilon\in I}$, $I = [-\varepsilon_0, \varepsilon_0]$ of a 2-dimensional diffeomorphism $f = f_0 \in \mathrm{Diff}^r(M)$ with homoclinic tangency and small $\varepsilon_0 > 0$ (see Fig. 1). Roughly speaking, $\varepsilon$ parametrizes oriented distance of the top tip of the unstable manifold to the stable manifold. Such a family is called an *unfolding of an HT*.

Robinson [37], adapting Newhouse's ideas [26,27], showed that for such an unfolding there is a sequence of open intervals converging to zero such that for a generic parameter from those intervals the corresponding diffeomorphism $f_\varepsilon$ has infinitely many coexisting sinks.

Assume that $f$ has a fixed saddle point $p_0 = f(p_0)$ and that the eigenvalues $\lambda, \mu$ of the linearization $Df(p_0)$, $0 < \lambda < 1 < \mu$, and they belong to the open dense set of pairs of full measure of eigenvalues for which Sternberg's linearization theorem holds. Then in a small neighborhood $\tilde{V}$ of $p_0$ there is a $C^r$ smooth normal coordinate system $(x, y) \in \tilde{V} \subset \mathbb{R}^2$ such that $f(x, y) = (\lambda x, \mu y)$. Suppose $q$ is the point of homoclinic tangency of $W^s(p_0)$ and $W^u(p_0)$ away from $\tilde{V}$, and let $\tilde{q} = f^{-1}(q)$ be its preimage. Extend the coordinate neighborhood $\tilde{V}$ by iterating forward and backward until first it contains $\tilde{q}$ and $f(q)$, respectively. Decreasing $\tilde{V}$ if necessary we can assume that there are no overlaps. Denote such a neighborhood by $V$ and call it *a normal neighborhood*. By definition $V$ does not contain $q$ (see Fig. 2). Consider a neighborhood $U$ (respectively $\tilde{U} \subset \hat{U}$) of $q$ (respectively $\tilde{q}$) such that $f(U) \cap U = \emptyset$ (respectively $f^{-1}(\hat{U}) \cap \hat{U} = \emptyset$), $f(\tilde{U}) \supset U$, and $f(\hat{U}) \cap V = \emptyset$. By rescaling coordinate axis one could set $q$ to have coordinates $(1, 0)$ and $\tilde{q}$ to have $(0, 1)$. Set $\mathcal{V} = V \cup U$. In what follows we *fix* a neighborhood $\mathcal{V}$ once and for all.

**Definition 1.** We call an invariant set of points *$\mathcal{V}$-localized* if it belongs to $\mathcal{V}$. In particular, any invariant set contained in

$$\Lambda_{\mathcal{V}} = \bigcap_{n\in\mathbb{Z}} f^n(\mathcal{V}) \tag{1}$$

is $\mathcal{V}$-localized. A periodic point $f^n(p) = p$, $n \in \mathbb{N}$, is called $\mathcal{V}$-localized if it belongs to $\Lambda_{\mathcal{V}}$ and is called $(\mathcal{V}, s)$-localized if its trajectory $\mathfrak{P} = \{f^k(p)\}_{k=1}^n$ visits $U$ exactly $s$ times. Call $s = s(\mathfrak{P})$ the *cyclicity* of a $\mathcal{V}$-localized periodic orbit.

The zoo of $\mathcal{V}$-localized invariant sets is incredibly rich. Below we just mention the authors favorite animals.

- *Smale's horseshoe* created by a small perturbation of an HT.
- *$\mathcal{V}$-localized sink of an arbitrary high period* (Gavrilov–Shilnikov [8] and Newhouse [27]).
- *Infinitely many coexisting $\mathcal{V}$-localized sinks*. Actually Newhouse [26] (see also Palis–Takens [33] for a simplified proof) proved that for a Baire generic set of diffeomorphisms in a Newhouse domain there are infinitely many coexisting sinks. However one can construct infinitely many of those as $\mathcal{V}$-localized.
- *Strange attractor* (Benedicks–Carleson [2], Mora–Viana [24], Young–Wang [41]). A strange attractor appears as an attractor for the return map of a certain rectangle localized in $U$ into itself. It would imply that a $\mathcal{V}$-localized set could contain a strange attractor.
- *Infinitely many coexisting strange attractors* (Colli [6]).
- *Arbitrarily degenerate periodic points of arbitrary high periods* (Gonchenko–Shilnikov–Turaev [13]).
- *Uniformly and non-uniformly hyperbolic horseshoes as maximal invariant sets $\Lambda_{\mathcal{V}}$* (Newhouse–Palis [28], for further generalizations see Palis–Takens [32], Palis–Yoccoz [34,35], and I. Rios [38]).

The first of our main results is the following

**Theorem 1.1.** *With the above notations, for a generic*[1] *1-parameter family $\{f_\varepsilon\}_{\varepsilon\in I}$ that unfolds an HT at $q$ there is a sequence of numbers $\{\mathbf{N}_s\}_{s\in\mathbb{N}}$ such that for almost every parameter $\varepsilon$ and any $D\in\mathbb{N}$ the corresponding $f_\varepsilon$ has only finitely many $\mathcal{V}$-localized sinks $\{\mathfrak{P}_j\}_{j\in J}$ whose cyclicity is bounded by $D$ or period exceeds $\mathbf{N}_{s_j}$, where $s_j = s(\mathfrak{P}_j) > D$ is cyclicity of a corresponding sink $\mathfrak{P}_j$. In other words, for almost every parameter $\varepsilon$ if there are infinitely many coexisting $\mathcal{V}$-localized sinks $\{\mathfrak{P}_j\}_{j\in J}$, then all but finitely many have cyclicity $s_j = s(\mathfrak{P}_j) > D$ and period $< \mathbf{N}_{s_j}$.*

**Remark 1.** For 1-loop periodic sinks a similar result is obtained by Tedeschini-Lalli–Yorke [23], see also [29]. Dynamical properties of periodic and homoclinic orbits of low cyclicity ($s = 1, 2, 3$) were studied in [11,12]. In particular, Gonchenko and Shilnikov found the relation between existence of the infinite number of 2-loop sinks and numerical properties of the invariants of smooth conjugacy [10]. For random maps the problem of finiteness of attractors was considered by Araujo [1].

**Remark 2.** We can choose $\mathbf{N}_s = s^{5s^2}$.

**Remark 3.** In a later publication of the authors on the subject of Newhouse phenomenon we modify arguments from Palis–Takens [32] and show that for a generic parameter $\varepsilon$ in a Newhouse interval (see the result of Robinson [37] stated above) $f_\varepsilon$ has infinitely many $\mathcal{V}$-localized sinks of $\{\mathcal{P}_j\}_j$ whose periods exceed the corresponding functions $\mathbf{N}_{s_j}$ of their cyclicity $s_j = s(\mathcal{P}_j)$. In particular, it implies that Theorem 1.1 gives another example of a topological generic phenom-

[1] Meaning of "generic" is in the sense of prevalence in the space of 1-parameter families, see Section 2.8 for a definition.

enon of zero measure in the space of parameters. For other examples of prevalent topologically negligible phenomena see [17,21,30].

Since a sink is the simplest example of an attractor, this result is a particular case of the following

**Conjecture 2.** (Palis) *With probability one a surface diffeomorphism has finitely many attractors.*

Our method provides significant additional information about hyperbolicity of corresponding localized periodic points, which is the second main result.

**Definition 2.** We say that a periodic (under the map $g$) point $p$ of period $n$ is $(\mu, \aleph)$-trace hyperbolic if

$$\left|\operatorname{Tr} Dg^n(p)\right| > \mu^{(1-\aleph)n}. \tag{2}$$

**Remark 4.** For a $\mathcal{V}$-localized periodic point $p$ of large period $n$ $(\gg s)$ we have that

$$\det Df^n(p) \approx (\lambda\mu)^n.$$

Hence if the product of eigenvalues $\lambda\mu \leqslant 1$, then the condition (2) implies hyperbolicity of $p$.

**Theorem 1.2.** *With the above notations of Theorem* 1.1*, for any* $\aleph > 0$ *there is a sequence of numbers* $\{\mathbf{N}_s(\aleph)\}_{s\in\mathbb{N}}$ *such that for almost every parameter* $\varepsilon$ *and any* $D \in \mathbb{N}$ *the corresponding* $f_\varepsilon$ *has only finitely many* $\mathcal{V}$*-localized not* $(\mu, \aleph)$*-trace hyperbolic periodic points* $\{\mathfrak{P}_j\}_{j\in J}$ *whose cyclicity is bounded by* $D$ *or period exceeds* $\mathbf{N}_{s_j}(\aleph)$*, where* $s_j = s(\mathfrak{P}_j) > D$ *is cyclicity of a corresponding sink* $\mathfrak{P}_j$*. In other words, for almost every parameter* $\varepsilon$ *if there are infinitely many coexisting not* $(\mu, \aleph)$*-trace hyperbolic* $\mathcal{V}$*-localized periodic points* $\{\mathfrak{P}_j\}_{j\in J}$*, then all but finitely many have cyclicity* $s_j = s(\mathfrak{P}_j) > D$ *and period* $< \mathbf{N}_{s_j}$*.*

**Remark 5.** Note that if $p$ is a periodic orbit of a planar diffeomorphism of period $n$ and $|\operatorname{Tr} Df^n_{\varepsilon}(p)| > 2$, then $p$ can not be a sink. Therefore Theorem 1.2 implies Theorem 1.1.

Palis–Takens [32] and Palis–Yoccoz [34,35] investigated generic unfolding of an HT not only for saddle periodic points but also for horseshoes. They investigated parameters *outside* of Newhouse domains. We obtain *less sharp properties* of the dynamics, but we treat *parameters inside* Newhouse domains too!

Strange attractors can be found as invariant sets of certain return maps of a subset $\Pi$ of $U$ into itself, i.e. for some $n$ we have $f^n : \Pi \to U$ is well defined. For trajectories in $\bigcap_{k\in\mathbb{Z}} f^{nk}(\Pi)$, the period grows linearly with cyclicity. The main result does not restrict attention to a subset $\Pi$ of $U$ but has to consider trajectories whose period grows superexponentially $(> s^{5s^2})$ with cyclicity $s$.

### 1.1. Main ideas of the proof of Theorem 1.2

The general idea of the method is described in non-technical terms in [14].

*1.1.1. Borel–Cantelli arguments*

Fix $\aleph > 0$, e.g. $\aleph = 1/2$. Let

$$B_{n,s} = \big\{\varepsilon \in I \mid f_\varepsilon \text{ has a not } (\mu, 1/2)\text{-trace hyperbolic periodic point of period } n \text{ and cyclicity } s\big\}.$$

Suppose

$$\sum_{n \in \mathbb{Z}_+} \mathrm{Leb}(B_{n,s}) < +\infty \quad \text{for any } s \in \mathbb{Z}_+,$$

where Leb is the Lebesgue measure. Then for almost every $\varepsilon \in I$ there are only finitely many not $(\mu, 1/2)$-trace hyperbolic periodic points of bounded cyclicity. The focal point of the proof is to estimate the measure of "bad" parameters $\mathrm{Leb}(B_{n,s})$.

*1.1.2. Trajectory type, hyperbolic and parabolic maps*

Any $(\mathcal{V}, s)$-localized periodic orbit, by definition, visits $U$ exactly $s$ times and spends $n_1, n_2, \ldots, n_s$ consecutive iterates in $V$, $n = n_1 + n_2 + \cdots + n_s + s$. We call an ordered sequence $(n_1, \ldots, n_s)$ *type* of a periodic orbit. For a given periodic orbit denote the points of intersection with $U$ by $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{s-1}$ and the corresponding points in $\tilde{U}$ by $\tilde{\mathbf{p}}_0 = f^{n_1}(\mathbf{p}_0)$, $\tilde{\mathbf{p}}_1 = f^{n_2}(\mathbf{p}_1), \ldots, \tilde{\mathbf{p}}_{s-1} = f^{n_s}(\mathbf{p}_{s-1})$.

Recall that $f$ is linear in $V \setminus \tilde{U}$ with eigenvalues $\lambda < 1 < \mu$, $f|_{V \setminus \tilde{U}}(x, y) = (\lambda x, \mu y)$. Call this linear map *hyperbolic*, denoted $L$, and $f|_{\tilde{U}}$ *parabolic*, denoted $\mathcal{P}$.

*1.1.3. Cone condition*

To estimate the measure of parameters for which a periodic orbit of a given type is not $(\mu, 1/2)$-trace hyperbolic, we introduce the following cone condition. Define at every point $p \in U$ a cone

$$K_A(p) = \big\{v = (v_x, v_y) \in T_p\mathcal{V} \simeq \mathbb{R}^2 \;\big|\; |v_y| \geqslant \mu^{-A}|v_x|\big\}.$$

To show that the periodic point $\mathbf{p}_0$ is hyperbolic it turns out that it suffices to find $0 < \alpha \ll 1$ independent of $n$ such that

$$Df^n_{\varepsilon,\mathbf{p}_0}\big(K_{\alpha n}(\mathbf{p}_0)\big) \subset K_{\alpha n}(\mathbf{p}_0). \tag{3}$$

To verify this condition directly does not seem possible in general. Our plan is to verify that for most parameters this cone condition holds after each loop:

$$Df^{n_i+1}_{\varepsilon,\mathbf{p}_i}\big(K_{\alpha n}(\mathbf{p}_i)\big) \subset K_{\alpha n}\big(\mathbf{p}_{i+1 \ (\mathrm{mod}\ s)}\big) \quad \text{for each } i = 0, \ldots, s-1. \tag{4}$$

See Fig. 3 for $s = 1$. This condition clearly implies (3), because the image of the first cone $K_{\alpha n}(\mathbf{p}_0)$ belongs to the second cone $K_{\alpha n}(\mathbf{p}_1)$. The image of the second one belongs to the third one and so on.

Fix $0 < \alpha \ll 1$. Notice that if all loops are *long*: $n_i > 3\alpha n$, then $L^{n_i} K_{\alpha n}(\mathbf{p}_i)$ is the cone of width angle $< 2\mu^{-\alpha n}$. Fix $1 \leqslant j \leqslant s$. To satisfy condition (4) for $j$ we need to avoid the intersection of the cone $Df_{\varepsilon,\tilde{\mathbf{p}}_j}(L^{n_j} K_{\alpha n}(\mathbf{p}_j))$ and the complement to $K_{\alpha n}(\mathbf{p}_{i+1})$. Assume that

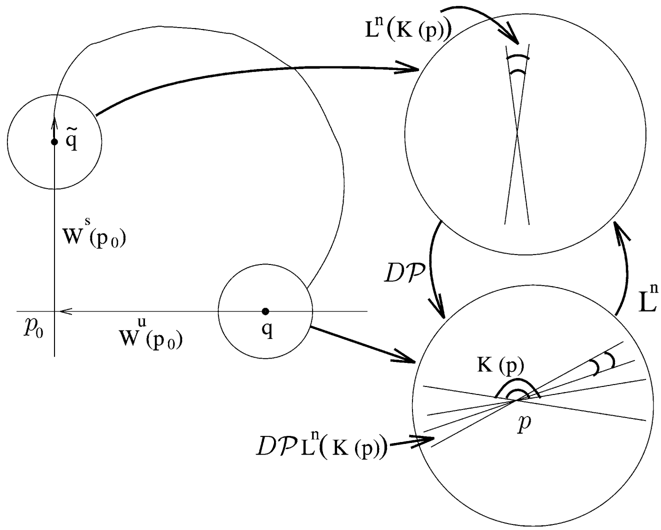


Fig. 3. Evolution of cones.

we can perturb $Df_{\varepsilon,\tilde{\mathbf{p}}_j}$ by composing with rotation and angle of rotation is a parameter. Then we need to avoid a phenomenon that has "probability" $\sim \mu^{-\alpha n}$. Taking the union over all types $\mathcal{N}_s$, $|\mathcal{N}_s| = n$ we get that probability to fail (4) for some $1 \leqslant i \leqslant s$ is $\sim n^s \mu^{-\alpha n}$. We avoid saying explicitly probability in what space, just assume that it is proportional to angle of rotation, and postpone the exact definition for further discussion.

However, it might happen that one of $n_i$'s is significantly smaller than $\alpha n$, e.g. $n_s \leqslant \ln n$. In this case, the above argument fails. Indeed, let $n_s = [\ln n]$, $n \gg 1$. Consider the image of the cone $K_{\alpha n}(\mathbf{p}_{s-1})$ after the last loop $L^{n_s} K_{\alpha n}(\mathbf{p}_{s-1})$. It is the cone, whose width angle is of order 1. Taking into account possibility that $Df_{\varepsilon,\tilde{\mathbf{p}}_{s-1}}$ rotates a vertical vector by $\pi/2$ it is certainly not possible to fulfill (4) by a small perturbation. The natural idea is *to avoid looking at condition* (4) *after "short" loops*. This leads to combinatorial analysis of type $\mathcal{N}_s$ of trajectories.

### *1.1.4. Combinatorial analysis of type $\mathcal{N}_s$ of s-loop trajectories*

Below we do not pay attention to dynamics of a trajectory of type $\mathcal{N}_s$ under consideration. We investigate only properties of $\mathcal{N}_s$.

- *Short and long loops.* We shall divide an $s$-tuple $\mathcal{N}_s = (n_1, \ldots, n_s)$ into two groups of *long* and *short* $n_i$'s, because they correspond to loops of a trajectory. After such a division long $n_i$'s should be *much longer* than short $n_i$'s. Denote by $t$ (respectively $s - t$) the number of long (respectively short) loops.
- *Generalized loops and essential returns.* Since we cannot fulfill (4) after short loops, we combine all loops into groups, called *generalized loops*. Each generalized loop starts with a long loop and is completed by all short loops following afterwards. Therefore, the number of generalized loops equals the number of long loops. Then we *verify* (4) *not after each loop, but after each generalized loop*. Denote by $P_0, \ldots, P_{t-1}$, $P_t = P_0 \subset U$ starting points of generalized loops, by $\tilde{P}_0, \ldots, \tilde{P}_{t-1}$, $\tilde{P}_t = \tilde{P}_0$, prestarting points of generalized loops, i.e.

$f(\tilde{P}_i) = P_{i+1}$, $i = 0, \dots, t-1$, and by $N_1, \dots, N_t$ their lengths, respectively. Then we modify (4) to

$$Df^{N_{i+1}}_{\varepsilon,\tilde{\mathbf{p}}_i}\big(K_{\alpha n}(P_i)\big) \subset K_{\alpha n}(P_{i+1}) \quad \text{for each } i = 0, \dots, t-1. \tag{5}$$

Now the idea presented above has a chance to work. Indeed, let $n_j$ be a long loop and $n_{j+1}, \dots, n_{j+j'}$ be short ones from the corresponding generalized loop. Consider the image of the corresponding cone $K_{\alpha n}(P_j)$ after the generalized loop. Notice that after the long loop $n_j$ the cone $L^{n_j} K_{\alpha n}(P_j)$ is the cone of width angle $< 2\mu^{-\alpha n}$. Since long $n_j$ is so much longer than short loops $n_{j+1}, \dots, n_{j+j'}$, the cone

$$\big(Df_{\varepsilon,\tilde{\mathbf{p}}_{j+j'}} \circ L^{n_{j+j'}} \circ \cdots \circ Df_{\varepsilon,\tilde{\mathbf{p}}_{j+1}} \circ L^{n_{j+1}}\big) \circ \big(Df_{\varepsilon,\tilde{\mathbf{p}}_j} \circ L^{n_j} K_{\alpha n}(\mathbf{p}_j)\big)$$

has width angle $< 3\mu^{-\alpha n}$. To satisfy condition (4) for $j + j'$ we need to avoid an interval of rotations (i.e. of parameters) of length $< 5\mu^{-\alpha n}$. This phenomenon still has "probability" $\sim \mu^{-\alpha n}$.

After this combinatorial analysis we face the next difficulty. *We cannot perturb $Df_{\tilde{\mathbf{p}}}$ and $Df_{\tilde{\mathbf{p}}'}$ independently at nearby points $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{p}}'$.*

### *1.1.5. Dynamical analysis of trajectories*

Assume for a moment that we are interested in properties of *scattered* periodic orbits, that is, such orbits that $P_0, \dots, P_{t-1}$ in $U$ are pairwise well spaced. In particular, it is always the case for 1-loop orbits. In this case the difficulty of nearby points is removed. Using Discretization Method and the cone condition (5) one can prove that for most parameters all but a finite number of the periodic orbits are hyperbolic saddles. Moreover, consider for $0 < \gamma' = \mu^{-\alpha' n} \ll \gamma'' = \mu^{-\alpha'' n}$ parameters for which a periodic not enough hyperbolic $\gamma''$-scattered $\gamma'$-pseudo-orbit of period $n$ exists. In fact, we can show that the measure of these parameters is small (see (28) for the formal statement). Now we are going to explain how this can be used to treat all periodic orbits, not necessarily scattered. Consider the 2-loop case for illustration. If starting points of loops $\mathbf{p}_0$ and $\mathbf{p}_1$ are far enough from each other, one can perturb differential of parabolic map at their preimages independently, and above arguments allow to estimate the measure of "bad" parameters. Otherwise a periodic orbit can be decomposed into a union of two 1-loop periodic pseudo-orbits, which have nearby endpoints in $U$. The cone condition (5) for each of these pseudo-orbits holds for most parameters, which implies (3).

Another illustration can be given by the case $t = 1$, i.e. we have one loop which is much longer than all the others. In this case the image of the cone $K_{\alpha n}(\mathbf{p}_0)$ after the application of differential of the map along the orbit has width angle $< 2\mu^{-\alpha n}$, as explained above. Point $\tilde{\mathbf{p}}_{s-1} = \tilde{P}_0 = f^{n-1}(\mathbf{p}_0) = f^{-1}(\mathbf{p}_0)$ cannot be too close to points $\tilde{\mathbf{p}}_0, \tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_{s-2}$. Indeed, the distance between $\mathbf{p}_i$ and $x$-axis is $(\mathbf{p}_i)_y \sim \mu^{-n_{i+1}}$. Since $n_1 \gg n_i$ we have $\mu^{-n_1} \ll \mu^{-n_i}$. Therefore the point $\mathbf{p}_0$ can not be too close to points $\mathbf{p}_1, \dots, \mathbf{p}_{s-1}$, and we can perturb $\phi(\tilde{\mathbf{p}}_{s-1}) = \phi(f^{-1}(\mathbf{p}_0))$ independently of $\phi(\tilde{\mathbf{p}}_0), \dots, \phi(\tilde{\mathbf{p}}_{s-2})$. This allows to estimate the measure of "bad" parameters.

To consider the general case we represent a periodic orbit as an oriented cyclic graph. Starting points of generalized loops are vertices of this graph, and vertices corresponding to a subsequent generalized loops are connected by an oriented edge (see Fig. 4, picture 1). It turns out that for some $\gamma' \ll \gamma''$ for any pair of points $(P_i, P_j)$ either $\operatorname{dist}(P_i, P_j) > \gamma''$ or $\operatorname{dist}(P_i, P_j) < \gamma'$. This is explained in details in Section 7. Therefore every pair of vertices is either $\gamma'$-close or $\gamma''$-far apart (see Fig. 4, picture 2). Now all the vertices can be divided into "clouds" or "clusters." Let us identify the vertices in the same cloud of nearby points, as shown on Fig. 4, picture 2. The

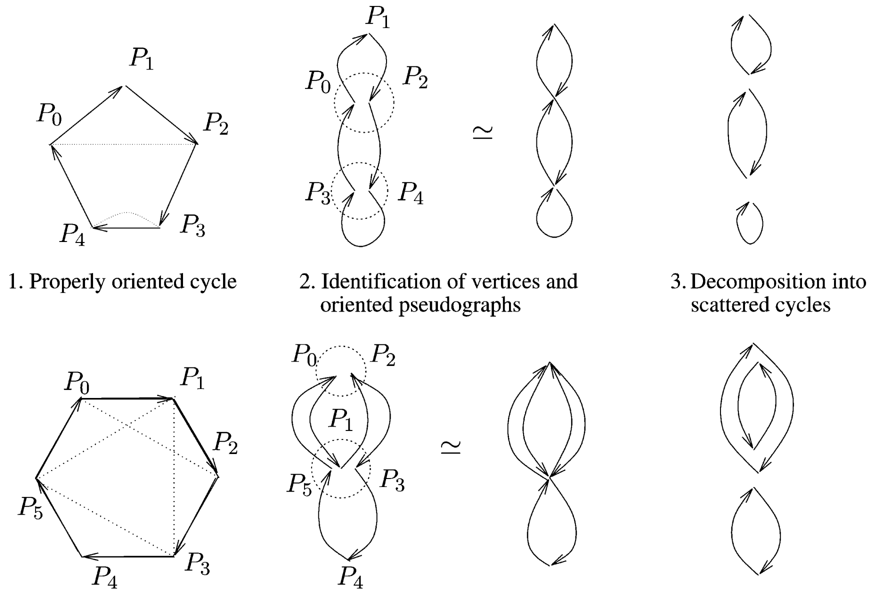


Fig. 4. Graph surgery.

initial cycle is transformed now into oriented pseudograph (see Definition 20) with the same number of ingoing and outgoing edges at each vertex. Such a pseudograph can be decomposed into the union of oriented cycles, see Lemma 7 and Fig. 4, picture 3. Each of cycles from this decomposition represents a $\gamma''$-scattered $\gamma'$-pseudo-orbit. Application of the arguments above to these pseudo-orbits gives inclusion (5) for most values of parameters and implies the cone condition (3) for the initial periodic orbit.

*1.1.6. Tools for measure estimates: Discretization Method and Newton Interpolation Polynomials*

To make perturbations in a described way and to estimate the measure in a space of parameters we use Discretization Method and Newton Interpolation Polynomials. This method was already successfully used in [22] and we strongly believe that it can be applied to wide range of problems in Dynamics. See Sections 9–11 of the present paper, Section 3 in [22] or [14] for discussion of the method.

The structure of the paper is the following. In Section 2 the exact statements of results are given. In Section 3 a model example is considered, a strategy of the proof is presented, and Fubini reduction is described. In Section 4 Auxiliary Theorems I and II are stated, and the results are reduced to those theorems. Sections 5–11 are devoted to the proof of Auxiliary Theorem I. Section 12 gives the proof of Auxiliary Theorem II. In Section 13 the results are extended to nonlinear situation and to periodic saddle with homoclinic orbit. Section 14 contains some technical proofs, including the proof of Addendum to Theorem B. In Appendix A we study relations between existence of infinite number of periodic orbits of bounded cyclicity and Kupka–Smale property. Appendix B provides the proof of the estimate of "non-hyperbolic" parameters that is used in the proof of Auxiliary Theorem II and in Section 13.

## 2. Statement of the results

### 2.1. Description of the initial map

Consider a linear map

$$L = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $0 < \lambda < 1 < \mu$. Set $\Im = -(\ln \lambda)/(\ln \mu) > 0$.

Denote $q = (1, 0)$ and $\tilde{q} = (0, 1)$. Fix small $\tilde{\delta}, \delta > 0$. Let $U$ be a neighborhood of point $q$, and let $\hat{U}$ and $\tilde{U}$ be neighborhoods of point $\tilde{q}$, such that

$$\begin{aligned} U &= \{(x, y) \mid x \in [1-\delta, 1+\delta],\ y \in [-\delta, \delta]\}, \\ \tilde{U} &= \{(x, y) \mid y \in [1-\tilde{\delta}, 1+\tilde{\delta}],\ x \in [-\tilde{\delta}, \tilde{\delta}]\}, \\ \hat{U} &= \{(x, y) \mid y \in [1-2\tilde{\delta}, 1+2\tilde{\delta}],\ x \in [-2\tilde{\delta}, 2\tilde{\delta}]\}, \end{aligned} \tag{6}$$

see Fig. 2. Take a $C^r$-diffeomorphism $G : \hat{U} \to \mathbb{R}^2$ such that $G(\tilde{U}) \supset U$ and $r \geqslant 2$. In particular, $G(\tilde{q})$ might be equal $q$, but it is not required. Since for homoclinic tangency or intersection $G(\tilde{q})$ indeed equals $q$, both are covered by our model.

Fix a small neighborhood $V$ of a set $\{(x, y) \mid x \in [0, \lambda], y = 0\} \cup \{(x, y) \mid y \in [0, 1], x = 0\}$. Picking small $\tilde{\delta}, \delta > 0$ we can assume that $L^{-1}(\hat{U}) \cap \hat{U} = \emptyset$, $L(U) \cap U = \emptyset$, and $U \cap V = \emptyset$. The map $f$ is defined in the following way:

$$f : \mathcal{V} \to \mathbb{R}^2, \qquad f(x, y) = \begin{cases} L(x, y), & \text{if } (x, y) \in \mathcal{V} \backslash \hat{U}, \\ G(x, y), & \text{if } (x, y) \in \hat{U}. \end{cases} \tag{7}$$

As a matter of fact trajectories we shall investigate will never visit $\hat{U} \setminus \tilde{U}$.

Informally we say that a map $f$ has a *hyperbolic* part $L$ and a *parabolic* part $G$.

### 2.2. Localized $s$-loop periodic orbits

In the introduction we defined periodic trajectories of special kind ($\mathcal{V}$-localized). We are going to investigate the behavior of those. Namely, fix $s \in \mathbb{N}$ and a neighborhoods $V$, $U$ and $\tilde{U}$. We shall consider periodic orbits which "go around" a neighborhood $V \cup U$ not more than $s$ times. Introduce a term "loop."

**Definition 3.** A sequence of points $\{p_0, \dots, p_{m-1}\} \subset V \cup U$ is called a loop (of length $m$), if $p_0 \in U$, $p_{m-1} \in \tilde{U}$, and for each $i = 0, \dots, m-2$ we have $L(p_i) = p_{i+1}$.

The following lemma is obvious.

**Lemma 1.** *Any $(\mathcal{V}, s)$-localized periodic orbit is a disjoint union of $s$ loops.*

In what follows an orbit consisting of $s$ loops is called also *an $s$-loop orbit*.

### 2.3. Description of perturbations

We shall consider an infinite parameter family of analytic perturbations of the map $f$ perturbing only parabolic part $G$ of the map. It is more natural to write these perturbations in a shifted coordinate system. Denote by $(\tilde{x}, \tilde{y}) = (x, y-1)$ coordinates in $\tilde{U}$. Consider a $C^\infty$-function $\rho$ identically 1 in $\tilde{U}$ and 0 outside $\hat{U}$. In these coordinates perturbed map has the following form:

$$G_{\vec{\varepsilon}}(\tilde{x}, \tilde{y}) = G(\tilde{x}, \tilde{y}) + \rho(\tilde{x}, \tilde{y}) \begin{pmatrix} \Phi^1_{\vec{\varepsilon}}(\tilde{x}, \tilde{y}) \\ \Phi^2_{\vec{\varepsilon}}(\tilde{x}, \tilde{y}) \end{pmatrix}, \tag{8}$$

where $\Phi^1_{\vec{\varepsilon}}$ and $\Phi^2_{\vec{\varepsilon}}$ are analytic functions in $\tilde{U}$,

$$\Phi^1_{\vec{\varepsilon}}(x, y) = \sum_{0 \leqslant i, j} \varepsilon^1_{ij} x^i y^j, \qquad \Phi^2_{\vec{\varepsilon}}(x, y) = \sum_{0 \leqslant i, j} \varepsilon^2_{ij} x^i y^j,$$

and

$$\vec{\varepsilon} = \left\{ \varepsilon^k_{ij} \in \mathbb{R} \;\middle|\; \left|\varepsilon^k_{ij}\right| < 1,\ k = 1, 2,\ 0 \leqslant i, j \right\}.$$

The family of maps $\{f_{\vec{\varepsilon}}\}$ we shall study is the following map $f_{\vec{\varepsilon}} : \mathcal{V} \to \mathbb{R}^2$

$$f_{\vec{\varepsilon}}\,(x, y) = \begin{cases} L(x, y), & \text{if } (x, y) \in \mathcal{V} \setminus \hat{U}, \\ G_{\vec{\varepsilon}}(x, y), & \text{if } (x, y) \in \hat{U}. \end{cases} \tag{9}$$

To make all the perturbations small we restrict size of the coefficients. Namely, take a small constant $\zeta$ and require $|\varepsilon^k_{ij}| \leqslant \zeta$ for $k = 1, 2, 0 \leqslant i, j$. So the space of coefficients is the following Hilbert cube:

$$HB(\zeta) = \left\{ \varepsilon^k_{ij} \in \mathbb{R} \;\middle|\; \left|\varepsilon^k_{ij}\right| \leqslant \zeta,\ k = 1, 2,\ 0 \leqslant i, j \right\}. \tag{10}$$

Take $\zeta$ small enough to guarantee that $G_{\vec{\varepsilon}}(\tilde{U}) \supset U$ for all $\vec{\varepsilon} \in HB(\zeta)$.

We shall also use the following constants:

$$M_1 = \sup_{\vec{\varepsilon} \in HB(\zeta)} \left\{ \|f_{\vec{\varepsilon}}\|_{C^1}, \left\|f^{-1}_{\vec{\varepsilon}}\right\|_{C^1} \right\}, \qquad M_2 = \sup_{\vec{\varepsilon} \in HB(\zeta)} \left\{ \|f_{\vec{\varepsilon}}\|_{C^2}, \left\|f^{-1}_{\vec{\varepsilon}}\right\|_{C^2} \right\}. \tag{11}$$

### 2.4. Product measure in the space of coefficients

To make any statements in terms of probability we need to choose a measure in the space of parameters. We do this in the following way.

Let $\mathrm{Leb}_\zeta$ be the Lebesgue measure on the interval $[-\zeta, \zeta]$. For each parameter $\varepsilon^k_{ij}$ we define a probability measure on this interval $\nu^k_{ij} = \frac{1}{2\zeta}\,\mathrm{Leb}_\zeta$ (all $\nu^k_{ij}$'s are the same, but indexes emphasize correspondence to different coefficients). Consider the normalized product Lebesgue measure $\nu$ in the space of coefficients $HB(\zeta)$:

$$\nu = \bigtimes_{0 \leqslant i, j} \left( \nu^1_{ij} \times \nu^2_{ij} \right). \tag{12}$$

*2.5. Statement of main results (unbounded cyclicity)*

**Theorem A.** *For $\nu$-almost every $\vec{\varepsilon} \in HB(\zeta)$ the corresponding map $f_{\vec{\varepsilon}}$ has only a finite number of localized sinks $\{\mathfrak{P}_j\}_j$ whose period exceeds*

$$\mathcal{A}s_j^{4s_j^2+4s_j+7}\big(10(1+\Im)\big)^{2s_j(s_j+1)},$$

*where $s_j = s(\mathfrak{P}_j)$ is cyclicity of a corresponding sink $\mathfrak{P}_j$ and $\mathcal{A}$ is a constant depending on parameters of the problem.*

**Remark 6.** According to the standard terminology this theorem says that for a prevalent diffeomorphism near HT there are only a finite number of localized sinks of large enough period compare to its cyclicity. Such a definition of prevalence is introduced in [17] and is used in [21] under similar circumstances. In [19] a different way to define prevalence is proposed.

Our method provides significant additional information about hyperbolicity of corresponding localized periodic points.

**Definition 4.** We say that a periodic (under the map $g$) point $p$ of period $n$ is $(\mu, \aleph)$-trace hyperbolic if

$$\big|\mathrm{Tr}\, Dg^n(p)\big| > \mu^{(1-\aleph)n}. \tag{13}$$

**Theorem B.** *For any $\aleph > 0$ there is a sequence of numbers $\{\mathbf{N}_s(\aleph)\}_{s\in\mathbb{N}}$ such that for $\nu$-almost every $\vec{\varepsilon} \in HB(\zeta)$ the corresponding map $f_{\vec{\varepsilon}}$ has only a finite number of $\mathcal{V}$-localized periodic points $\{\mathfrak{P}_j\}$ of period greater than $\mathbf{N}_{s_j}(\aleph)$ that are not $(\mu, \aleph)$-trace hyperbolic, where $s_j = s(\mathfrak{P}_j)$ is cyclicity of the corresponding sink $\mathfrak{P}_j$.*

**Addendum 2.1.** *One can take*

$$\mathbf{N}_s(\aleph) = 3s\big(\mathcal{B}s^4 - \big(2s^2+2s+1\big)(s+1)\ln\aleph\big)\frac{(5s^2\aleph^{-1}(1+\Im))^{2s^2+2s+1}}{\ln\mu},$$

*where $\mathcal{B}$ is a constant depending on parameters of the problem. Notice that $\mathbf{N}_s(\aleph) < s^{5s^2}$ for large $s$.*

**Remark 7.** Note that if $p$ is a periodic orbit of a planar diffeomorphism of period $n$ and $|\,\mathrm{Tr}\, Df_{\vec{\varepsilon}}^n(p)| > 2$, then $p$ can not be a sink. Therefore Theorem B implies Theorem A (see Section 4 for more details).

To state corresponding theorems for periodic trajectories of bounded cyclicity we need to introduce families of polynomial perturbations of bounded degree.

## 2.6. Description of polynomial perturbations

Now we consider a finite-parameter family of polynomial perturbations of the map $f$ perturbing only parabolic part $G$ of the map. As above $(\tilde{x}, \tilde{y}) = (x, y-1)$ denotes the shifted coordinates in $\tilde{U}$. In these coordinates perturbed map has the following form:

$$G_{\vec{\varepsilon}_{<2s}}(\tilde{x}, \tilde{y}) = G(\tilde{x}, \tilde{y}) + \rho(\tilde{x}, \tilde{y}) \begin{pmatrix} \Phi^1_{\vec{\varepsilon}_{<2s}}(\tilde{x}, \tilde{y}) \\ \Phi^2_{\vec{\varepsilon}_{<2s}}(\tilde{x}, \tilde{y}) \end{pmatrix}, \tag{14}$$

where $\Phi^1_{\vec{\varepsilon}_{<2s}}$ and $\Phi^2_{\vec{\varepsilon}_{<2s}}$ are polynomials,

$$\Phi^1_{\vec{\varepsilon}_{<2s}}(x, y) = \sum_{0 \leqslant i, j, i+j<2s} \varepsilon^1_{ij} x^i y^j, \qquad \Phi^2_{\vec{\varepsilon}_{<2s}}(x, y) = \sum_{0 \leqslant i, j, i+j<2s} \varepsilon^2_{ij} x^i y^j,$$

and

$$\vec{\varepsilon}_{<2s} = \left\{ \varepsilon^k_{ij} \in \mathbb{R} \mid k = 1, 2, \ 0 \leqslant i, j, \ i + j < 2s \right\}.$$

The family of maps $\{f_{\vec{\varepsilon}_{<2s}}\}$ we shall study is the following map $f_{\vec{\varepsilon}_{<2s}} : \mathcal{V} \to \mathbb{R}^2$

$$f_{\vec{\varepsilon}_{<2s}}(x, y) = \begin{cases} L(x, y), & \text{if } (x, y) \in \mathcal{V} \setminus \hat{U}, \\ G_{\vec{\varepsilon}_{<2s}}(x, y), & \text{if } (x, y) \in \hat{U}. \end{cases}$$

We put the same restriction on size of the coefficients as for analytic perturbations. Namely, we require $|\varepsilon^k_{ij}| \leqslant \zeta$ for $k = 1, 2, \ 0 \leqslant i, j, \ i + j < 2s$. So the space of coefficients is the following cube:

$$HB_{<2s}(\zeta) = \left\{ \varepsilon^k_{ij} \in \mathbb{R} \ \middle| \ \left|\varepsilon^k_{ij}\right| \leqslant \zeta, \ k = 1, 2, \ 0 \leqslant i, j, \ i + j < 2s \right\}. \tag{15}$$

Constant $\zeta$ was chosen small enough to have $G_{\vec{\varepsilon}_{<2s}}(\tilde{U}) \supset U$ and $G_{\vec{\varepsilon}_{<2s}}(\hat{U}) \cap V = \emptyset$ for all $\vec{\varepsilon}_{<2s} \in HB_{<2s}(\zeta)$. Constants $M_1$ and $M_2$ as well as $\zeta$ can be chosen the same as above. Product measure in a space of coefficients is as follows

$$\nu_{<2s} = \bigtimes_{0 \leqslant i, j, i+j<2s} \left( \nu^1_{ij} \times \nu^2_{ij} \right). \tag{16}$$

## 2.7. Statement of main results (bounded cyclicity)

**Theorem A′.** *Fix $s \in \mathbb{N}$. For $\nu_{<2s}$-almost every $\vec{\varepsilon}_{<2s}$ from $HB_{<2s}(\zeta)$ the corresponding map $f_{\vec{\varepsilon}_{<2s}}$ has only a finite number of localized sinks of cyclicity at most $s$.*

If we restrict ourselves to a fixed number of loops $s$, then existence of an infinite number of $s$-loop sinks implies that a corresponding diffeomorphism is not Kupka–Smale[2] (see Appendix A

[2] We are grateful to D. Turaev for this remark.

for a precise statement and a proof). Therefore, this result should be expected. Indeed, in [19] it was shown that Kupka–Smale systems are prevalent (although the notion of prevalence is different from ours).

**Theorem B′.** *Fix $s \in \mathbb{N}$. For any $\aleph > 0$ and $\nu_{<2s}$-almost every $\vec{\varepsilon}_{<2s} \in HB_{<2s}(\zeta)$ the corresponding map $f_{\vec{\varepsilon}_{<2s}}$ has only a finite number of $\mathcal{V}$-localized periodic points of cyclicity at most $s$ that are not $(\mu, \aleph)$-trace hyperbolic.*

### 2.8. Prevalence in the space of families

In this section we define the notion of prevalence in the space of 1-parameter families and then show how Theorem A implies Theorem 1.1. This notion was introduced by Hunt–Sauer–Yorke [17] for linear spaces and by Christensen [5] for Polish spaces.

Consider the space of $C^r$-smooth 1-parameter families of diffeomorphisms $\{f_\kappa : W \to \mathbb{R}^2\}_{\kappa \in I}$ of an open set $W \subset \mathbb{R}^2$ such that $f = f_0$ has a non-resonant[3] saddle periodic point $p = f^k(p)$ which has an HT at some points $q$ and $f^{-k}(q) = \tilde{q}$. Denote this space $C^r_{HT_n}(W \times I, \mathbb{R}^2)$. Choose small $I = [-\varepsilon_0, \varepsilon_0]$ and small neighborhoods $\tilde{U} \subset \hat{U} \subset W$ and $U \subset W$ of points $\tilde{q}$ and $U$ respectively so that $f^k(U) \cap U = \emptyset$, $f^{-k}(\hat{U}) \cap \hat{U} = \emptyset$, $f^k_\kappa(\tilde{U}) \supset U$ and $f^k_\kappa(\hat{U}) \cap V = \emptyset$ for all $\kappa \in I$ as we do in front of Definition 1. Due to theory of normal forms (see, e.g., [18]) there are linearizing normal coordinates in a neighborhood $\tilde{V}$ of a saddle $p$. Consider a family of analytic perturbations (8) of each family inside of $\hat{U}$. This gives the new family

$$\{f_{\kappa,\vec{\varepsilon}}\}_{(\kappa,\vec{\varepsilon}) \in I \times HB(\zeta)}.$$

**Definition 5.** A set of families $\mathcal{U}$ in $C^r_{HT_n}(W \times I, \mathbb{R}^2)$ is called prevalent if for any family $\{f_\kappa\}_{\kappa \in I}$ for $\nu$-almost every $\vec{\varepsilon}_*$ the family $\{f_{\kappa,\vec{\varepsilon}_*}\}_{\kappa \in I}$ belongs to $\mathcal{U}$.

Show that the set of families satisfying Theorem A is prevalent. Consider the set of families $\mathcal{U}$ in $C^r_{HT_n}(W \times I, \mathbb{R}^2)$ such that for any $\{f_\kappa\}_{\kappa \in I} \in \mathcal{U}$ we have that for almost every $\kappa \in I$ the diffeomorphism $f_\kappa$ satisfies Theorem A. Show that $\mathcal{U}$ is prevalent.

Apply to each $f_\kappa$ Theorem A (see Section 13 for additional arguments required to apply Theorem A). It gives that for any $\kappa \in I$ and $\nu$-almost every $\vec{\varepsilon}$ we have that $f_{\kappa,\vec{\varepsilon}}$ has only finitely many sinks of sufficiently large period compare to its cyclicity, i.e. satisfying conditions of Theorem A.

By Fubini Theorem it implies that for $\nu$-almost every $\vec{\varepsilon}$ we have that the family $\{f_{\kappa,\vec{\varepsilon}}\}_{\kappa \in I}$ satisfies conclusion of Theorem A for almost every $\kappa$. Therefore, $\nu$-almost every family is in $\mathcal{U}$.

In the next sections we shall be proving Theorem B′ and extract Theorem B in the process of the proof. Theorems B and B′ imply Theorems A and A′, respectively, see Section 4 for details.

## 3. Scheme of the proof

### 3.1. Strategy

Here we give a complete scheme of the proof omitting involved definitions. Detailed definitions, exact intermediate statements and their proofs are given in subsequent sections.

[3] We hope to get rid of non-resonance condition in a future publication.

**Step I** *(Sorting by type). Reduction to a uniform* (*over all types of a fixed length*) *estimate of the measure of "bad" parameters associated with periodic orbits of a given type.*[4]

Set initial parameters of the problem

$$\mathfrak{W} = \left\{\mu, \lambda, M_1, M_2, V, \delta, \zeta, \Im = -\frac{\ln\lambda}{\ln\mu}\right\}. \tag{17}$$

In what follows these parameters stay fixed.

Define the following sets in the spaces of parameters $HB(\zeta)$ and $HB_{<2s}(\zeta)$:

$$B^{\mathrm{trace}}_{\mathfrak{W},\infty}[f, \mathcal{N}_s, \aleph] = \big\{\vec{\varepsilon} \in HB(\zeta) \bigm| f_{\vec{\varepsilon}} \text{ has a } \mathcal{V}\text{-localized periodic not } (\mu,\aleph)\text{-trace hyperbolic orbit of type } \mathcal{N}_s\big\}, \tag{18}$$

and

$$B^{\mathrm{trace}}_{\mathfrak{W}}[s, f, \mathcal{N}_s, \aleph] = \big\{\vec{\varepsilon}_{<2s} \in HB_{<2s}(\zeta) \bigm| f_{\vec{\varepsilon}_{<2s}} \text{ has a } \mathcal{V}\text{-localized periodic not } (\mu,\aleph)\text{-trace hyperbolic orbit of type } \mathcal{N}_s\big\}. \tag{19}$$

To apply standard Borel–Cantelli argument (as it is done in Section 4) to prove Theorem B′ (which implies Theorem A′) we need to prove that the following series is convergent:

$$\sum_{\mathcal{N}_s} \nu_{<2s}\big\{B^{\mathrm{trace}}_{\mathfrak{W}}[s, f, \mathcal{N}_s, \aleph]\big\} < \infty. \tag{20}$$

Since $\#\{\mathcal{N}_s \mid |\mathcal{N}_s| = n_1 + \cdots + n_s + s = n\} < n^s$, to prove (20) it is enough to show that the following estimate holds true:

$$\nu_{<2s}\big\{B^{\mathrm{trace}}_{\mathfrak{W}}[s, f, \mathcal{N}_s, \aleph]\big\} \leqslant C_s \mu^{-\mathbf{h}_s|\mathcal{N}_s|}, \tag{21}$$

where $\mathcal{N}_s = (n_1, \ldots, n_s)$, $|\mathcal{N}_s| = n_1 + \cdots + n_s + s$, and positive constants $\mathbf{h}_s$ and $C_s$ are uniform over all types $\mathcal{N}_s$ of large enough length $|\mathcal{N}_s|$.

In order to prove Theorems A and B we need to choose (see Sections 4.5 and 14.1) a sequence $\{\mathbf{N}_s(\aleph)\}_s$ such that the following series is convergent:

$$\sum_{s\in\mathbb{N}} \sum_{\mathcal{N}_s,\, |\mathcal{N}_s| \geqslant \mathbf{N}_s(\aleph)} \nu\big\{B^{\mathrm{trace}}_{\mathfrak{W},\infty}[f, \mathcal{N}_s, \aleph]\big\} < \infty. \tag{22}$$

Due to Fubini reduction argument (Section 3.2) estimate (21) implies the same estimate for $\nu\{B^{\mathrm{trace}}_{\mathfrak{W},\infty}[f, \mathcal{N}_s, \aleph]\}$. Therefore we reduced the proof to estimate (21).

---

[4] See Definition 12 for a definition of type.

Set $\beta = \aleph/(5s^2(1+\Im))$. This parameter will be responsible for definitions of short and long loops, sizes of cones, etc. When we need to include $s$, $\beta$ and $\aleph$ to the set of parameters of the problem, we use the notation

$$\mathfrak{Q} = \mathfrak{W} \cup \{s, \beta, \aleph\} = \left\{\mu, \lambda, M_1, M_2, V, \delta, \zeta,\ \Im = -\frac{\ln\lambda}{\ln\mu}, s, \beta, \aleph\right\}. \tag{23}$$

At this moment we fix the extended set of parameters $\mathfrak{Q}$.

**Step II** *(Sorting by shape). Reduction to uniform* (*over all types of given shape and a fixed length*) *estimate of the measure of "bad" parameters associated with periodic orbits of a given type.*

In Section 5.1 we shall introduce notion of shape $l = l(\mathcal{N}_s)$ (see Definition 14). To show that (21) holds it is enough to prove the following estimate:

$$\nu_{<2s}\left\{B^{\mathrm{trace}}_{\mathfrak{W}}[s, f, \mathcal{N}_s, \aleph]\right\} \leqslant C_s \mu^{-h_l|\mathcal{N}_s|}, \tag{24}$$

where $C_s > 0$ and $h_l = h_{l(\mathcal{N}_s)} > 0$ is uniform over all types $\mathcal{N}_s$ having shape $l$. Indeed, if this estimate holds, one can set $\mathbf{h}_s = \min(h_1, \dots, h_s)$.

**Step III.** *From a* $(\mu, \aleph)$*-trace hyperbolicity condition to a generalized loop cone condition.*

Introduce the generalized loop cone condition with constants $(\theta, \xi)$ (see Sections 5.2 and 6.1 for complete definitions).

**Definition 6.** Consider an $s$-loop periodic orbit $\mathfrak{P} = \{p_0, \dots, p_{n-1}\}$ of a map $f_{\vec{\varepsilon}_{<2s}}$ of type $\mathcal{N}_s$, $|\mathcal{N}_s| = n$. We say that a generalized loop cone condition with constants $(\theta, \xi)$ holds if

$$K_{\xi n}(P_i) \hookrightarrow_{Df^{N_i}_{\vec{\varepsilon}_{<2s}}(P_i)} K_{\theta n}(P_{i+1}) \quad \text{for each } i = 0, \dots, t(\mathcal{N}_s) - 1,$$

where $P_i$ is the starting point and $N_i$ is a length of $i$th generalized loop for each $i = 0, \dots, t(\mathcal{N}_s) - 1$, respectively.

Denote this property for periodic orbits by $(f_{\vec{\varepsilon}_{<2s}}, \mathfrak{P}) \in \mathcal{K}\{\mathfrak{Q}, \mathcal{N}_s, n; (\theta, \xi)\}$. Indeed, it depends on $\mathfrak{P}$ and the linearization of $f_{\vec{\varepsilon}_{<2s}}$ at all points from $\mathfrak{P}$.

Define the following sets:

$$B^{\mathrm{cone}}_{\mathfrak{Q}}\big[s, f, \mathcal{N}_s; (\theta, \xi)\big] = \big\{\vec{\varepsilon}_{<2s} \in HB_{<2s}(\zeta) \bigm| f_{\vec{\varepsilon}_{<2s}} \text{ has a } (\mathcal{V}, s)\text{-localized periodic orbit } \mathfrak{P} \text{ of type } \mathcal{N}_s \text{ with } (f_{\vec{\varepsilon}}, \mathfrak{P}) \notin \mathcal{K}\big\{\mathfrak{Q}, \mathcal{N}_s, |\mathcal{N}_s|; (\theta, \xi)\big\}\big\}.$$

We prove (see Lemma 3) that if $n = |\mathcal{N}_s|$ is large enough, then

$$B^{\mathrm{trace}}_{\mathfrak{W}}[s, f, \mathcal{N}_s, \aleph] \subset B^{\mathrm{cone}}_{\mathfrak{Q}}\big[s, f, \mathcal{N}_s; (\theta, \xi)\big], \tag{25}$$

for $\aleph = 2s\theta + 2(1+\Im)(s-1)\beta$ and $0 < \theta \leqslant \xi$. Therefore to prove (24) it is enough to prove that for small $0 < \theta_l \leqslant \xi_l$, $l = 1, \dots, s$,

$$\nu\big\{B^{\mathrm{cone}}_{\mathfrak{Q}}\big[s, f, \mathcal{N}_s; (\theta_{l(\mathcal{N}_s)}, \xi_{l(\mathcal{N}_s)})\big]\big\} \leqslant C_s \mu^{-h_{l(\mathcal{N}_s)}|\mathcal{N}_s|}. \tag{26}$$

**Step IV** *(Partition into non-recurrent parts). Cloud decomposition and reduction to estimate of the measure of "bad" parameters associated with a scattered pseudotrajectory of a given type.*

First we need several definitions.

**Definition 7.** A sequence of points $\mathcal{Z} = \{z_0, \ldots, z_{\mathbf{n}-1}\} \subset \mathcal{V}$ is called a $k$-loop periodic $\gamma$-pseudo-orbit of the map $f_{\vec{\varepsilon}_{<2s}}$ if it intersects $\tilde{U}$ at exactly $k$ points and

(1) if $z_j \notin \tilde{U}$, then $f_{\vec{\varepsilon}_{<2s}}(z_j) = z_{j+1}$;
(2) if $z_j \in \tilde{U}$, then $\operatorname{dist}(G_{\vec{\varepsilon}_{<2s}}(\tilde{z}_j), z_{j+1 \pmod{\mathbf{n}}}) \leqslant \gamma$.

**Remark 8.** We consider pseudo-orbits, for which the image of a point may differ from the next point only for parabolic part of the map $f_{\vec{\varepsilon}_{<2s}}$. Similarly to $s$-loop periodic orbits, any $k$-loop periodic pseudo-orbit is a disjoint union of $k$ loops.

In Section 5.1 we introduce collection of positive rapidly decreasing to zero numbers $\{d_l\}_{l=1}^{s+1}$.

**Definition 8.** A pseudotrajectory has shape $(l, n)$, if all loops have length $\geqslant d_l n$ (long loops) or $< d_{l+1} n$ (short loops).

Notions of type and generalized loop can be introduced for periodic pseudo-orbits in the same way as for periodic $s$-loop orbits. We will use the notations similar to the notations from Sections 4.1 and 5.2 for pseudo-orbits.

**Definition 9.** Given a periodic pseudo-orbit $\mathcal{Z} = \{z_0, \ldots, z_{\mathbf{n}-1}\}$, let us denote by $F^N_{\mathcal{Z}}(z_m) : \mathbb{R}^2 \to \mathbb{R}^2$ the following linear map:

$$F^N_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(z_m) = Df_{\vec{\varepsilon}_{<2s}}(z_{m+N-1}) \circ Df_{\vec{\varepsilon}_{<2s}}(z_{m+N-2}) \circ \cdots \circ Df_{\vec{\varepsilon}_{<2s}}(z_{m+1}) \circ Df_{\vec{\varepsilon}_{<2s}}(z_m).$$

Note that we naturally identify $T_{z_i}V$ with $\mathbb{R}^2$.

Note that if $\mathfrak{P} = \{p_0, \ldots, p_{n-1}\}$ is an orbit (not just pseudo-orbit) of the map $f_{\vec{\varepsilon}_{<2s}}$, then $F^N_{\mathfrak{P},\vec{\varepsilon}_{<2s}}(p_m) = Df^N_{\vec{\varepsilon}_{<2s}}(p_m)$. If $F^N_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(z_m)(K_{A_1}(z_m)) \subset K_{A_1}(z_{m+N})$, we will write

$$K_{A_1}(z_m) \hookrightarrow_{F^N_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(z_m)} K_{A_2}(z_{m+N}).$$

Again, we identify tangent spaces at any two points of $\mathcal{V}$ in a natural way.

**Definition 10.** Consider a $k$-loop periodic pseudo-orbit $\mathcal{Z} = \{z_0, \ldots, z_{\mathbf{n}-1}\}$ of a map $f_{\vec{\varepsilon}_{<2s}}$ of type $\mathcal{N}_k$ having shape $(l, n)$, $|\mathcal{N}_k| = \mathbf{n} \leqslant n$. We say that an $(l, n)$-generalized loop cone condition with constants $(\theta, \xi)$ holds if

$$K_{\xi n}(Z_i) \hookrightarrow_{F^{N_i}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_i)} K_{\theta n}(Z_{i+1}) \quad \text{for each } i = 0, \ldots, \tau - 1,$$

where $\tau = t(\mathcal{N}_k)$ is the number of generalized loops and $Z_i$ is the starting point and $N_i$ is a length of $i$th generalized loop for each $i = 0, \ldots, \tau - 1$ respectively.

Denote this property for periodic pseudo-orbits by $(f_{\vec{\varepsilon}_{<2s}}, \mathcal{Z}) \in \mathcal{K}\{\mathfrak{Q}, \mathcal{N}_k, n; (\theta, \xi)\}$. Indeed, it depends on $\mathcal{Z}$ and the linearization of $f_{\vec{\varepsilon}_{<2s}}$ at all points from $\mathcal{Z}$.

**Definition 11.** A $k$-loop periodic orbit is called $\varrho$-scattered, if for any two generalized loops the distance between starting points of these generalized loops is at least $\varrho$.

Fix a shape $l$. For any sequence $\{\alpha_{l,m}\}_{m=1}^s$, $0 < \alpha_{l,1} < \alpha_{l,2} < \cdots < \alpha_{l,s}$, one can define the following sets in the space of parameters.

Recall that for any type $\mathcal{N}_k$ with $1 \leqslant k \leqslant s$ having shape $(l, n)$ we have $d_l n \leqslant |\mathcal{N}_k| \leqslant n$, a loop is long if length is $\geqslant d_l n$ and short if $< d_{l+1} n$ respectively and $\mathcal{N}_k \subseteq_l \mathcal{N}_s$ denotes $\mathcal{N}_k$ being an $l$-subtype of $\mathcal{N}_s$ (see Definition 21). Suppose $\mathcal{N}_k$ has $\tau = t(\mathcal{N}_k)$ long and $(k - \tau)$ short loops, then for any $0 < \tau \leqslant m \leqslant s$ we define

$$B_{\mathfrak{Q}}^{\text{scatt}}[f, \mathcal{N}_k, n, l, m] = \Big\{\vec{\varepsilon} \in HB_{<2s}(\zeta) \ \Big|\ f_{\vec{\varepsilon}_{<2s}} \text{ has a } k\text{-loop } s^{-1}\mu^{-\alpha_{l,m-1}n}\text{-scattered periodic } \mu^{-\alpha_{l,m}n}\text{-pseudotrajectory } \mathcal{Z} \text{ of type } \mathcal{N}_k \text{ having shape } (l,n) \text{ with } (f_{\vec{\varepsilon}_{<2s}}, \mathcal{Z}) \notin \mathcal{K}\big\{\mathfrak{Q}, \mathcal{N}_k, n; (2\theta_{l,m}, \xi_l)\big\}\Big\}.$$

Existence of a decomposition of any $s$-loop orbit into the union of some scattered pseudotrajectories (see Lemma 8) allows to claim the following. One can choose $\theta_l$ and $\{\theta_{l,m}\}_{m=1}^s$ in such a way that the following inclusion holds for any type $\mathcal{N}_s$ having shape $l$:

$$B_{\mathfrak{Q}}^{\text{cone}}\big[s, f, \mathcal{N}_s; (\theta_l, \xi_l)\big] \subset \bigcup_{\mathcal{N}_k \subseteq_l \mathcal{N}_s} \ \bigcup_{t(\mathcal{N}_k) \leqslant m \leqslant s} B_{\mathfrak{Q}}^{\text{scatt}}\big[s, f, \mathcal{N}_k, |\mathcal{N}_s|, l, m\big]. \tag{27}$$

If we prove that

$$\nu_{<2s}\big\{B_{\mathfrak{Q}}^{\text{scatt}}[s, f, \mathcal{N}_k, n, l, m]\big\} \leqslant C_s^* \mu^{-h_l n} \tag{28}$$

then we get estimate

$$\nu_{<2s}\big\{B_{\mathfrak{W}}^{\text{trace}}[s, f, \mathcal{N}_s, \aleph]\big\} \leqslant \nu_{<2s}\big\{B_{\mathfrak{Q}}^{\text{cone}}\big[s, f, \mathcal{N}_s; (\theta_l, \xi_l)\big]\big\} \leqslant C_s \mu^{-h_l |\mathcal{N}_s|},$$

where one can take $C_s = (s!2^s s)C_s^*$.

**Step V** *(Discretization). Reduction to estimate of the measure of "bad" parameters associated with a scattered admissible pseudotrajectory of a given type.*

After discretization procedure (see Sections 8.1 and 8.2 for the definitions of grids) and the construction of admissible pseudo-orbits (see Definition 23) we introduce one more family of sets in a space of parameters.

One can choose constants $\{\theta_{l,m}\}_{m=1}^s$, $0 < \theta_{l,m} < \theta_l$, in such a way that the following holds. Define the following sets of parameters:

$$B_{\mathfrak{Q}}^{\text{adm}}[s, f, \mathcal{N}_k, n, l, m] = \Big\{\vec{\varepsilon}_{<2s} \in HB_{<2s}(\zeta) \ \Big|\ f_{\vec{\varepsilon}_{<2s}} \text{ has a } k\text{-loop } (2s)^{-1}\mu^{-\alpha_{l,m-1}n}\text{-scattered } \mu^{-\alpha_{l,m}n}\text{-admissible pseudotrajectory } \mathfrak{R} \text{ of type } \mathcal{N}_k \text{ having shape } (l,n) \text{ with } (f_{\vec{\varepsilon}_{<2s}}, \mathfrak{R}) \notin \mathcal{K}\big\{\mathfrak{Q}, \mathcal{N}_k, n; (\theta_{l,m}, \xi_l)\big\}\Big\}. \tag{29}$$

One can show that for an appropriate choice of $\{\theta_{l,m}\}_{m=1}^{s}$ (see Lemma 4) we have the inclusion:

$$B_{\mathfrak{Q}}^{\text{scatt}}[s, f, \mathcal{N}_k, n, l, m] \subset B_{\mathfrak{Q}}^{\text{adm}}[s, f, \mathcal{N}_k, n, l, m]. \tag{30}$$

Therefore to prove (28) we need to justify the estimate

$$\nu_{<2s}\bigl\{B_{\mathfrak{Q}}^{\text{adm}}[s, f, \mathcal{N}_k, n, l, m]\bigr\} \leqslant C_s^* \mu^{-h_l n}. \tag{31}$$

**Step VI** *(Newton Interpolation Polynomials). Proving estimate* (31) *of the measure of the set* $B_{\mathfrak{Q}}^{\text{adm}}[s, f, \mathcal{N}_k, n, l, m]$.

This step is carried out in Section 11, where we prove estimate (31) of the measure of a “bad” set. Preliminary discussions are in Sections 9 and 10. The informal presentation of the general method of investigation of prevalent dynamical properties with the use of Newton Interpolation Polynomials that we apply here can be found in [21, part II, Section 4].

### 3.2. *Fubini reduction from* $HB(\zeta)$ *to* $HB_{<2s}(\zeta)$

We need to estimate $\nu\{B_{\mathfrak{W},\infty}^{\text{trace}}[s, f, \mathcal{N}_s, \aleph]\}$. One can reduce this estimate from infinite-dimensional Hilbert cube of parameters to $(2s+4s^2)$-dimensional cube, that is to the estimate of $\nu_{<2s}\{B_{\mathfrak{W}}^{\text{trace}}[s, f, \mathcal{N}_s, \aleph]\}$, using simple Fubini arguments. Namely, decompose the set of parameters in the following way:

$$\begin{gathered} HB_{\geqslant 2s}(\zeta) = \bigl\{\varepsilon_{ij}^q \in \mathbb{R} \bigm| \bigl|\varepsilon_{ij}^q\bigr| \leqslant \zeta,\ q = 1, 2,\ 0 < i, j,\ 2s \leqslant i + j\bigr\}, \\ HB(\zeta) = HB_{<2s}(\zeta) \oplus HB_{\geqslant 2s}(\zeta). \end{gathered} \tag{32}$$

Each parameter $\vec{\varepsilon} \in HB(\zeta)$ has a unique decomposition into

$$\begin{aligned} &\vec{\varepsilon} = (\vec{\varepsilon}_{<2s}, \vec{\varepsilon}_{\geqslant 2s}) \in HB_{<2s}(\zeta) \oplus HB_{\geqslant 2s}(\zeta), \\ &\Phi_{\vec{\varepsilon}}^q(x, y) = \Phi_{\vec{\varepsilon}_{<2s}}^q(x, y) + \Phi_{\vec{\varepsilon}_{\geqslant 2s}}^q(x, y) = \sum_{0\leqslant i,j,i+j<2s} \varepsilon_{ij}^q x^i y^j + \sum_{0\leqslant i,j,2s\leqslant i+j} \varepsilon_{ij}^q x^i y^j. \end{aligned} \tag{33}$$

Decompose the product measure $\nu$ in the space of parameters, defined in (12), into the direct products

$$\nu = \nu_{<2s} \times \nu_{\geqslant 2s}, \quad \text{where} \quad \nu_{\geqslant 2s} = \mathop{\times}_{0\leqslant i,j,\,2s\leqslant i+j} \bigl(\nu_{ij}^1 \times \nu_{ij}^2\bigr). \tag{34}$$

Thus, each component of the decomposition of the space of parameters is supplied with the Lebesgue product probability measure. Suppose we can get an estimate

$$\nu_{<2s}\bigl\{B_{\mathfrak{W}}^{\text{trace}}[s, f, \mathcal{N}_s, \aleph, \vec{\varepsilon}_{\geqslant 2s}]\bigr\} < C_s \mu^{-\mathbf{h}_s|\mathcal{N}_s|} \tag{35}$$

of the measure of the “bad” set

$$B^{\text{trace}}_{\mathfrak{W}}[s,\, f, \mathcal{N}_s, \aleph, \vec{\varepsilon}_{\geqslant 2s}] = \big\{\vec{\varepsilon}_{<2s} \in HB_{<2s}(\zeta) \mid f_{\vec{\varepsilon}} = f_{\vec{\varepsilon}_{<2s}\oplus\vec{\varepsilon}_{\geqslant 2s}} \text{ has an } s\text{-loop periodic}$$
$$\text{not } (\mu, \aleph)\text{-trace hyperbolic orbit of type } \mathcal{N}_s\big\} \tag{36}$$

in each slice $HB_{<2s}(\zeta)\times\{\vec{\varepsilon}_{\geqslant 2s}\}\subset HB(\zeta)$ uniformly over all parameters $\vec{\varepsilon}_{\geqslant 2s}$ in $HB_{\geqslant 2s}(\zeta)$. Then by Fubini Theorem and choice of the probability measure (34), estimate (35) implies the same estimate of the measure $\nu\{B^{\text{trace}}_{\mathfrak{W},\infty}[s,\, f, \mathcal{N}_s, \aleph]\}$.

Fix a parameter value $\vec{\varepsilon}_{\geqslant 2s} \in HB_{\geqslant 2s}(\zeta)$ and the corresponding parameter slice $HB_{<2k}(\zeta)\times\{\vec{\varepsilon}_{\geqslant 2s}\}$ in the space of parameters $HB(\zeta)$. Let $\tilde{f} = f_{(0,\vec{\varepsilon}_{\geqslant 2s})}$ be the center of this slice. In this slice we have the family

$$\{\tilde{f}_{\vec{\varepsilon}_{<2s}}\}_{\vec{\varepsilon}_{<2s}\in HB_{<2s}(\zeta)} = \{f_{(\vec{\varepsilon}_{<2s},\vec{\varepsilon}_{\geqslant 2s})}\}_{\vec{\varepsilon}_{<2s}\in HB_{<2s}(\zeta)} \tag{37}$$

of perturbations by polynomials of degree $2s-1$. This is the family for which we shall investigate the measure of "bad" parameters. Re-denote the set of "bad" parameters $B^{\text{trace}}_{\mathfrak{W}}[s,\, f, \mathcal{N}_s, \aleph, \vec{\varepsilon}_{\geqslant 2s}]$ by $B^{\text{trace}}_{\mathfrak{W}}[s,\, \tilde{f}, \mathcal{N}_s, \aleph]$.

## 4. Reduction to Auxiliary Theorems

In this section we state two Auxiliary Theorems and reduce the proof of Theorems A, B, A$'$ and B$'$ to the proof of those two. Recall that parameters of the problem $\mathfrak{W} = \{\mu, \lambda, M_1, M_2, V, \delta, \zeta\}$ are fixed once and for all.

### *4.1. Types of localized s-loop periodic orbits*

Let us consider a $(\mathcal{V}, s)$-localized periodic orbit $\mathfrak{P} = \{p_0, \dots, p_{n-1}\}\subset\mathcal{V}$ of a diffeomorphism $f_{\vec{\varepsilon}}$ (of period $n$). This orbit (by definition) meets a neighborhood $\tilde{U}$ (respectively $U$) at exactly $s$ points. Denote those points by $\tilde{\mathbf{p}}_0, \tilde{\mathbf{p}}_2, \dots, \tilde{\mathbf{p}}_{s-1}$ (in $\tilde{U}$) and $\mathbf{p}_0, \mathbf{p}_2, \dots, \mathbf{p}_{s-1}$ (in $U$) in such a way that

$$\tilde{\mathbf{p}}_0 = L^{n_1}(\mathbf{p}_0),\quad \dots,\quad \tilde{\mathbf{p}}_i = L^{n_{i+1}}(\mathbf{p}_i),\quad \dots,\quad \tilde{\mathbf{p}}_{s-1} = L^{n_s}(\mathbf{p}_{s-1}),\quad \text{and}$$
$$\mathbf{p}_1 = G_{\vec{\varepsilon}}(\tilde{\mathbf{p}}_0),\quad \dots,\quad \mathbf{p}_{i+1} = G_{\vec{\varepsilon}}(\tilde{\mathbf{p}}_i),\quad \dots,\quad \mathbf{p}_0 = G_{\vec{\varepsilon}}(\tilde{\mathbf{p}}_{s-1}). \tag{38}$$

Similar notations can be used for diffeomorphisms $f_{\vec{\varepsilon}_{<2s}}$ too.

Note that $n = n_1 + n_2 + \dots + n_s + s$.

**Definition 12.** We say that a $(\mathcal{V}, s)$-localized periodic orbit described above has *type* $\mathcal{N}_s = (n_1, \dots, n_s)$.

### *4.2. Auxiliary Theorem I*

To prove Theorems A, B, A$'$ and B$'$ we apply the standard Borel–Cantelli argument. Essential ingredient of these type of arguments is estimates of the measure where certain "bad" phenomenon occurs. The following result provides such an estimate.

**Auxiliary Theorem I.** *Given $s \in \mathbb{N}$. For any $\aleph > 0$ and a sufficiently large $\mathbf{N}^*(s, \aleph)$ there are positive constants $h_s = h_s(\aleph)$ and $C_s$ such that for any type $\mathcal{N}_s = (n_1, \ldots, n_s)$, $n = n_1 + \cdots + n_s + s$, $n > \mathbf{N}^*(s, \aleph)$, we have estimates*

$$\nu\big\{B^{\mathrm{trace}}_{\mathfrak{W},\infty}[f, \mathcal{N}_s, \aleph]\big\} \leqslant C_s \mu^{-\mathbf{h}_s n},$$
$$\nu_{<2s}\big\{B^{\mathrm{trace}}_{\mathfrak{W}}[s, f, \mathcal{N}_s, \aleph]\big\} \leqslant C_s \mu^{-\mathbf{h}_s n}. \tag{39}$$

*Moreover, one can take $\mathbf{N}^*(s, \aleph) = Bs(5s^2\aleph^{-1}(1 + \Im))^{2s^2+2s+1}$, $C_s = \exp(s^2(A + 9\ln s))$, $\mathbf{h}_s = \mathbf{h}_s(\aleph) = (5s^2\aleph^{-1}(1 + \Im))^{-(2s^2+2s+1)}$, where $B = B(\mathfrak{W})$ and $A = A(\mathfrak{W})$ depend on parameters $\mathfrak{W}$ only.*

### *4.3. Auxiliary Theorem II*

Recall that a periodic orbit is called non-hyperbolic if one of its linearization eigenvalues has an absolute value equal to 1.

**Auxiliary Theorem II.** *For $\nu$-almost every $\vec{\varepsilon} \in HB(\zeta)$ the map $f_{\vec{\varepsilon}}$ has no non-hyperbolic $\mathcal{V}$-localized periodic orbits.*

*For any $s \in \mathbb{N}$, for all $0 < s' \leqslant s$ and $\nu_{<2s}$-almost every $\vec{\varepsilon}_{<2s} \in HB_{<2s}(\zeta)$ the map $f_{\vec{\varepsilon}_{<2s}}$ has no non-hyperbolic $(\mathcal{V}, s')$-localized periodic orbits.*

**Remark 9.** Note that due to Fubini reduction argument (Section 3.2) the first part of Auxiliary Theorem II is a consequence of the second part.

### *4.4. Auxiliary Theorems imply Theorems A′ and B′*

Assume that Auxiliary Theorems hold. We show that this implies Theorem B′.

Take a small $r > 0$. Show that

$$\nu_{<2s}\Big\{\vec{\varepsilon}_{<2s} \in HB_{<2s}(\zeta) \,\Big|\, f_{\vec{\varepsilon}_{<2s}} \text{ has an infinite number of } (\mathcal{V}, s)\text{-localized not } (\mu, \aleph)\text{-trace hyperbolic periodic points of period greater than } \mathbf{N}^*(s, \aleph)\Big\} \leqslant b. \tag{40}$$

Suppose that for some $b > 0$ the inequality (40) fails. For a given period $n$ there are at most $n^s$ different types of $(\mathcal{V}, s)$-localized periodic orbits of this period. Therefore, Auxiliary Theorem I implies that the measure of parameters $\vec{\varepsilon}_{<2s} \in HB_{<2s}(\zeta)$, for which the corresponding map $f_{\vec{\varepsilon}_{<2s}}$ has a $(\mathcal{V}, s)$-localized not $(\mu, \aleph)$-trace hyperbolic periodic orbit of period $n$, is not greater than $C_s n^s \mu^{-\mathbf{h}_s n}$.

Note that the series

$$\sum_{n=1}^{\infty} C_s n^s \mu^{-\mathbf{h}_s n}$$

is convergent.

Take $N_0$ so large that the following inequality holds:

$$\sum_{n=N_0}^{\infty} C_s n^s \mu^{-\mathbf{h}_s n} < \frac{b}{2}.$$

Now Auxiliary Theorem I implies that for a set of parameters of the measure at least $b/2$ there is an infinite number of not $(\mu, \aleph)$-trace hyperbolic periodic orbits of period $\leqslant N_0$. This implies that for some $N_1 < N_0$ and for a set of parameters of the measure at least $b/(2N_0)$ there is an infinite number of periodic orbits of the same period $N_1$ (an absence of trace hyperbolicity is not essential now).

**Proposition 1.** *If the map $f_{\vec{\varepsilon}_{<2s}}$ has an infinite number of $(\mathcal{V}, s)$-localized periodic points of the same period $N_1$, then $f_{\vec{\varepsilon}}$ has a non-hyperbolic $(\mathcal{V}, s')$-localized periodic orbit, $s' \leqslant s$, of period not greater than $N_1$.*

**Proof.** Since period is bounded, any limit point of those periodic points has to be a non-hyperbolic periodic point. □

This contradicts Auxiliary Theorem II and, therefore, proves (40). Since $b$ here can be taken arbitrary small, Proposition 1 and Auxiliary Theorem II imply that for almost every $\vec{\varepsilon}_{<2s}$ the map $f_{\vec{\varepsilon}_{<2s}}$ has only finite number of $s$-loop orbits of period $\leqslant \mathbf{N}^*(s, \aleph)$. This proves Theorem B′.

**Proof of Theorem A′.** This theorem is an immediate consequence of Theorem B′ and Auxiliary Theorem II. Indeed, apply Theorem B′ with $\aleph = 1/2$. Note that if $n > [(2\ln 2)/(\ln \mu)] + 1$ then $\mu^{(1-\aleph)n} = \mu^{n/2} > 2$. Due to Auxiliary Theorem II and Proposition 1 for almost every $\vec{\varepsilon}_{<2s} \in HB_{<2s}(\zeta)$ a map $f_{\vec{\varepsilon}_{<2s}}$ has only finite number of periodic $(\mathcal{V}, s)$-localized orbits of period $\leqslant [(2\ln 2)/(\ln \mu)] + 1$. Now Theorem A′ follows from Theorem B′ and Remark 7. □

*4.5. Derivation of Theorems A and B from Auxiliary Theorems*

**Prove of Theorem B.** Take $\aleph \in (0, 1)$. Auxiliary Theorem I claims that for any type $\mathcal{N}_s$, $|\mathcal{N}_s| > \mathbf{N}^*(s, \aleph)$,

$$\begin{aligned} &\nu\bigl\{\vec{\varepsilon} \in HB(\zeta) \mid f_{\vec{\varepsilon}} \text{ has a } \mathcal{V}\text{-localized not } (\mu, \aleph)\text{-trace hyperbolic orbit of type } \mathcal{N}_s\bigr\} \\ &\quad \leqslant C_s \mu^{-\mathbf{h}_s |\mathcal{N}_s|}, \end{aligned} \tag{41}$$

where $C_s = \exp(s^2(A + 9\ln s))$ and $\mathbf{h}_s = \mathbf{h}_s(\aleph) = (5s^2\aleph^{-1}(1 + \Im))^{-(2s^2+2s+1)}$.

Consider a sequence $\{\mathbf{N}_s(\aleph)\}_s$ chosen in such a way that $\mathbf{N}_s(\aleph) > \mathbf{N}^*(s, \aleph)$, and the series

$$\sum_{s\in\mathbb{N}} \sum_{\mathcal{N}_s,\ |\mathcal{N}_s| \geqslant \mathbf{N}_s(\aleph)} \nu\bigl(B^{\text{trace}}_{\mathfrak{W},\infty}[f, \mathcal{N}_s, \aleph]\bigr) \leqslant \sum_{s\in\mathbb{N}} \sum_{\mathcal{N}_s,\ |\mathcal{N}_s| \geqslant \mathbf{N}_s(\aleph)} C_s \mu^{-\mathbf{h}_s |\mathcal{N}_s|}. \tag{42}$$

For any period $n$ and cyclicity $s$ there exists at most $n^s$ different types $\mathcal{N}_s$ such that $|\mathcal{N}_s| = n$. So if $\mathbf{N}_s(\aleph)$ grows fast enough with $s$, this series converges, and due to Borel–Cantelli argument almost every $\vec{\varepsilon} \in HB(\zeta)$ belongs to only a finite number of sets $B^{\text{trace}}_{\mathfrak{W},\infty}[f, \mathcal{N}_s, \aleph]$, $|\mathcal{N}_s| \geqslant \mathbf{N}_s(\aleph)$.

By Auxiliary Theorem II for almost every $\vec{\varepsilon} \in HB(\zeta)$ and any type $\mathcal{N}_s$ the map $f_{\vec{\varepsilon}}$ can have only finite number of $\mathcal{V}$-localized periodic orbits of type $\mathcal{N}_s$. This implies Theorem B. $\square$

Quantitative estimates showing how fast $\mathbf{N}_s(\aleph)$ should grow with $s$ and the proof of Addendum 2.1 are presented in Section 14.

Theorem A follows from Theorem B and Addendum 2.1 in the same way as Theorem A′ follows from Theorem B′.

## 5. Combinatorics of the loops

Here we define the notions of long and short loops, generalized loop, shape, and make a reduction to the estimate (24).

### 5.1. "Short" and "long" loops

Consider a $(\mathcal{V}, s)$-localized periodic orbit $\{p_0, \ldots, p_{n-1}\} \subset \mathcal{V} = V \cup U$ of type $\mathcal{N}_s = (n_1, \ldots, n_s)$. It is the union of $s$ loops of lengths $(n_1 + 1, \ldots, n_s + 1)$. Some of loops can be much longer than others. As we have seen in the model example, it is essential to treat some of these loops as "long" and others as "short." Below we give an algorithm of division.

Take a small constant $0 < \beta \ll 1/s$ to be determined later. A decision whether a particular loop is short or long depends on a type $\mathcal{N}_s$ and $\beta$.

Introduce the following constants $\{d_i\}_{i=1}^{s+1}$:

$$d_{s+1} = \beta^{(2s+2)s+1},\ d_s = \beta^{(2s+2)(s-1)+1},\ \ldots,\ d_i = \beta^{(2s+2)(i-1)+1},\ \ldots,\ d_1 = \beta. \tag{43}$$

**Definition 13.** A loop of length $(n_i + 1)$ of a $(\mathcal{V}, s)$-localized periodic orbit of period $n$ is called $d$-long (respectively $d$-short), if $(n_i + 1) > dn$ (respectively $(n_i + 1) \leqslant dn$).

**Lemma 2.** *For any type $\mathcal{N}_s = (n_1, \ldots, n_s)$, $n_1 + \cdots + n_s + s = n$, there is $j \in \{1, \ldots, s\}$ such that any loop of any $s$-loop periodic orbit of type $\mathcal{N}_s$ is either $d_j$-long or $d_{j+1}$-short.*

**Proof.** Since $0 < \beta < 1/s$, there is at least one loop of length greater than $d_1 n = \beta n$. Consider intervals

$$I_s = (d_{s+1}n, d_s n], \quad \ldots, \quad I_i = (d_{i+1}n, d_i n], \quad \ldots, \quad I_1 = (d_2 n, d_1 n].$$

At most $(s - 1)$ loops have lengths which belong to one of these intervals. By the Pigeon hole principle at least one interval is "empty." If interval $I_j$ is "empty," then any loop is either $d_j$-long or $d_{j+1}$-short. Lemma 2 is proved. $\square$

**Definition 14.** We say that an $s$-loop periodic orbit of type $\mathcal{N}_s = (n_1, \ldots, n_s)$, $n = n_1 + \cdots + n_s + s$, has shape $l$ if

$$l = \min\bigl\{j \in \{1, \ldots, s\} \mid \text{any loop is either } d_j\text{-long or } d_{j+1}\text{-short}\bigr\}.$$

**Remark 10.** A $(\mathcal{V}, s)$-localized orbit of shape $l$ has at least $l$ loops which are $d_l$-long (or, equivalently, it has at most $(s - l)$ loops which are $d_{l+1}$-short).

*5.2. Generalized loops*

Consider an $s$-loop periodic orbit of type $\mathcal{N}_s$. We need to mark all $d_l$-long loops. By definition all $d_l$-long loops are so much longer than $d_{l+1}$-short ones that we regroup them into generalized loops. Each generalized loop starts at one $d_l$-long loop and ends right before the next $d_l$-long one. In other words, we attach to each $d_l$-long loop following afterwards $d_{l+1}$-short loops. Formal definition of generalized loop is the following. Consider an $s$-loop periodic orbit $\{p_0, \dots, p_{n-1}\}$ of type $\mathcal{N}_s = (n_1, \dots, n_s)$ and shape $l$. It is the union of $s$ loops, cyclically ordered in a natural way.

**Definition 15.** A generalized loop is the union of a $d_l$-long loop and all (if there are any) consecutive $d_{l+1}$-short loops following afterwards.

Introduce notations:

- $t$ – the number of generalized loops (= the number of $d_l$-long loops);

Each $j$th generalized loop, $j = 1, \dots, t$, has the following characteristics:

- length $N_j$;
- $n_j^* + 1$ – length of the corresponding $j$th $d_l$-long loop;
- $P_{j-1}$ – its starting point. Set $P_t = P_0$;
- $\tilde{P}_{j-1}$ – its ending point. Set $\tilde{P}_t = \tilde{P}_0$;
- $h_j$ – the number of $d_{l+1}$-short loops after the $j$th $d_l$-long loop.

Note that $\tilde{P}_{j-1} = f_{\vec{\varepsilon}}^{N_j-1} P_{j-1} \in \tilde{U}$ and $\{P_0, \dots, P_{t-1}\} \subseteq \{\mathbf{p}_0, \dots, \mathbf{p}_{s-1}\}$, $\{\tilde{P}_0, \dots, \tilde{P}_{t-1}\} \subseteq \{\tilde{\mathbf{p}}_0, \dots, \tilde{\mathbf{p}}_{s-1}\}$.

The following proposition is a direct consequence of definitions.

**Proposition 2.**

- *Total number of short loops* $\sum_{j=1}^t h_j = s - t \leqslant s - 1$.
- *Total length of short loops* $n - \sum_{j=1}^t n_j^* - t \leqslant (s-t) d_{l+1} n$.
- *Length of $j$th generalized loop* $N_j \leqslant n_j^* + 1 + (s-1) d_{l+1} n$. (44)

*5.3. Restriction to the case of a given shape*

Recall that the set of parameters $\mathfrak{Q} = \{\mu, \lambda, M_1, M_2, V, \delta, \zeta, \Im = -(\ln\lambda)/(\ln\mu), s, \beta, \aleph\}$ is fixed. Take any shape $l \in \{1, \dots, s\}$.

We will prove the following estimate for any type $\mathcal{N}_s = (n_1, \dots, n_s)$ of shape $l$:

$$\begin{aligned} \nu\big\{\vec{\varepsilon} \;\big|\; & f_{\vec{\varepsilon}} \text{ has a } (\mathcal{V}, s)\text{-localized periodic orbit of type } \mathcal{N}_s, \\ & \text{which is not } (\mu, \aleph)\text{-trace hyperbolic}\big\} \leqslant C_s \mu^{-h_l n}, \end{aligned} \tag{45}$$

where $h_l > 0$ and $C_s > 0$ do not depend on type $\mathcal{N}_s$ and $n = n_1 + \dots + n_s + s$.

Taking $\mathbf{h}_s = \min_l h_l$, we have a uniform estimate (i.e. Auxiliary Theorem I):

$$\nu\Big\{\vec{\varepsilon}\ \Big|\ f_{\vec{\varepsilon}} \text{ has a } (\mathcal{V}, s)\text{-localized periodic orbit of type } \mathcal{N}_s = (n_1, \dots, n_s), \\ \text{which is not } (\mu, \aleph)\text{-trace hyperbolic}\Big\} \leqslant C_s \mu^{-\mathbf{h}_s n}, \tag{46}$$

where $n = n_1 + \cdots + n_s + s$.

## 6. Cones and trace hyperbolicity

In this section we prove the inclusion (25).

### 6.1. Cones and generalized loop cone condition

Instead of checking $(\mu, \aleph)$-trace hyperbolicity (13), we will check another, more geometric condition. We have already discussed the latter (cone) condition in Section 3.1 (see Definition 10).

**Definition 16.** Denote by $K_A(p)$ a vertical cone at point $p \in \mathcal{V}$ (i.e. in $T_p\mathcal{V}$) of the following form:

$$K_A(p) = \big\{\bar{v} = (v_x, v_y) \in T_p\mathcal{V}\ \big|\ |v_y| \geqslant \mu^{-A}|v_x|\big\}.$$

**Definition 17.** Assume that $f^m_{\vec{\varepsilon}_{<2s}}(p_1) = p_2$. If $Df^m_{\vec{\varepsilon}_{<2s}}(p_1)(K_{A_1}(p_1)) \subset K_{A_2}(p_2)$, we will write

$$K_{A_1}(p_1) \hookrightarrow_{Df^m_{\vec{\varepsilon}_{<2s}}} K_{A_2}(p_2).$$

**Definition 18.** We will say that generalized loop cone condition with constants $(\theta, \xi)$, $0 < \theta \leqslant \xi$, holds for an $s$-loop periodic orbit $\mathfrak{P} = \{p_1, \dots, p_n\}$ if for each $i = 0, \dots, t-1$ we have

$$K_{\xi n}(P_i) \hookrightarrow_{Df^{N_{i+1}}_{\vec{\varepsilon}_{<2s}}} K_{\theta n}(P_{i+1}).$$

### 6.2. Generalized loop cone condition implies trace hyperbolicity

The following lemma shows that the cone condition implies trace hyperbolicity.

**Lemma 3.** *Let $\beta, \theta$ and $\xi$ be sufficiently small, $0 < \theta \leqslant \xi$, and let $\mathbf{N}$ be sufficiently large. Suppose a $(\mathcal{V}, s)$-localized periodic orbit of $f_{\vec{\varepsilon}_{<2s}}$ of period $n > \mathbf{N}$ satisfies the generalized loop cone condition with constants $(\theta, \xi)$. Then it is $(\mu, \aleph)$-trace hyperbolic, $\aleph = 2s\theta + 2(1 + \Im)(s-1)\beta$.*

**Proof.** Let us denote $g = f_{\vec{\varepsilon}_{<2s}}$ for brevity. We use the notations from Section 5.2. Since for each $i = 0, \dots, t-1$ the inclusion $K_{\theta n}(P_i) \subset K_{\xi n}(P_i)$ holds, we have

$$K_{\xi n}(P_0) \hookrightarrow_{Dg^n} K_{\theta n}(P_0).$$

It means that there is a unit eigenvector of a linear map $Dg^n(P_0)$ in $K_{\theta n}(P_0)$, denoted by $\vec{w} \in K_{\theta n}(P_0)$, $|\vec{w}| = 1$.

Denote an eigenvalue, corresponding to $\vec{w}$, by $\sigma_1 \in \mathbb{R}$. The second eigenvalue of $Dg^n(P_0)$ also has to be real and is denoted by $\sigma_2 \in \mathbb{R}$.

We can estimate product of eigenvalues $|\sigma_1 \cdot \sigma_2| = |\det Dg^n(P_0)|$. Type of $(\mathcal{V}, s)$-localized trajectories and the form of $Df$ (see (9)) we have

$$|\sigma_1 \cdot \sigma_2| = \left|\det Dg^n(P_0)\right| \leqslant \lambda^{n-s}\mu^{n-s}M_1^s.$$

In order to estimate the sum $|\sigma_1 + \sigma_2|$ estimate $|\sigma_1|$ first.

For each $i = 1, \ldots, t+1$ set $\vec{w}_i = Dg^{N_1+\cdots+N_{i-1}}(\vec{w})$, that is, $\vec{w}_1 = \vec{w}$, $\vec{w}_2 = Dg^{N_1}(\vec{w}) \in T_{P_1}V, \ldots, \vec{w}_t = Dg^{n-N_t}(\vec{w}) \in T_{P_{t-1}}V$, $\vec{w}_{t+1} = Dg^n(\vec{w}) = \sigma_1\vec{w}$.

Estimate $y$-component of these vectors $(\vec{w}_1)_y, \ldots, (\vec{w}_t)_y, (\vec{w}_{t+1})_y$. First of all, note that since $\vec{w}_i \in K_{\theta n}(P_i)$, we have $|(\vec{w}_i)_y| \geqslant \mu^{-\theta n}|(\vec{w}_i)_x|$. Since $|(\vec{w}_i)_x| + |(\vec{w}_i)_y| \geqslant |\vec{w}_i|$, we have $(1+\mu^{\theta n})|(\vec{w}_i)_y| \geqslant |(\vec{w}_i)_x| + |(\vec{w}_i)_y| \geqslant |\vec{w}_i|$, which implies

$$\left|(\vec{w}_i)_y\right| \geqslant \frac{1}{1+\mu^{\theta n}}|\vec{w}_i| > \frac{1}{2}\mu^{-\theta n}|\vec{w}_i|.$$

Now in notations of Section 5.2 we have

$$\begin{aligned}|\vec{w}_{i+1}| &= \left|Dg^{N_i}(\vec{w}_i)\right| = \left|Dg^{N_i-n_i^*}\left(L^{n_i^*}(\vec{w}_i)\right)\right| \geqslant M_1^{-h_i-1}\lambda^{N_i-n_i^*}\left|L^{n_i^*}(\vec{w}_i)\right| \\ &\geqslant M_1^{-h_i-1}\lambda^{N_i-n_i^*}\mu^{n_i^*}\left|(\vec{w}_i)_y\right| > \frac{1}{2}M_1^{-h_i-1}\lambda^{N_i-n_i^*}\mu^{n_i^*}\mu^{-\theta n}|\vec{w}_i|. \end{aligned} \tag{47}$$

After counting over all the generalized loops and applying Proposition 2 we have:

$$\begin{aligned}|\sigma_1||\vec{w}| = |\vec{w}_{t+1}| &\geqslant \frac{1}{2^t}M_1^{-t-\sum_{i=1}^t h_i}\lambda^{\sum_{i=1}^t(N_i-n_t^*)}\mu^{\sum_{i=1}^t n_i^*}\mu^{-t\theta n}|\vec{w}| \\ &\geqslant \frac{1}{2^s}M_1^{-s}\lambda^{s+(s-1)d_{l+1}n}\mu^{n-s-(s-1)d_{l+1}n}\mu^{-s\theta n}|\vec{w}| \\ &= \left[2^{-s}M_1^{-s}\mu^{-(1+\Im)s}\right]\mu^{(1-s\theta-(1+\Im)(s-1)d_{l+1})n}|\vec{w}|. \end{aligned} \tag{48}$$

Since $|\sigma_1 \cdot \sigma_2| \leqslant \lambda^{n-s}\mu^{n-s}M_1^s = M_1^s\mu^{(\Im-1)s}\mu^{(1-\Im)n}$, we have

$$|\sigma_2| \leqslant \frac{1}{|\sigma_1|}M_1^s\mu^{(\Im-1)s}\mu^{(1-\Im)n} \leqslant \left[2^s M_1^{2s}\mu^{2\Im s}\right]\mu^{(-\Im+s\theta+(1+\Im)(s-1)d_{l+1})n}.$$

Now we can estimate $|\sigma_1 + \sigma_2|$:

$$\begin{aligned}|\sigma_1+\sigma_2| \geqslant |\sigma_1| - |\sigma_2| &\geqslant \left[2^{-s}M_1^{-s}\mu^{-(1+\Im)s}\right]\mu^{(1-s\theta-(1+\Im)(s-1)d_{l+1})n} \\ &\quad - \left[2^s M_1^{2s}\mu^{2\Im s}\right]\mu^{(-\Im+s\theta+(1+\Im)(s-1)d_{l+1})n} \\ &\geqslant \left[2^{-s}M_1^{-s}\mu^{-(1+\Im)s}\right]\mu^{(1-s\theta-(1+\Im)(s-1)d_{l+1})n} \\ &\quad \times \left[1 - \left(2^{2s}M_1^{3s}\mu^{s+3\Im s}\right)\mu^{(-1-\Im+2s\theta+2(1+\Im)(s-1)d_{l+1})n}\right]. \end{aligned} \tag{49}$$

If $\theta$ and $\beta$ are small enough, and $n$ is large enough, then

$$\left[1 - \left(2^{2s}M_1^{3s}\mu^{s+3\Im s}\right)\mu^{(-1-\Im+2s\theta+2(1+\Im)(s-1)d_{l+1})n}\right] > \frac{1}{2},$$

and we have

$$
\begin{aligned}
|\sigma_1+\sigma_2| &> \left[2^{-s-1}M_1^{-s}\mu^{-(1+\Im)s}\right]\mu^{(1-s\theta-(1+\Im)(s-1)d_{l+1})n} \\
&\geqslant \left[2^{-s-1}M_1^{-s}\mu^{-(1+\Im)s}\right]\mu^{(1-s\theta-(1+\Im)(s-1)\beta)n} \\
&= \left[2^{-s-1}M_1^{-s}\mu^{-(1+\Im)s}\mu^{(s\theta+(1+\Im)(s-1)\beta)n}\right]\mu^{(1-2s\theta-2(1+\Im)(s-1)\beta)n}. \qquad (50)
\end{aligned}
$$

For large $n$

$$\left[2^{-s-1}M_1^{-s}\mu^{-(1+\Im)s}\mu^{(s\theta+(1+\Im)(s-1)\beta)n}\right]\geqslant 1,$$

therefore we have

$$\left|\operatorname{Tr} Dg^n(P_0)\right| = |\sigma_1+\sigma_2| > \mu^{(1-2s\theta-2(1+\Im)(s-1)\beta)n}.$$

Lemma 3 is proven. □

## 7. Decomposition into scattered pseudotrajectories

In this section we introduce combinatorial constants, decompose $s$-loop periodic orbits into the union of some scattered pseudo-orbits and, finally, prove the inclusion (27).

### *7.1. Choice of combinatorial constants and cone characteristics*

Here we provide exact values for the following set of constants:

- $\{(\theta_l,\xi_l)\}_{l=1}^s$ – sizes of cones for pseudotrajectories;
- $\{\theta_{l,m}\}_{l,m}^s$ – sizes of cones for admissible pseudotrajectories;
- $\{\alpha_{l,m}\}_{l,m=1}^s$ – exponents of sizes of grids and scales;
- $\{h_l\}_{l=1}^s$ – exponents in upper bounds of the measure of sets of “bad” parameters.

**Definition 19.** Introduce the following notations:

- $\xi_l=\theta_l=d_l\beta=\beta^{(2s+2)(l-1)+2}$, $l=1,\dots,s$;
- $\theta_{l,m}=\xi_l\beta^{2(s-m)+2}=\beta^{2(sl+l-m+1)}$, $l,m=1,\dots,s$;
- $\alpha_{l,m}=\xi_l\beta^{2(s-m)+1}=\beta^{2(sl+l-m)+1}$, $l,m=1,\dots,s$;
- $h_l=d_{l+1}=d_l\beta^{2s+2}=\beta^{(2s+2)l+1}$, $l=1,\dots,s$. (51)

**Remark 11.** Since $\beta$ is small, we have

$$h_l=d_{l+1}\ll\theta_{l,1}\ll\alpha_{l,1}\ll\theta_{l,2}\ll\cdots\ll\theta_{l,s}\ll\alpha_{l,s}\ll\theta_l=\xi_l\ll d_l.$$

The following lemma is a direct consequence of Definition 19.

**Lemma 4.** *If* $0 < \beta < 1/(3s^2 + 2 + (1 + \Im)(s - 1))$, *then the following inequalities hold for all* $l, m = 1, \dots, s$:

$$\begin{gathered} 2(s^2 - 1)d_{l+1} + (s^2 - 1)\alpha_{l,m-1} - \theta_{l,m} < -h_l, \\ 2(s^2 - 1)d_{l+1} + (s^2 - 1)\alpha_{l,m-1} + \xi_l - (1 + \Im)(d_l - (s - 1)d_{l+1}) < -h_l, \\ \theta_{l,m} < \alpha_{l,m} - (1 + \Im)(s - 1)d_{l+1} - h_l, \\ \xi_l < (1 + \Im)d_l - \theta_{l,m} - (1 + \Im)(s - 1)d_{l+1} - h_l. \end{gathered} \tag{52}$$

*7.2. Cloud decomposition*

**Lemma 5.** *Consider a* $t$*-tuple of points* $\mathcal{P} = \{P_0, \dots, P_{t-1}\}$ *in a metric space* $\mathcal{M}$. *Fix a sequence of numbers* $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{t-1}$ *and a number* $0 < c \leqslant 1/(t - 1)$. *For some* $1 \leqslant m \leqslant t$ *there is a decomposition of* $\mathcal{P}$ *into disjoint union of* $k'$ ($k' \leqslant m$ *and* $k' = t$ *if* $m = t$) *subsets*

$$\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2 \sqcup \cdots \sqcup \mathcal{P}_{k'}$$

*with the property*

$$\operatorname{dist}(\mathcal{P}_i, \mathcal{P}_j) > c\gamma_{t-m+1} \quad \textit{for } m > 1 \tag{53}$$

*and*

$$\operatorname{diam}(\mathcal{P}_i) \leqslant \gamma_{t-m} \quad \textit{for } m < t. \tag{54}$$

**Remark 12.** We shall apply this lemma for a rapidly decreasing with $j$ sequence of $\gamma_j$'s. Informally this means that a finite set of points can be decomposed into disjoint union of subsets ("clouds") such that distance between any pair of points in the same "cloud" are much smaller then distances between "clouds." In the case $k' = 1$ this decomposition contains just one "cloud" and inequality (53) does not apply. Also in the case $k' = t$ each "cloud" contains only one point and inequality (54) does not apply.

Note also that Lemma 5 provides such a decomposition even though in general it is not unique. We shall apply this lemma with $\gamma_i = \mu^{-\alpha_{l,t-i} n}$, where $(l, n)$ is fixed and $i = 1, \dots, t - 1$.

In this section (in the proof of Lemma 5) and in the next two sections we need some notions from *graph theory*. Recall them in the following

**Definition 20.**

1. Graph is a collection of points (vertices) and lines (edges) connecting some of them.
2. A graph is a multigraph if multiple edges are allowed between vertices.
3. An edge of a graph which joins a vertex to itself is called a graph loop.
4. A graph is a simple graph if it contains no multiple edges and no graph loops.
5. A graph is a pseudograph if multiple edges and graph loops are allowed.
6. An oriented graph is a graph in which each edge is oriented.
7. A cycle is a graph which forms a closed path.

8. A cycle (or a circuit) of a graph is a subset of the graph edge-set which forms a closed path with pairwise distinct vertices.
9. An oriented cycle is called properly oriented if each of its vertices has one ingoing edge and another outgoing.

**Proof of Lemma 5.** Consider a sequence of simple graphs $\Gamma_1, \ldots, \Gamma_{t-1}$. For each $\Gamma_m$ its vertices are points $P_0, \ldots, P_{t-1}$, and two vertices are connected by an edge if and only if the distance between them $\leqslant c\gamma_m$. Note that the distance between any two points from the same connected component of $\Gamma_m$ is not greater than $(t-1)c\gamma_m \leqslant \gamma_m$.

Let $g_m$ be a number of connected components of $\Gamma_m$. Since a graph $\Gamma_{m+1}$ contains all edges of the graph $\Gamma_m$, we have $g_{m+1} \leqslant g_m$. Set $g_0 = t$ and $g_t = 1$. We have

$$1 = g_t \leqslant g_{t-1} \leqslant \cdots \leqslant g_1 \leqslant g_0 = t.$$

By the Pigeon hole principle some of the numbers $\{g_0, g_1, \ldots, g_t\}$ must coincide. Suppose $\tilde{k} \in \{1, \ldots, t\}$ is a minimal index such that $g_{\tilde{k}} = g_{\tilde{k}-1}$. Then

$$g_{\tilde{k}} = g_{\tilde{k}-1} < g_{\tilde{k}-2} < \cdots < g_2 < g_1 < g_0 = t,$$

hence $g_{\tilde{k}} = g_{\tilde{k}-1} \leqslant t - \tilde{k} + 1$. Set $k' = g_{\tilde{k}}$ and $m = t - \tilde{k} + 1$. We have $k' \leqslant m$, and if $m = t$ then $k' = t$.

Decomposition $\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2 \sqcup \cdots \sqcup \mathcal{P}_{k'}$ into $k'$ connected components of $\Gamma_{\tilde{k}-1}$ satisfies (53) and (54). Indeed, if $\tilde{k} > 1$ (i.e. $m < t$), then diameter of any connected component of $\Gamma_{\tilde{k}-1}$ is not greater than $(t-1)\gamma_{\tilde{k}-1} \leqslant \gamma_{\tilde{k}-1} = \gamma_{t-m}$. It implies (54). If $\tilde{k} < t$ (i.e. $m > 1$), then distance between any points from different connected components of $\Gamma_{\tilde{k}}$ is at least $c\gamma_{\tilde{k}} = c\gamma_{t-m+1}$. It implies (53). Lemma 5 is proven. □

*7.3. Decomposition of oriented pseudographs*

Now we state two simple lemmas about oriented pseudographs. It is helpful to look at Fig. 4. The following lemma is obvious.

**Lemma 6.** *Consider a properly oriented cycle. After identification of some vertices we get a connected oriented pseudograph with the following property: At each vertex the numbers of ingoing and outgoing edges are the same.*

**Remark 13.** Note that the converse is also true, namely any connected oriented pseudograph such that at each vertex the number of ingoing edges is equal to the number of outgoing edges can be represented as a properly oriented cycle with some vertices identified.

**Lemma 7.** *Consider a connected oriented pseudograph such that at each vertex the number of ingoing edges is equal to the number of outgoing edges. Then it can be decomposed into the union of oriented cycles in the following way:*

- *each cycle is a subgraph of the initial graph*;
- *each cycle is properly oriented*;
- *each edge belongs to only one cycle* (*different cycles could have common vertices*).

The proof of this lemma is in Section 14.2.

### 7.4. Decomposition into scattered cycles

**Definition 21.** Consider a type $\mathcal{N}_s = (n_1, \dots, n_s)$, $|\mathcal{N}_s| = n = n_1 + \dots + n_s + s$, having shape $l$ (in particular, $n_1 \geqslant d_l n$). Set $n_{s+1} = n_1$ and denote $1 = i_0 < i_1 < \dots < i_{t(\mathcal{N}_s)-1} \leqslant s$ indices of $d_l$-long loops, i.e. $P_j = \mathbf{p}_{i_j-1}$. An $l$-generalized element of $\mathcal{N}_s$ is a sequence of consecutive $n_{i_\omega}, \dots, n_{i_{\omega+1}-1}$ bounded by adjacent $d_l$-long $n_i$'s. Call this sequence the $\omega$th generalized element and denote it by

$$\mathcal{N}_s^\omega = (n_{i_\omega}, \dots, n_{i_{\omega+1}-1}).$$

We say $\mathcal{N}_k = (n'_1, \dots, n'_k)$ is an $l$-subtype of $\mathcal{N}_s$ if for some $\tau \leqslant t$ it consists of some $l$-generalized elements of $\mathcal{N}_s$ (possibly permuted), i.e. for a subset

$$\{\omega_0, \omega_1, \dots, \omega_{\tau-1}\} \subseteq \{0, 1, \dots, t(\mathcal{N}_s) - 1\},$$

a permutation on $\tau$ elements $\sigma : S_\tau \to S_\tau$, we have

$$\mathcal{N}_k = \left(\mathcal{N}_s^{\omega_{\sigma(1)}}, \dots, \mathcal{N}_s^{\omega_{\sigma(\tau)}}\right).$$

If $\mathcal{N}_k = (n'_1, \dots, n'_k)$ is an $l$-subtype of $\mathcal{N}_s$ we denote

$$\mathcal{N}_k \subseteq_l \mathcal{N}_s.$$

Notice also that if $\mathcal{N}_k$ is an $l$-subtype of $\mathcal{N}_s$, then it has shape $(l(\mathcal{N}_s), |\mathcal{N}_s|)$ (see Definition 8).

**Lemma 8.** *Any $(\mathcal{V}, s)$-localized orbit of period $n$ and shape $l$ is*
- *either a union of some periodic $\mu^{-\alpha_{l,1}n}$-pseudo-orbits. Moreover, each one is a generalized loop of the initial $s$-loop periodic orbit. Set $m = 1$ in this case*;
- *or a union of some $(\frac{1}{s}\mu^{-\alpha_{l,m-1}n})$-scattered periodic $\mu^{-\alpha_{l,m}n}$-pseudo-orbits for some $m \in \{2, 3, \dots, s\}$. Moreover, each of these pseudo-orbits consists of at most $m$ generalized loops of the initial $s$-loop periodic orbit.*

*Each of these periodic $\mu^{-\alpha_{l,m}n}$-pseudo-orbits has a type which is an $l$-subtype of $\mathcal{N}_s$ and, therefore, has $(l, n)$-shape.*

**Proof.** Consider a set $\mathcal{P} = \{P_0, \dots, P_{t-1}\}$ of starting points of generalized loops. Let the constant $c = 1/s$ and the sequence $0 < \gamma_1 < \gamma_2 < \dots < \gamma_{t-1}$ is given by $\gamma_i = \mu^{-\alpha_{l,t-i}n}$. Apply Lemma 5 with these constants to $\mathcal{P}$. For a corresponding decomposition $\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2 \sqcup \dots \sqcup \mathcal{P}_{k'}$ into $k'$ subsets for some $k' \leqslant m \leqslant t$ we have

$$\operatorname{dist}(\mathcal{P}_i, \mathcal{P}_j) > c\gamma_{t-m+1} = \frac{1}{s}\mu^{-\alpha_{l,m-1}n} \quad \text{if } k' > 1, \quad \text{and}$$
$$\operatorname{diam}(\mathcal{P}_i) \leqslant \gamma_{t-m} = \mu^{-\alpha_{l,m}n} \quad \text{for } m < t.$$

Consider the graph $\Gamma$ whose vertices are points $P_0, \dots, P_{t-1}$ and two vertices are connected by an edge if and only if they are starting point of consecutive generalized loops. An orientation is introduced in the natural way. Notice that the graph $\Gamma$ is a properly oriented cycle. Identifying vertices that belong to the same set of the decomposition, we get an oriented pseudograph. Apply Lemma 7 to this pseudograph and decompose the graph $\Gamma$ into union of properly oriented cycles. Due to inequalities above, each cycle from this decomposition represents $(\frac{1}{s}\mu^{-\alpha_{l,m-1}n})$-scattered periodic $\mu^{-\alpha_{l,m}n}$-pseudo-orbit, if $k' > 1$. If $k' = 1$, graph $\Gamma$ has just one vertex, and any generalized loop is periodic $\mu^{-\alpha_{l,m}n}$-pseudo-orbit (hence, periodic $\mu^{-\alpha_{l,1}n}$-pseudo-orbit).

Each of these periodic $\mu^{-\alpha_{l,m}n}$-pseudo-orbits represented by cycles from decomposition consists of one or several generalized loops of the initial periodic orbit of type $\mathcal{N}_s$, and, therefore, has a type which is an $l$-subtype of $\mathcal{N}_s$. This completes the proof of Lemma 8. □

### 7.5. *Cone fitting*

Here we prove that Lemma 8 implies the inclusion (27). Indeed, assume that there exists $\vec{\varepsilon}_{<2s} \in HB_{<2s}(\zeta)$ such that

$$\vec{\varepsilon}_{<2s} \in B^{\text{cone}}_{\mathfrak{Q}}\big[s, f, \mathcal{N}_s; (\theta_l, \xi_l)\big] \quad \text{and}$$

$$\vec{\varepsilon}_{<2s} \notin \bigcup_{\mathcal{N}_k \subseteq_l \mathcal{N}_s} \bigcup_{t(\mathcal{N}_k) \leqslant m \leqslant s} B^{\text{scatt}}_{\mathfrak{Q}}\big[s, f, \mathcal{N}_k, |\mathcal{N}_s|, l, m\big].$$

This means that $f_{\vec{\varepsilon}_{<2s}}$ has an $s$-loop periodic orbit $\mathfrak{P} = (p_0, \ldots, p_{n-1})$ of period $n = |\mathcal{N}_s|$ of type $\mathcal{N}_s$ (having shape $l = l(\mathcal{N}_s)$) such that a generalized loop cone condition with constants $(\theta_l, \xi_l)$ does not hold. That is, for some $\mathbf{i} = 0, \ldots, t(\mathcal{N}_s) - 1$

$$\text{the inclusion } K_{\xi_l n}(P_{\mathbf{i}}) \hookrightarrow_{Df^{N_i}_{\vec{\varepsilon}_{<2s}}(P_{\mathbf{i}})} K_{\theta_l n}(P_{\mathbf{i}+1}) \text{ does not hold.} \tag{55}$$

Consider the decomposition of $\mathfrak{P}$ into the union of periodic $\mu^{-\alpha_{l,m}n}$-pseudo-orbits (which exists for some $1 \leqslant m \leqslant s$ due to Lemma 8). Take a $k$-loop ($k \leqslant s$) $\mu^{-\alpha_{l,m}}$-pseudo-orbit $\mathcal{Z}$ which contains a point $P_{\mathbf{i}}$ (and, therefore, the generalized loop of $\mathfrak{P}$ that begins at $P_{\mathbf{i}}$). This pseudo-orbit $\mathcal{Z}$ has type $\mathcal{N}_k \subseteq_l \mathcal{N}_s$ and has $t(\mathcal{N}_k) \leqslant k' \leqslant m$ generalized loops. Since we assume that

$$\vec{\varepsilon}_{<2s} \notin \bigcup_{\mathcal{N}_k \subseteq_l \mathcal{N}_s} \bigcup_{t(\mathcal{N}_k) \leqslant m \leqslant s} B^{\text{scatt}}_{\mathfrak{Q}}[s, f, \mathcal{N}_k, n, l, m],$$

the $(l, n)$-generalized loop cone condition with constants $(2\theta_{l,m}, \xi_l)$ (see Definition 10) holds for pseudo-orbit $\mathcal{Z}$. In particular (since $P_{\mathbf{i}} = Z_j$ is a starting point of a generalized loop from $\mathcal{Z}$),

$$K_{\xi_l n}(P_{\mathbf{i}}) \hookrightarrow_{F^{N_i}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(P_{\mathbf{i}})} K_{2\theta_{l,m}n}(Z_{j+1}).$$

Note that $F^{N_i}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(P_{\mathbf{i}}) = Df^{N_i}_{\vec{\varepsilon}_{<2s}}(P_{\mathbf{i}})$ (we identify tangent spaces at $P_{\mathbf{i}+1}$ and $Z_{j+1}$). Constants $\theta_l$ and $\theta_{l,m}$ were chosen in such a way that $2\theta_{l,m} < \theta_l$ (see Definition 19), therefore $K_{2\theta_{l,m}n} \subset K_{\theta_l n}$. Finally we have

$$K_{\xi_l n}(P_{\mathbf{i}}) \hookrightarrow_{Df^{N_i}_{\vec{\varepsilon}_{<2s}}(P_{\mathbf{i}})} K_{2\theta_{l,m}n}(P_{\mathbf{i}+1}) \subset K_{\theta_l n}(P_{\mathbf{i}+1})$$

and get a contradiction with (55). This proves inclusion (27).

## 8. Class of admissible pseudotrajectories and discretization

So far we reduced the proof to the estimate (28) of the measure of "bad" parameters $B^{\mathrm{scatt}}_{\mathfrak{Q}}[s, f, \mathcal{N}_k, n, l, m]$. Recall that this is the set of parameters for which there is a $k$-loop periodic scattered (non-recurrent) $\mu^{-\alpha_{l,m}}$-pseudotrajectory of type $\mathcal{N}_k$ having shape $(l, n)$, such that an $(l, n)$-generalized loop cone condition with constants $(2\theta_{l,m}, \xi_l)$ fails. In this section, we replace this set of "bad" parameters by the set $B^{\mathrm{adm}}_{\mathfrak{Q}}[s, f, \mathcal{N}_k, n, l, m]$ (that is, we prove the inclusion (30)), see next several sections and Section 3.1 for definitions and description of the discretization procedure. The advantage of this replacement is that there are only finite number of admissible pseudo-orbits of a given type for all possible values of parameters. In subsequent parts of the proof we shall estimate the measure of "bad" parameters associated with a *particular* admissible pseudotrajectory and extend this estimate to the set of "bad" parameters associated with *all possible* admissible pseudotrajectories of a given type.

### 8.1. Testing rectangles

For some points from $U$ we can a priori be sure that these points cannot belong to any loop of given length $q + 1$. Consider for each $q \in \mathbb{N}$ testing rectangles $\Pi_q$ and $\tilde{\Pi}_q$.

**Definition 22.** A testing rectangle $\Pi_m$ is a rectangle

$$\Pi_q = \left\{(x, y) \,\middle|\, x \in [1 - \delta, 1 + \delta],\ y \in \left[(1 - \delta)\mu^{-q}, (1 + \delta)\mu^{-q}\right]\right\} \subset U.$$

A testing rectangle $\tilde{\Pi}_m$ is a rectangle

$$\tilde{\Pi}_q = \left\{(x, y) \,\middle|\, x \in \left[(1 - \delta)\lambda^q, (1 + \delta)\lambda^q\right],\ y \in [1 - \delta, 1 + \delta]\right\} \subset \tilde{U}.$$

**Remark 14.** Note that $L^q(\Pi_q) = \tilde{\Pi}_q$ for every $q$.

The reason why we want to consider these rectangles is the following obvious lemma.

**Lemma 9.** *Let $\{p_0, \dots, p_{n-1}\}$ be an $s$-loop periodic orbit of type $(n_1, \dots, n_s)$, $n_1 + \cdots + n_s + s = n$. Denote the points of this orbit, which belong to $\tilde{U}$ and $U$, by $\tilde{\mathbf{p}}_0, \tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_{s-1}$ and $\mathbf{p}_0, \mathbf{p}_2, \dots, \mathbf{p}_{s-1}$ respectively. Then*

$$\tilde{\mathbf{p}}_{i-1} \in \tilde{\Pi}_{n_i}, \quad \mathbf{p}_{i-1} \in \Pi_{n_i}, \quad i = 1, \dots, s.$$

### 8.2. Grids in $\Pi_q$ and in $\tilde{\Pi}_q$

Let us take a small $0 < \gamma \ll \delta$, and consider a grid in $\Pi_q$ of size $\gamma\mu^{-q}$ in vertical direction and of size $\gamma$ in horizontal direction. Denote this grid by $\boldsymbol{\Pi}_q(\gamma)$. Let us also consider a grid in $\tilde{\Pi}_q$ of size $\gamma$ in vertical direction and of size $\gamma\lambda^q$ in horizontal direction. Let us denote this grid by $\tilde{\boldsymbol{\Pi}}_q(\gamma)$.

It is clear that the following lemmas hold.

**Lemma 10.** *The number of grid points in $\Pi_q(\gamma)$, as well as the number of grid points in $\tilde{\boldsymbol{\Pi}}_q(\gamma)$, is not greater than*

$$\left(\frac{2\delta}{\gamma}+1\right)^2=(2\delta+\gamma)^2\gamma^{-2}<9\delta^2\gamma^{-2}.$$

**Lemma 11.** *The hyperbolic map $L^q$ sends the grid $\boldsymbol{\Pi}_q(\gamma)$ into the grid $\tilde{\boldsymbol{\Pi}}_q(\gamma)$:*

$$L^q\big(\boldsymbol{\Pi}_q(\gamma)\big)=\tilde{\boldsymbol{\Pi}}_q(\gamma).$$

*8.3. Admissible pseudo-orbits*

**Definition 23.** A sequence of points $\mathfrak{R}=\{r_0,r_2,\dots,r_{\mathbf{n}-1}\}\subset V$ is called a $\gamma$-admissible $k$-loop pseudo-orbit of type $\mathcal{N}_k=(n_1,\dots,n_k)$, $|\mathcal{N}_k|=n_1+\cdots+n_k+k=\mathbf{n}$, associated to $\vec{\varepsilon}$ (or to the map $f_{\vec{\varepsilon}}$) if

(1) it is a disjoint union of $k$ loops of lengths $n_1+1,n_2+1,\dots,n_k+1$. Denote the starting points of these loops by $\mathbf{r}_0,\dots,\mathbf{r}_{k-1}\in U$, and the ending points by $\tilde{\mathbf{r}}_0,\dots,\tilde{\mathbf{r}}_{k-1}\in\tilde{U}$, in such a way that $\tilde{\mathbf{r}}_i=L^{n_i}(\mathbf{r}_i)$;
(2) $\mathbf{r}_i\in\boldsymbol{\Pi}_{n_i}(\gamma)$, $\tilde{\mathbf{r}}_i\in\tilde{\boldsymbol{\Pi}}_{n_i}(\gamma)$;
(3) $\operatorname{dist}(G_{\vec{\varepsilon}}(\tilde{\mathbf{r}}_i),\mathbf{r}_{i+1\ (\mathrm{mod}\ k)})\leqslant(3+2M_1)\gamma$ for each $i=0,1,\dots,k-1$.

In other words, $\gamma$-admissible pseudo-orbit is a periodic $(3+2M_1)\gamma$-pseudo-orbit which loops begin in vertices of corresponding grids.

**Definition 24.** A $\gamma$-admissible pseudo-orbit is $\varrho$-scattered, if the distance between starting points of any two loops is at least $\varrho$.

The definition of admissible $k$-loop pseudo-orbit is motivated by the following proposition.

**Proposition 3.** *For any $k$-loop periodic $\gamma$-pseudo-orbit $\{z_0,\dots,z_{\mathbf{n}-1}\}$ of a map $f_{\vec{\varepsilon}}$ there exists a $\gamma$-admissible $k$-loop pseudo-orbit $\{r_0,\dots,r_{\mathbf{n}-1}\}$ of the same type, such that* $\operatorname{dist}(r_i,z_i)\leqslant 2\gamma$. *If the initial periodic $\gamma$-pseudo-orbit is $\varrho$-scattered, then a corresponding $\gamma$-admissible pseudo-orbit is $(\varrho-4\gamma)$-scattered.*

**Proof.** Consider points of the orbit $\{z_0,\dots,z_{\mathbf{n}-1}\}$ which belong to $U$, denote them by $\mathbf{z}_0,\dots,\mathbf{z}_{k-1}$. By Lemma 9 we have $\mathbf{z}_i\in\Pi_{n_i}$. Since for each $i=1,\dots,k$ the rectangle $\Pi_{n_i}$ is divided into the union of small rectangles (of size $\gamma\times\gamma\mu^{-n_i}$), we can take a small rectangle $\pi_i\subset\Pi_{n_i}$ which contains a point $\mathbf{z}_i$. Take one of the vertices of $\pi_i$ (let it be left upper vertex, for example) and denote it by $\mathbf{r}_i$. Now take a pseudo-orbit $\{r_0,\dots,r_{\mathbf{n}-1}\}$ as a union of loops $\{\mathbf{r}_0,L\mathbf{r}_0,\dots,L^{n_1}\mathbf{r}_0\}$, $\{\mathbf{r}_1,L\mathbf{r}_1,\dots,L^{n_2}\mathbf{r}_1\}$, $\dots$, $\{\mathbf{r}_{k-1},L\mathbf{r}_{k-1},\dots,L^{n_k}\mathbf{r}_{k-1}\}$. Since for each $q=0,\dots,n_i$ we have $L^q\mathbf{r}_i\in L^q\pi_i$, $L^q\mathbf{z}_i\in L^q\pi_i$, and $\operatorname{diam}(L^q\pi_i)<2\gamma$, the property $\operatorname{dist}(r_i,z_i)\leqslant 2\gamma$ holds.

Show that $\{r_1,\dots,r_{\mathbf{n}}\}$ is a $\gamma$-admissible pseudo-orbit. Property (1) is guaranteed by our construction. Property (2) is a consequence of Proposition 11. We just need to check property (3).

Take $\tilde{\mathbf{r}}_0 = L^{n_1}\tilde{\mathbf{r}}_0, \ldots, \tilde{\mathbf{r}}_{k-1} = L^{n_k}\mathbf{r}_{k-1}$. By our construction $\operatorname{dist}(\tilde{\mathbf{r}}_i, \tilde{\mathbf{z}}_i) \leqslant 2\gamma$, therefore (if we set $\mathbf{z}_k = \mathbf{z}_0$)

$$\operatorname{dist}\bigl(G_{\vec{\varepsilon}}(\tilde{\mathbf{r}}_i), \mathbf{z}_{i+1}\bigr) \leqslant \operatorname{dist}\bigl(G_{\vec{\varepsilon}}(\tilde{\mathbf{r}}_i), G_{\vec{\varepsilon}}(\tilde{\mathbf{z}}_i)\bigr) + \gamma \leqslant 2M_1\gamma + \gamma.$$

Since $\operatorname{dist}(\mathbf{r}_{i+1}, \mathbf{z}_{i+1}) \leqslant 2\gamma$, we finally have

$$\operatorname{dist}\bigl(G_{\vec{\varepsilon}}(\tilde{\mathbf{r}}_i), \mathbf{r}_{i+1}\bigr) \leqslant \operatorname{dist}\bigl(G_{\vec{\varepsilon}}(\tilde{\mathbf{r}}_i), \mathbf{z}_{i+1}\bigr) + \operatorname{dist}(\mathbf{z}_{i+1}, \mathbf{r}_{i+1}) \leqslant (3 + 2M_1)\gamma.$$

The second part of the statement of Proposition 3 is obvious.

Proposition 3 is proved. □

### *8.4. Discretization of a generalized loop cone condition*

A generalized loop cone condition for pseudo-orbits was introduced in Definitions 9 and 10.

To check an $(l, n)$-generalized loop cone condition for a $k$-loop periodic $\gamma$-pseudo-orbit (having shape $(l, n)$) we just need to check an $(l, n)$-generalized loop cone condition for a corresponding $\gamma$-admissible $k$-loop pseudo-orbit.

**Proposition 4.** *Fix $s, \mu, \lambda, M_1, M_2$, and $\beta < 1/(3s^2 + 2 + (1 + \Im)(s - 1))$, where $\Im = -(\ln\lambda)/(\ln\mu)$. Take any shape $l \in \{1, \ldots, s\}$. For any constants $\theta, \theta', \alpha, \xi$, such that*

$$\theta < \theta', \qquad \theta < \alpha - (1 + \Im)(s - 1)d_{l+1}, \qquad \xi < (1 + \Im)d_l - \theta - (1 + \Im)(s - 1)d_{l+1} \tag{56}$$

*(where $d_l = \beta^{(2s+2)(l-1)+1}$ and $d_{l+1} = \beta^{(2s+2)l+1}$), and sufficiently large $\mathbf{N}$ the following holds. For any given type $\mathcal{N}_k = (n_1, \ldots, n_k)$, $n_1 + \cdots + n_k + k = \mathbf{n}$, having shape $(l, n)$, $n \geqslant \mathbf{N}$, and for $\gamma = \mu^{-\alpha n}$, an existence of a $\gamma$-admissible $k$-loop pseudo-orbit $\mathfrak{R} = \{r_0, \ldots, r_{\mathbf{n}-1}\}$ that satisfies $(l, n)$-generalized loop cone condition with constants $(\theta, \xi)$ implies that any $k$-loop periodic $\gamma$-pseudo-orbit $\mathcal{Z} = \{z_0, \ldots, z_{\mathbf{n}-1}\}$ such that* $\operatorname{dist}(z_i, r_i) \leqslant 2\gamma$ *for all $i = 0, \ldots, \mathbf{n} - 1$, satisfy the $(l, n)$-generalized loop cone condition with constants $(\theta', \xi)$.*

**Proof.** The proof is technical but more or less straightforward. We are going to estimate the size of an image of cone $K_{\xi n}(Z_i)$ under the map $F^{N_i}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_i)$, which is going to be small. Also we are going to estimate angle between images of the vertical line under $F^{N_i}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_i)$ and under $F^{N_i}_{\mathfrak{R},\vec{\varepsilon}_{<2s}}(R_i)$. It is also going to be small. Since $\theta < \theta'$, and we know that an image of the vertical line under $F^{N_i}_{\mathfrak{R},\vec{\varepsilon}_{<2s}}(R_i)$ belongs to $K_{\theta n}(R_{i+1})$, those estimates allow to claim that

$$K_{\xi n}(Z_i) \hookrightarrow_{F^{N_i}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_i)} K_{\theta' n}(Z_{i+1}).$$

First of all we are going to investigate how the size of a cone changes under iterations of a map. To do that let us define a cone in more general way than it was done by Definition 16.

**Definition 25.** A cone $K = K(v_1, v_2)$ between two nonzero vectors $v_1, v_2 \in \mathbb{R}^2$is the following set:

$$K = \{\bar{v} \mid \bar{v} = a_1 v_1 + a_2 v_2,\ a_1 a_2 \geqslant 0\}.$$

The size of this cone is an angle between $v_1$ and $v_2$. Let us denote it by $\angle K$.

**Lemma 12.** *For any cone $K \subset \mathbb{R}^2$ and any linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ the following inequality holds*:

$$\sin \angle A(K) \leqslant \|A\| \cdot \|A^{-1}\| \cdot |\sin \angle K|.$$

**Proof.** Take two vectors $v_1$ and $v_2$ such that $v_2 \perp (v_1 - v_2)$ and $K = K(v_1, v_2)$. In this case $|\sin \angle K| = |v_1 - v_2|/|v_1|$. Since $A(K) = K(Av_1, Av_2)$, we have

$$\sin \angle A(K) \leqslant \frac{|Av_1 - Av_2|}{|Av_1|} \leqslant \frac{\|A\||v_1 - v_2|}{\|A^{-1}\|^{-1}|v_1|} = \|A\| \cdot \|A^{-1}\| \cdot |\sin \angle K|.$$

Lemma 12 is proved. □

We need to consider just one generalized loop. Without loss of generality we can proceed with the first one. Recall that it is length is $N_1$, the length of the first (long) loop is $n_1 + 1$, the number of short loops is $h_1$. Therefore, the lengths of loops in this generalized loop are $n_1 + 1, n_2 + 1, \ldots, n_{h_1+1} + 1$. Note also that $n_1 > d_l n$ and $n_i \leqslant d_{l+1} n$ for $i = 2, \ldots, h_1 + 1$.

The proof of the following lemma is straightforward.

**Lemma 13.** *Size of the cone $K_{\xi n}$ is equal to* $2\arctan(\mu^{\xi n})$. *Under the iterations of the map $L$ a cone $K_{\xi n}$ changes in the following way*:

$$K_{\xi n} \hookrightarrow_{L^{n_1}} K_{\xi n-(1+\Im)n_1}.$$

Therefore

$$\angle L^{n_1}\big(K_{\xi n}(Z_0)\big) = \angle K_{\xi n-(1+\Im)n_1}(\tilde{Z}_0) = 2\arctan\big(\mu^{\xi n-(1+\Im)n_1}\big).$$

Since $\sin\varphi = (2\cot(\varphi/2))/(1 + \cot^2(\varphi/2))$, we have

$$\sin \angle K_{\xi n-(1+\Im)n_1}(\tilde{Z}_0) = \frac{2\mu^{(1+\Im)n_1-\xi n}}{1 + \mu^{2((1+\Im)n_1-\xi n)}} < 2\mu^{\xi n-(1+\Im)n_1}.$$

Now by Lemma 12 we have

$$\begin{aligned}
&\sin \angle\big(F^{N_i-n_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}\big(K_{\xi n-(1+\Im)n_1}(\tilde{Z}_0)\big)\big) \\
&\quad\leqslant M_1^{2h_1+2}\left(\frac{\mu}{\lambda}\right)^{n_2+n_3+\cdots+n_{h_1+1}} \sin \angle K_{\xi n-(1+\Im)n_1}(\tilde{Z}_0) \\
&\quad< 2M_1^{2h_1+2}\mu^{(1+\Im)(n_2+n_3+\cdots+n_{h_1+1})} \cdot \mu^{\xi n-(1+\Im)n_1}.
\end{aligned} \tag{57}$$

This implies that (since $\varphi \in (0, \pi/2) \Rightarrow \varphi < \frac{\pi}{2}\sin\varphi$)

$$\angle\big(F^{N_i}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}\big(K_{\xi n}(Z_0)\big)\big) < \pi M_1^{2h_1+2}\mu^{(1+\Im)(\sum_{i=1}^{h_1} n_{i+1}-n_1)+\xi n}. \tag{58}$$

Take $v=(0,1)$ and estimate angle between $F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v)$ and $F^{N_1}_{\mathfrak{R},\vec{\varepsilon}_{<2s}}(R_0)(v)$. To do that we use the inequality

$$\sin\angle K\left(F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v), F^{N_1}_{\mathfrak{R},\vec{\varepsilon}_{<2s}}(R_0)(v)\right) \leqslant \frac{|F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v) - F^{N_1}_{\mathfrak{R},\vec{\varepsilon}_{<2s}}(R_0)(v)|}{|F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v)|}.$$

We need to estimate $|F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v)|$ from below and $|F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v) - F^{N_1}_{\mathfrak{R},\vec{\varepsilon}_{<2s}}(R_0)(v)|$ from above. It is easy to estimate $|F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v)|$:

$$\begin{gathered}
\left|L^{n_1}(v)\right| = \mu^{n_1}, \\
M_1^{-1}\mu^{n_1} \leqslant \left|Df_{\vec{\varepsilon}}(\tilde{Z}_0)\circ L^{n_1}(v)\right| \leqslant M_1\mu^{n_1}, \\
M_1^{-1}\mu^{n_1}\lambda^{n_2} \leqslant \left|L^{n_2}\circ Df_{\vec{\varepsilon}}(\tilde{Z}_0)\circ L^{n_1}(v)\right| \leqslant M_1\mu^{n_1+n_2}, \\
\dots \\
M_1^{-i}\mu^{n_1}\lambda^{n_2+\dots+n_i} \leqslant \left|Df_{\vec{\varepsilon}}(\tilde{Z}_i)\circ L^{n_i}\circ\dots\circ Df_{\vec{\varepsilon}}(\tilde{Z}_0)\circ L^{n_1}(v)\right| \leqslant M_1^i\mu^{n_1+n_2+\dots+n_i}, \\
M_1^{-i}\mu^{n_1}\lambda^{n_2+\dots+n_{i+1}} \leqslant \left|L^{n_{i+1}}\circ Df_{\vec{\varepsilon}}(\tilde{Z}_i)\circ\dots\circ Df_{\vec{\varepsilon}}(\tilde{Z}_0)\circ L^{n_1}(v)\right| \leqslant M_1^i\mu^{n_1+\dots+n_i+n_{i+1}}, \\
\dots \\
M_1^{-h_1-1}\mu^{n_1}\lambda^{n_2+\dots+n_{h_1+1}} \leqslant \left|F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v)\right| \leqslant M_1^{h_1+1}\mu^{n_1+n_2+\dots+n_{h_1+1}}.
\end{gathered}\tag{59}$$

To estimate $|F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v) - F^{N_1}_{\mathfrak{R},\vec{\varepsilon}_{<2s}}(R_0)(v)|$ we use the following obvious lemma.

**Lemma 14.** *For any linear maps $A_1$ and $A_2$ and vectors $v_1$ and $v_2$ the following inequality holds*:

$$|A_1v_1 - A_2v_2| \leqslant \|A_1\||v_1-v_2| + \|A_1-A_2\||v_2|.$$

Since $\mathrm{dist}(\tilde{\mathbf{z}}_i,\tilde{\mathbf{r}}_i)\leqslant 2\gamma$, we have an estimate $\|Df_{\vec{\varepsilon}}(\tilde{\mathbf{z}}_i) - Df_{\vec{\varepsilon}}(\tilde{\mathbf{r}}_i)\| \leqslant 2\gamma M_2$. Together with Lemma 14 this gives us:

$$\begin{gathered}
\left|Df_{\vec{\varepsilon}}(\tilde{Z}_0)\circ L^{n_1}(v) - Df_{\vec{\varepsilon}}(\tilde{R}_0)\circ L^{n_1}(v)\right| \leqslant 2\gamma M_2\mu^{n_1}, \\
\left|Df_{\vec{\varepsilon}}(\tilde{Z}_1)\circ L^{n_2}\circ Df_{\vec{\varepsilon}}(\tilde{Z}_0)\circ L^{n_1}(v) - Df_{\vec{\varepsilon}}(\tilde{R}_1)\circ L^{n_2}\circ Df_{\vec{\varepsilon}}(\tilde{R}_0)\circ L^{n_1}(v)\right| \\
\leqslant M_1\cdot\left[2\gamma M_2\mu^{n_1+n_2}\right] + 2\gamma M_2\cdot\left[M_1\mu^{n_1+n_2}\right] = 4\gamma M_1M_2\mu^{n_1+n_2}, \\
\dots \\
\left|F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v) - F^{N_1}_{\mathfrak{R},\vec{\varepsilon}_{<2s}}(R_0)(v)\right| \leqslant 2(h_1+1)\gamma M_1^{h_1}M_2\mu^{n_1+n_2+\dots+n_{h_1+1}}.
\end{gathered}\tag{60}$$

This implies

$$\begin{aligned}
\sin\angle K\left(F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v), F^{N_1}_{\mathfrak{R},\vec{\varepsilon}_{<2s}}(R_0)(v)\right) &\leqslant \frac{2(h_1+1)\gamma M_1^{h_1}M_2\mu^{\sum_{i=1}^{h_1+1}n_i}}{M_1^{-h_1-1}\mu^{n_1}\lambda^{n_2+\dots+n_{h_1+1}}} \\
&= 2(h_1+1)\gamma M_1^{2h_1+1}\mu^{(1+\Im)(n_2+\dots+n_{h_1+1})}.
\end{aligned}\tag{61}$$

Therefore

$$\angle K\big(F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v), F^{N_1}_{\mathfrak{R},\vec{\varepsilon}_{<2s}}(R_0)(v)\big) \leqslant \pi(h_1+1)\gamma M_1^{2h_1+1}\mu^{(1+\Im)(n_2+\cdots+n_{h_1+1})}. \tag{62}$$

We want to check that

$$F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}\big(K_{\xi n}(Z_0)\big) \subset K_{\theta' n}(Z_1). \tag{63}$$

We know that $F_{N_1}(R_0)(v) \in K_{\theta n}(R_1)$, so to check the inclusion (63) it is enough to check that

$$\begin{aligned}&\angle K\big(F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v), F^{N_1}_{\mathfrak{R},\vec{\varepsilon}_{<2s}}(R_0)(v)\big) + \angle\big(F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}\big(K_{\xi n}(Z_0)\big)\big)\\ &\quad < \frac{1}{2}\big(\angle K_{\theta' n}(Z_1) - \angle K_{\theta n}(R_1)\big).\end{aligned} \tag{64}$$

We have the following estimate (since $\varphi \in (0,\pi/4) \ \Rightarrow\ \varphi \geqslant \frac{\pi}{4}\tan\varphi$)

$$\tan\left(\frac{\pi}{2} - \frac{1}{2}\angle K_{\theta n}(R_1)\right) = \mu^{-\theta n} \quad \Rightarrow \quad \frac{\pi}{2} - \frac{1}{2}\angle K_{\theta n}(R_1) \geqslant \frac{\pi}{4}\mu^{-\theta n}.$$

Also we have (since $\varphi \in (0,\pi/2) \ \Rightarrow\ \varphi \leqslant \tan\varphi$)

$$\tan\left(\frac{\pi}{2} - \frac{1}{2}\angle K_{\theta' n}(Z_1)\right) = \mu^{-\theta' n} \quad \Rightarrow \quad \frac{\pi}{2} - \frac{1}{2}\angle K_{\theta' n}(Z_1) \leqslant \mu^{-\theta' n}.$$

Therefore

$$\begin{aligned}\frac{1}{2}\big(\angle K_{\theta' n}(Z_1) - \angle K_{\theta n}(R_1)\big) &= \left(\frac{\pi}{2} - \frac{1}{2}\angle K_{\theta n}(R_1)\right) - \left(\frac{\pi}{2} - \frac{1}{2}\angle K_{\theta' n}(Z_1)\right)\\ &\geqslant \frac{\pi}{4}\mu^{-\theta n} - \mu^{-\theta' n} = \mu^{-\theta n}\left(\frac{\pi}{4} - \mu^{-(\theta'-\theta)n}\right).\end{aligned} \tag{65}$$

We have also the following estimate (from (58) and (62)):

$$\begin{aligned}&\angle K\big(F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}(Z_0)(v), F^{N_1}_{\mathfrak{R},\vec{\varepsilon}_{<2s}}(R_0)(v)\big) + \angle\big(F^{N_1}_{\mathcal{Z},\vec{\varepsilon}_{<2s}}\big(K_{\xi n}(Z_0)\big)\big)\\ &\quad \leqslant \pi\big((h_1+1)\gamma M_1^{2h_1+1}\mu^{(1+\Im)(\sum_{i=1}^{h_1} n_{i+1})} + M_1^{2h_1+2}\mu^{(1+\Im)(\sum_{i=1}^{h_1} n_{i+1} - n_1)+\xi n}\big)\\ &\quad \leqslant \pi M_1^{2s-2l+1}\big((s-l+1)\mu^{-\alpha n}\mu^{(1+\Im)(s-l)d_{l+1}n} + M_1\mu^{(1+\Im)((s-l)d_{l+1}n - d_l n)+\xi n}\big).\end{aligned} \tag{66}$$

Now inequalities (56) imply that

$$\begin{aligned}&\mu^{\theta n}\big(\pi(s-l+1)\mu^{-\alpha n}M_1^{2s-2l+1}\mu^{(1+\Im)(s-l)d_{l+1}n}\\ &\quad + \pi M_1^{2s-2l+2}\mu^{(1+\Im)((s-l)d_{l+1}n - d_l n)+\xi n}\big) \to 0 \quad \text{as } n \to +\infty\end{aligned} \tag{67}$$

and

$$\frac{\pi}{4} - \mu^{-(\theta'-\theta)n} \to \frac{\pi}{4} \quad \text{as } n \to +\infty, \tag{68}$$

hence

$$\mu^{-\theta n}\left(\frac{\pi}{4}-\mu^{-(\theta'-\theta)n}\right)$$
$$>\pi M_1^{2s-2l+1}\left((s-l+1)\mu^{-\alpha n}\mu^{(1+\Im)(s-l)d_{l+1}n}+\mu^{(1+\Im)((s-l)d_{l+1}n-d_l n)+\xi n}\right) \tag{69}$$

for all large $n$, which implies an inequality (64) for large $n$.

Proposition 4 is proved. □

### 8.5. *Decomposition into admissible pseudotrajectories*

Here we show that Propositions 3 and 4 imply the inclusion (30).

Take $\vec{\varepsilon}_{<2s}\in B_{\mathfrak{Q}}^{\mathrm{scatt}}[s,f,\mathcal{N}_k,n,l,m]$. Consider the case $m>1$ (the case $m=1$ is similar, we just need to omit the scattering condition). This means that $f_{\vec{\varepsilon}_{<2s}}$ has a $k$-loop periodic $\frac{1}{s}\mu^{-\alpha_{l,m-1}n}$-scattered $\mu^{-\alpha_{l,m}n}$-pseudotrajectory $\mathcal{Z}$ of type $\mathcal{N}_k$ such that the $(l,n)$-generalized loop cone condition with constants $(2\theta_{l,m},\xi_l)$ fails. Consider the $\mu^{-\alpha_{l,m}n}$-admissible pseudotrajectory $\mathfrak{R}$ the same as in Proposition 3. It has the same type $\mathcal{N}_k$ and (by Proposition 3) is $(\frac{1}{s}\mu^{-\alpha_{l,m-1}n}-4\mu^{-\alpha_{l,m}n})$-scattered. Since $\alpha_{l,m-1}\ll\alpha_{l,m}$ (see Remark 11), for large enough $n$ we have

$$\left(\frac{1}{s}\mu^{-\alpha_{l,m-1}n}-4\mu^{-\alpha_{l,m}n}\right)>\frac{1}{2s}\mu^{-\alpha_{l,m-1}n}. \tag{70}$$

Therefore $\mathfrak{R}$ is $\frac{1}{2s}\mu^{-\alpha_{l,m-1}n}$-scattered $\mu^{-\alpha_{l,m}n}$-admissible periodic pseudotrajectory. Since $\mathfrak{R}$ is $2\mu^{-\alpha_{l,m}n}$-close to $\mathcal{Z}$, the $(l,n)$-generalized loop cone condition with constants $(\theta_{l,m},\xi_l)$ fails for $\mathfrak{R}$. Indeed, note that inequalities (56) hold if we choose $\{\theta_{l,m},2\theta_{l,m},\alpha_{l,m},\xi_l\}$ equal $\{\theta,\theta',\alpha,\xi\}$ (this is guaranteed by Lemma 4). Hence, if the $(l,n)$-generalized loop cone condition with constants $(\theta_{l,m},\xi_l)$ holds for $\mathfrak{R}$, if $n$ is large by Proposition 4 the $(l,n)$-generalized loop cone condition with constants $(2\theta_{l,m},\xi_l)$ holds for $\mathcal{Z}$, which contradicts to the choice of pseudotrajectory $\mathcal{Z}$. Thus, $\vec{\varepsilon}_{<2s}\in B_{\mathfrak{Q}}^{\mathrm{adm}}[s,f,\mathcal{N}_k,n,l,m]$.

Inclusion (30) is proved.

## 9. Newton Interpolation Polynomials and blow-up along the diagonal in multijet space

Now we present a construction due to Grigoriev and Yakovenko [15] of choosing a convenient for dynamics basis in the space of polynomials. The exposition is closely related to [22, Section 2.2]. This construction is an interpretation of Newton Interpolation Polynomials as an algebraic blow-up along the diagonal in the multijet space. In order to keep the notations and formulas simple and put the main ideas in evidence in this section we consider only the 1-dimensional case. See [22, Section 3] for more detailed description.

Fix a positive integer $k$. Consider the $2k$-parameter family of perturbations of a $C^1$ map $f:I\to I$ by polynomials of degree $2k-1$

$$f_\varepsilon(x)=f(x)+\phi_\varepsilon(x),\quad \phi_\varepsilon(x)=\sum_{j=0}^{2k-1}\varepsilon_j x^j, \tag{71}$$

where $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{2k-1}) \in \mathbb{R}^{2k}$. Suppose the perturbation vector $\varepsilon$ consists of coordinates from the cube $HB^1_{<2k}(\zeta) = \{\varepsilon_j \in \mathbb{R} \mid |\varepsilon_j| < \zeta,\ j = 0, 1, \dots, 2k-1\}$.

Given $n > 0$ and a $C^1$ function $f : I \to \mathbb{R}$ we define an associated function $j^{1,k} f : I^k \to I^k \times \mathbb{R}^{2k}$ by

$$j^{1,k} f(x_0, \dots, x_{k-1}) = \big(x_0, \dots, x_{k-1}, f(x_0), \dots, f(x_{k-1}), f'(x_0), \dots, f'(x_{k-1})\big). \tag{72}$$

In singularity theory this function is called the *$k$-tuple* 1*-jet* of $f$. The ordinary 1-jet of $f$, usually denoted by $j^1 f(x) = (x, f(x), f'(x))$, maps $I$ to the 1*-jet space* $\mathcal{J}^1(I, \mathbb{R}) \simeq I \times \mathbb{R}^2$. The product of $k$ copies of $\mathcal{J}^1(I, \mathbb{R})$, called the *multijet space*, is denoted by

$$\mathcal{J}^{1,k}(I, \mathbb{R}) = \underbrace{\mathcal{J}^1(I, \mathbb{R}) \times \cdots \times \mathcal{J}^1(I, \mathbb{R})}_{k \text{ times}}, \tag{73}$$

and is equivalent to $I^k \times \mathbb{R}^{2k}$ after rearranging coordinates. The $k$-tuple 1-jet of $f$ associates with each $k$-tuple of points in $I^k$ all the information necessary to determine how close the $k$-tuple is to being a periodic orbit, and if so, how much hyperbolicity does the linearization has.

The set

$$\Delta_n(I) = \big\{\{x_0, \dots, x_{n-1}\} \times I^n \times \mathbb{R}^n \subset \mathcal{J}^{1,n}(I, \mathbb{R}) \mid \exists\, i \neq j \text{ such that } x_i = x_j\big\} \tag{74}$$

is called the *diagonal* (or sometimes the *generalized diagonal*) in the space of multijets. In singularity theory the space of multijets is defined outside of the diagonal $\Delta_n(I)$ and is usually denoted by $\mathcal{J}^1_n(I, \mathbb{R}) = \mathcal{J}^{1,n}(I, \mathbb{R}) \setminus \Delta_n(I)$ (see, e.g., [9]). It is easy to see that *a recurrent trajectory $\{x_k\}_{k\in\mathbb{N}}$ is located in a neighborhood of the diagonal $\Delta_n(I) \subset \mathcal{J}^{1,n}(I, \mathbb{R})$ in the space of multijets for a sufficiently large $n$*. If $\{x_k\}_{k=0}^{n-1}$ is a part of a recurrent trajectory of length $n$, then the product of distances along the trajectory

$$\prod_{k=0}^{n-2} |x_{n-1} - x_k| \tag{75}$$

measures how close $\{x_k\}_{k=0}^{n-1}$ to the diagonal $\Delta_n(I)$, or how independently one can perturb points of a trajectory. One can also say that (75) is a quantitative characteristic of how recurrent a trajectory of length $n$ is. This product is introduced in [22] and is extremely important quantity for our analysis.

Our goal now is to describe how such perturbations affect the $k$-tuple 1-jet of $f$, and since the operator $j^{1,k}$ is linear in $f$, for the time being we consider only the perturbations $\phi_\varepsilon$ and their $k$-tuple 1-jets. For each $k$-tuple $\{x_j\}_{j=0}^{k-1}$ there is a natural transformation $\mathcal{J}^{1,k} : I^k \times \mathbb{R}^{2k} \to \mathcal{J}^{1,k}(I, \mathbb{R})$ from $\varepsilon$-coordinates to jet-coordinates, given by

$$\mathcal{J}^{1,s}(x_0, \dots, x_{k-1}, \varepsilon) = j^{1,k} \phi_\varepsilon(x_0, \dots, x_{k-1}). \tag{76}$$

Instead of working directly with the transformation $\mathcal{J}^{1,k}$, we introduce intermediate $u$-coordinates based on Newton interpolation polynomials. The relation between $\varepsilon$-coordinates

$$
\begin{tikzcd}
 & \mathcal{DD}^{1,k}(I,\mathbb{R}) = \underbrace{I\times\cdots\times I}_{k\ \text{times}}\times\mathbb{R}^{2k} \arrow[d, "\pi^{1,k}"] \\
\underbrace{I\times\cdots\times I}_{k\ \text{times}}\times\mathbb{R}^{2k} \arrow[ur, "\mathcal{D}^{1,k}"] \arrow[r, "\mathcal{J}^{1,k}"] & \mathcal{J}^{1,k}(I,\mathbb{R}) = \underbrace{I\times\cdots\times I}_{k\ \text{times}}\times\mathbb{R}^{2k}
\end{tikzcd}
$$

Fig. 5. Algebraic blow-up along the diagonal $\Delta_k(I)$.

and $u$-coordinates is given implicitly by

$$\phi_\varepsilon(x) = \sum_{j=0}^{2k-1} \varepsilon_j x^j = \sum_{j=0}^{2k-1} u_j \prod_{i=0}^{j-1} (x - x_{i \ (\mathrm{mod}\ k)}). \tag{77}$$

Based on this identity, we will define functions $\mathbb{D}^{1,k}: I^k \times \mathbb{R}^{2k} \to I^k \times \mathbb{R}^{2k}$ and $\pi^{1,k}: I^k \times \mathbb{R}^{2k} \to \mathcal{J}^{1,k}(I,\mathbb{R})$ so that $\mathcal{J}^{1,k} = \pi^{1,k} \circ \mathbb{D}^{1,k}$, or in other words the diagram in Fig. 5 commutes. We will show later that $\mathbb{D}^{1,k}$ is invertible, while $\pi^{1,k}$ is invertible away from the diagonal $\Delta_k(I)$ and defines a blow-up along it in the space of multijets $\mathcal{J}^{1,k}(I,\mathbb{R})$.

The intermediate space, which we denote by $\mathcal{DD}^{1,k}(I,\mathbb{R})$, is called *the space of divided differences* and consists of $k$-tuples of points $\{x_j\}_{j=0}^{k-1}$ and $2k$ real coefficients $\{u_j\}_{j=0}^{2k-1}$. Here are explicit coordinate-by-coordinate formulas defining

$$\pi^{1,k}: \mathcal{DD}^{1,k}(I,\mathbb{R}) \to \mathcal{J}^{1,k}(I,\mathbb{R}).$$

This mapping is given by

$$\begin{aligned}
&\pi^{1,k}(x_0,\dots,x_{k-1},u_0,\dots,u_{2k-1}) \\
&\quad = \big(x_0,\dots,x_{k-1},\phi_\varepsilon(x_0),\dots,\phi_\varepsilon(x_{k-1}),\phi'_\varepsilon(x_0),\dots,\phi'_\varepsilon(x_{k-1})\big),
\end{aligned} \tag{78}$$

where

$$\begin{aligned}
\phi_\varepsilon(x_0) &= u_0, \\
\phi_\varepsilon(x_1) &= u_0 + u_1(x_1 - x_0), \\
\phi_\varepsilon(x_2) &= u_0 + u_1(x_2 - x_0) + u_2(x_2 - x_0)(x_2 - x_1), \\
&\ \ \vdots \\
\phi_\varepsilon(x_{k-1}) &= u_0 + u_1(x_{k-1} - x_0) + \cdots \\
&\qquad + u_{k-1}(x_{k-1} - x_0)\cdots(x_{k-1} - x_{k-2}),
\end{aligned}$$

$$\phi_\varepsilon'(x_0) = \frac{\partial}{\partial x}\left(\sum_{j=0}^{2k-1} u_j \prod_{i=0}^{j}(x - x_{i \pmod k})\right)\bigg|_{x=x_0},$$

$$\vdots$$

$$\phi_\varepsilon'(x_{k-1}) = \frac{\partial}{\partial x}\left(\sum_{j=0}^{2k-1} u_k \prod_{i=0}^{j}(x - x_{i \pmod k})\right)\bigg|_{x=x_{k-1}}. \tag{79}$$

These formulas are very useful for dynamics. For a given base map $f$ and initial point $x_0$, the image $f_\varepsilon(x_0) = f(x_0) + \phi_\varepsilon(x_0)$ of $x_0$ depends only on $u_0$. Furthermore the image can be set to any desired point by choosing $u_0$ appropriately—we say then that it depends only and non-trivially on $u_0$. If $x_0$, $x_1$, and $u_0$ are fixed, the image $f_\varepsilon(x_1)$ of $x_1$ depends only on $u_1$, and as long as $x_0 \neq x_1$ it depends non-trivially on $u_1$. More generally for $0 \leqslant j \leqslant k-1$, if distinct points $\{x_j\}_{j=0}^{k}$ and coefficients $\{u_i\}_{i=0}^{j-1}$ are fixed, then the image $f_\varepsilon(x_j)$ of $x_j$ depends only and non-trivially on $u_j$.

Suppose now that an $k$-tuple of points $\{x_i\}_{i=0}^{k}$ not on the diagonal $\Delta_k(I)$ and Newton coefficients $\{u_i\}_{i=0}^{k-1}$ are fixed. Then derivative $f_\varepsilon'(x_0)$ at $x_0$ depends only and non-trivially on $u_n$. Likewise for $0 \leqslant j \leqslant k-1$, if distinct points $\{x_j\}_{j=0}^{k-1}$ and Newton coefficients $\{u_i\}_{i=0}^{k+j-1}$ are fixed, then the derivative $f_\varepsilon'(x_j)$ at $x_j$ depends only and non-trivially on $u_{k+j}$.

As Fig. 6 illustrates, these considerations show that for any map $f$ and any desired trajectory of distinct points with any given derivatives along it, one can choose Newton coefficients $\{u_j\}_{j=0}^{2k-1}$ and explicitly construct a map $f_\varepsilon = f + \phi_\varepsilon$ with such a trajectory. Thus we have shown that $\pi^{1,k}$ is invertible away from the diagonal $\Delta_k(I)$ and defines a blow-up along it in the space of multijets $\mathcal{J}^{1,k}(I, \mathbb{R})$.

Next we define the function $\mathbb{D}^{1,k} : I^k \times \mathbb{R}^{2k} \to \mathcal{DD}^{1,k}(I, \mathbb{R})$ explicitly using so-called divided differences. Let $g : \mathbb{R} \to \mathbb{R}$ be a $C^r$ function of one real variable.

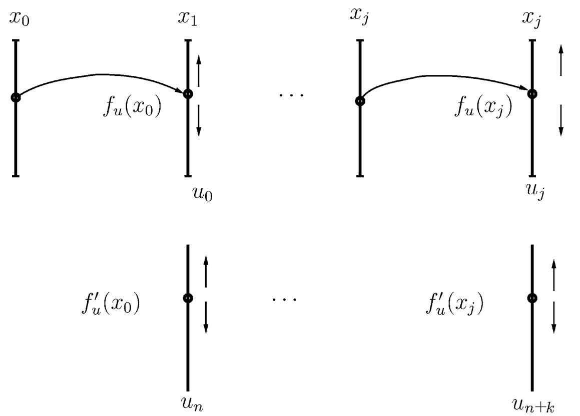


Fig. 6. Newton coefficients and their action.

**Definition 26.** The first order divided difference of $g$ is defined as

$$\Delta g(x_0, x_1) = \frac{g(x_1) - g(x_0)}{x_1 - x_0} \tag{80}$$

for $x_1 \neq x_0$ and extended by its limit value as $g'(x_0)$ for $x_1 = x_0$. Iterating this construction we define divided differences of the $m$th order for $2 \leqslant m \leqslant r$,

$$\Delta^m g(x_0, \dots, x_m) = \frac{\Delta^{m-1} g(x_0, \dots, x_{m-2}, x_m) - \Delta^{m-1} g(x_0, \dots, x_{m-2}, x_{m-1})}{x_m - x_{m-1}} \tag{81}$$

for $x_{m-1} \neq x_m$ and extended by its limit value for $x_{m-1} = x_m$.

A function loses at most one derivative of smoothness with each application of $\Delta$, so $\Delta^m g$ is at least $C^{r-m}$ if $g$ is $C^r$. Notice that $\Delta^m$ is linear as a function of $g$, and one can show that it is a symmetric function of $x_0, \dots, x_m$; in fact, by induction it follows that

$$\Delta^m g(x_0, \dots, x_m) = \sum_{i=0}^{m} \frac{g(x_i)}{\prod_{j \neq i}(x_i - x_j)}. \tag{82}$$

Another identity that is proved by induction will be more important for us, namely

$$\Delta^m\, x^j(x_0, \dots, x_m) = p_{j,m}(x_0, \dots, x_m), \tag{83}$$

where $p_{j,m}(x_0, \dots, x_m)$ is 0 for $m > j$ and for $m \leqslant j$ is the sum of all degree $j - m$ monomials in $x_0, \dots, x_m$ with unit coefficients,

$$p_{j,m}(x_0, \dots, x_m) = \sum_{r_0 + \dots + r_m = j-m} \prod_{i=0}^{m} x_i^{r_i}. \tag{84}$$

The divided differences form coefficients for the Newton interpolation formula. For all $C^\infty$ functions $g : \mathbb{R} \to \mathbb{R}$ we have

$$\begin{aligned} g(x) = {} & \Delta^0 g(x_0) + \Delta^1 g(x_0, x_1)(x - x_0) + \cdots \\ & + \Delta^{k-1} g(x_0, \dots, x_{k-1})(x - x_0) \cdots (x - x_{k-2}) \\ & + \Delta^k g(x_0, \dots, x_{k-1}, x)(x - x_0) \cdots (x - x_{k-1}) \end{aligned} \tag{85}$$

identically for all values of $x, x_0, \dots, x_{k-1}$. All terms of this representation are polynomial in $x$ except for the last one which we view as a remainder term. The sum of the polynomial terms is the degree $(k-1)$ *Newton interpolation polynomial* for $g$ at $\{x_j\}_{j=0}^{k-1}$. To obtain the degree $(2k-1)$ interpolation polynomial for $g$ and its derivative at $\{x_j\}_{j=0}^{k-1}$, we simply use (85) with $k$ replaced by $2k$ and the $2k$-tuple of points $\{x_{j \pmod k}\}_{j=0}^{2k-1}$.

Recall that $\mathbb{D}^{1,k}$ was defined implicitly by (77). We have described how to use divided differences to construct a degree $2k - 1$ interpolating polynomial of the form on the right-hand side of (77) for an arbitrary $C^\infty$ function $g$. Our interest then is in the case $g = \phi_\varepsilon$, which as a degree

$2k-1$ polynomial itself will have no remainder term and coincide exactly with the interpolating polynomial. Thus $\mathbb{D}^{1,k}$ is given coordinate-by-coordinate by

$$\begin{aligned} u_m &= \Delta^m\left(\sum_{j=0}^{2k-1}\varepsilon_j x^j\right)(x_0,\dots,x_{m\ (\mathrm{mod}\ k)}) \\ &= \varepsilon_m + \sum_{j=m+1}^{2k-1}\varepsilon_j p_{j,m}(x_0,\dots,x_{m\ (\mathrm{mod}\ s)}) \end{aligned} \tag{86}$$

for $m = 0,\dots,2k-1$.

Equation (86) defines a transformation $(u_0,\dots,u_{2k-1}) = \mathcal{L}^1_{\mathbf{X}_k}(\varepsilon)$ on $\mathbb{R}^{2k}$, where $\mathbf{X}_k = (x_0,\dots,x_{k-1}) \in I^k$. We call $\mathcal{L}^1_{\mathbf{X}_k}$ the *Newton map*. This map is simply a restriction of $\mathbb{D}^{1,k}$ to its final $2k$ coordinates:

$$\mathbb{D}^{1,k}(\mathbf{X}_k,\varepsilon) = \big(\mathbf{X}_k, \mathcal{L}^1_{\mathbf{X}_k}(\varepsilon)\big). \tag{87}$$

Notice that for fixed $\mathbf{X}_k$, the Newton map is linear and given by an upper triangular matrix with units on the diagonal. Hence it is Lebesgue measure-preserving and invertible, whether or not $\mathbf{X}_k$ lies on the diagonal $\Delta_k(I)$.

Furthermore, the Newton map $\mathcal{L}^1_{\mathbf{X}_k}$ preserves the class of *scaled Lebesgue product measures*. In general, we define

**Definition 27.** A measure $\mu$ on $\mathbb{R}^{2k}$ is called a scaled Lebesgue product measure if it is the product $\mu = \mu_0 \times \cdots \times \mu_{2k-1}$, where each $\mu_j$ is Lebesgue measure on $\mathbb{R}$ scaled by a constant factor (which may depend on the coordinate $j$).

Since the $\mathcal{L}^1_{\mathbf{X}_k}$ only shears in coordinate directions, we have the following lemma.

**Lemma 15.** *The Newton map $\mathcal{L}^1_{\mathbf{X}_k}$ given by* (86) *preserves all scaled Lebesgue product measures.*

Extension of this lemma to 2-dimensional case will be used in Section 10.

We call the basis of monomials

$$\prod_{i=0}^{j}(x - x_{i\ (\mathrm{mod}\ k)}) \quad \text{for } j = 0,\dots,2k-1 \tag{88}$$

in the space of polynomials of degree $2k-1$ the *Newton basis* defined by the $k$-tuple $\{x_j\}_{j=0}^{k-1}$. The Newton map and the Newton basis, and their analogues in dimension 2, are useful tools for perturbing trajectories and estimating the measure of "bad" parameter values $\vec{\varepsilon} \in HB^2_{<2k}(\zeta)$.

## 10. The multidimensional space of divided differences and dynamically essential parameters

### 10.1. Dynamically essential parameters

In Section 9 we defined the space of divided differences $\mathcal{DD}^{1,k}(I,\mathbb{R}) = I^k \times \mathbb{R}^k$ in the 1-dimensional case, where $I$ can be the interval $[-1,1]$. In this case in [22], we develop a method of estimating the measure of "bad" parameters (see Sections 3.3–3.4 there).

Similarly to notations of Section 9, in 2-dimensional case we define the space of divided differences

$$\begin{aligned}\mathcal{DD}^{2,k}\big(B^2_\delta,\mathbb{R}^2\big) &= \Big\{\big(p_0,p_1,\dots,p_{k-1};\{\vec{u}_\alpha\}_{|\alpha|=0},\dots,\{\vec{u}_\alpha\}_{|\alpha|=2k-1}\big)\\ &\qquad \in \underbrace{B^2_\delta\times\dots\times B^2_\delta}_{k\text{ times}}\times\mathbb{R}^{\nu(0,2)}\times\dots\times\mathbb{R}^{\nu(2k-1,2)}\Big\}\\ &= \underbrace{B^2_\delta\times\dots\times B^2_\delta}_{k\text{ times}}\times W^{u,\mathbf{P}_0}_{0,2}\times W^{u,\mathbf{P}_1}_{1,2}\times\dots\times W^{u,\mathbf{P}_{k-1}}_{k-1,2}\times W^{u,\mathbf{P}_k}_{k,2}\\ &\qquad\times W^{u,\mathbf{P}_k}_{k+1,2}\times\dots\times W^{u,\mathbf{P}_k}_{2k-1,2},\end{aligned}\tag{89}$$

where $B^2_\delta$ is the 2-dimensional ball of radius $\delta$, $\nu(j,2)$ is double of the number of multiindices $\alpha=(\alpha_1,\alpha_2)$ with $|\alpha| = j < 2k$,

$$\mathbf{P}_j = \{p_0,p_1,\dots,p_{j-1\ (\mathrm{mod}\ k)}\},\qquad \mathbf{P}_0=\emptyset,$$

and $W^{u,\mathbf{P}_{\min\{k,j\}}}_{j,2}$ is the space of homogeneous polynomials of degree $j$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ with the Newton basis defined below. There are two issues we face that were not a concern for the 1-dimensional Newton basis (88).

#### 10.1.1. Non-uniqueness

It turns out that the choice of a basis in the space of divided differences $\mathcal{DD}^{2,k}(B^2_\delta,\mathbb{R}^2)$ and the definition of the Newton map

$$\mathcal{L}^2_{\mathbf{P}_k}:\big(\{\vec{\varepsilon}_\alpha\}_{|\alpha|<2k}\big)\to\big(\{\vec{u}_\alpha\}_{|\alpha|<2k}\big)\tag{90}$$

(defined by (86) in the 1-dimensional case) for a multiindex $\alpha\in\mathbb{Z}^2_+$ is far from unique. In the 1-dimensional case, the standard basis is $\{x^j\}_{j=0}^{k-1}$ and the Newton basis is

$$\left\{\prod_{j=0}^{r-1}(x-x_j)\right\}_{r=0}^{k-1}.$$

In the 2-dimensional case, $(p-p_j)=(x-x_j,y-y_j)\in\mathbb{R}^2$ is a 2-dimensional vector. For a fixed coordinate system in $\mathbb{R}^2$, let $(p-p_j)_i$ denote the $i$th coordinate of the vector $(p-p_j)$.

The number of different monomials of the form

$$\left\{ \prod_{j=0}^{k-1} (p - p_j)_{i(j)} \right\}_{\{i(0),\dots,i(k-1)\}\in\{1,2\}^k} \tag{91}$$

is $2^k$. However the number of homogeneous monomials in 2 variables of degree $k$, i.e. $\{p^\alpha\}_{|\alpha|=k}$, is equal to $k+1$, which is much smaller than $2^k$ for $k \gg 2$.

Therefore, among the monomials (91) we need to choose an appropriate basis and define an appropriate Newton map $\mathcal{L}^2_{\mathbf{P}_k}$. The *standard* way to choose a Newton basis (see, e.g., [15]) is as follows. For $\alpha \in \mathbb{Z}^2_+$, let the Newton basis monomial for the multiindex $\alpha$ be

$$(p; p_0, \dots, p_{|\alpha|-1 \pmod k})^\alpha = \prod_{i_1=0}^{\alpha_1-1} (x - x_{i_1}) \prod_{i_2=0}^{\alpha_2-1} (y - y_{\alpha_1+i_2}). \tag{92}$$

The Newton basis for $W^{u,\mathbf{P}_k}_{<2k,2}$, the space of homogeneous vector-polynomials of degree $< 2k$, consists of 2 such monomials (one for each basis vector of $\mathbb{R}^2$) for each $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| < 2k$. By analogy with identity (77) and definition (86) we implicitly define the *2-dimensional Newton map* $\mathcal{L}^2_{\mathbf{P}_k}$ by identity

$$\sum_{|\alpha|<2k} \vec{\varepsilon}_\alpha p^\alpha \equiv \sum_{|\alpha|<2k} \vec{u}_\alpha (p; p_0, \dots, p_{|\alpha|-1 \pmod k})^\alpha. \tag{93}$$

The explicit formula involves taking divided differences with respect to both $x$ and $y$ as in (80) and will be given by (93).

The standard Newton basis does not fit purposes, as the following example illustrates: $p_0 = (1,0)$, $p_1 = (0,1)$, $p_2 = (1,1)$. Then for all $\alpha$ with $|\alpha| = 2$, we have $(p_2; p_0, p_1)^\alpha = 0$. Thus, the monomial $(p; p_0, p_1)^\alpha$ is useless to perturb the image of $p_2$. So we need to define the basis differently depending on the given sequence $\mathbf{P}_k$.

*10.1.2. Dynamically essential coordinates/monomials*

After a Newton basis is chosen, one needs to make sure that it is effective for dynamical purposes. In Section 9 we noticed that in order to perturb by Newton Interpolation Polynomials in an effective way, we need to make sure that the product of distances $\prod_{j=0}^{k-2} |p_{k-1} - p_j|$ is not too small. Similarly, in the multidimensional case we need at least one Newton monomial $(p; p_0, \dots, p_{k-2})^\alpha$ with $|\alpha| = k-1$ not to be too small. The most natural way to choose a "good" monomial is by taking the maximal coordinates of corresponding vectors. Let $v = (v_1, v_2) \in \mathbb{R}^2$ be a nonzero vector. Set

$$m(v) = \begin{cases} 1, & \text{if} |v_1| \geqslant |v_2|, \\ 2, & \text{otherwise.} \end{cases}$$

Then

$$\left| \prod_{j=0}^{k-2} (p_{k-1} - p_j)_{m(p_{k-1}-p_j)} \right| \geqslant 2^{-(k-1)/2} \prod_{j=0}^{k-2} |p_{k-1} - p_j|. \tag{94}$$

This is a satisfactory estimate, because $k$ (the number of loops) is bounded. For a given $\mathbf{P}_k = \{p_0, \dots, p_{k-1}\}$ we can neglect uniformly bounded distortion factors. Given $\mathbf{P}_k = \{p_i\}_{i=0}^{k-1}$ of pairwise distinct points, we call the monomials

$$Q^{\mathrm{dyn}}(p, \mathbf{P}_j) = \prod_{i=0}^{j-1} (p - p_i)_{m(p_j - p_i)}, \quad j = 0, \dots, k-1, \tag{95}$$

*dynamically essential*. These Newton monomials control *periodicity* (see Fig. 6, line 1). Denote by $\mathbf{P}'_j = \{p_{j+1}, \dots, p_{k-1}, p_j\}$ for $j = 0, \dots, k-1$, where $\mathbf{P}'_k = \emptyset$ and $Q^{\mathrm{dyn}}(p, \mathbf{P}'_k) \equiv 1$. Then for each $m = 1, 2$ and $j = 0, \dots, k-1$ set

$$Q^{\mathrm{dyn}}_m(p, \mathbf{P}_j, \mathbf{P}'_j) = (p - p_j)_m \left(Q^{\mathrm{dyn}}(p, \mathbf{P}_j)\right)^2 Q^{\mathrm{dyn}}(p, \mathbf{P}'_j). \tag{96}$$

These Newton monomials control *hyperbolicity* (see Fig. 6, line 2).

We use these Newton monomials to estimate the measure of "bad" $u$-parameters.[5]

*10.1.3. Complete set of dynamically essential coordinates/monomials*

Dynamically essential Newton monomials introduced in (95) control position of trajectories (see Fig. 6, line 1) and those in (96) control properties of the linearization (see Fig. 6, line 2).

**Definition 28.** The complete set of dynamically essential Newton monomials associated with a $k$-tuple $\mathbf{P}_k = \{p_0, \dots, p_{k-1}\}$ is a collection of $3k$ pairs of monomials (one for each basis vector of $\mathbb{R}^2$) given by (95)–(96).

*10.1.4. An algorithm of constructing Newton basises*

Now we present an elementary scheme of constructing a vast family of Newton basises in the space of polynomials of given degree and show that at least one of them contains all dynamically essential monomials.

Consider the positive octant of planar integer grid $\mathbb{Z}^2_+ = \{\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2 \mid \alpha_1 \geqslant 0,\ \alpha_2 \geqslant 0\}$. Denote $|\alpha| = \alpha_1 + \alpha_2$. We say that a grid point $\alpha \in \mathbb{Z}^2_+$ is *equipped* if there is a oriented path $\Gamma_\alpha$ from the origin to $\alpha$ consisting of oriented unit segments $\Gamma_{\alpha,j}$ connecting a vertex $\alpha'$ with either $\alpha' + (1, 0)$ or $\alpha' + (0, 1)$. Represent $\Gamma_\alpha = \bigcup_{j=1}^{|\alpha|} \Gamma_{\alpha,j}$ as the ordered union of these unit segments. Denote an *equipped point* by $\alpha(\Gamma)$, where this symbol is a point in $\mathbb{Z}^2_+$ along with an oriented path connecting it to the origin. See Fig. 7.

Let $\mathbf{P}_k = \{p_0, \dots, p_{k-1}\} \subset B^2_\delta$ be a $k$-tuple of points and let $|\alpha| < 2k$. A *Newton monomial associated to an equipped vertex* $\alpha(\Gamma)$ is defined as follows

$$(p; p_0, p_1 \dots, p_{|\alpha|-1 \pmod k})^{\alpha(\Gamma)} = \prod_{j=1}^{|\alpha|} (p - p_{j-1 \pmod k})_{j(\Gamma)}, \tag{97}$$

where $j(\Gamma)$ is 1 if $\Gamma_j$ is horizontal and 2 otherwise.

Important to notice that properly choosing equipment $\Gamma$ we could obtain the complete set of dynamically essential monomials associated with any $k$-tuple $\mathbf{P}_k$ of pairwise distinct points.

[5] We shall imitate estimates (3.26)–(3.34) from [22].

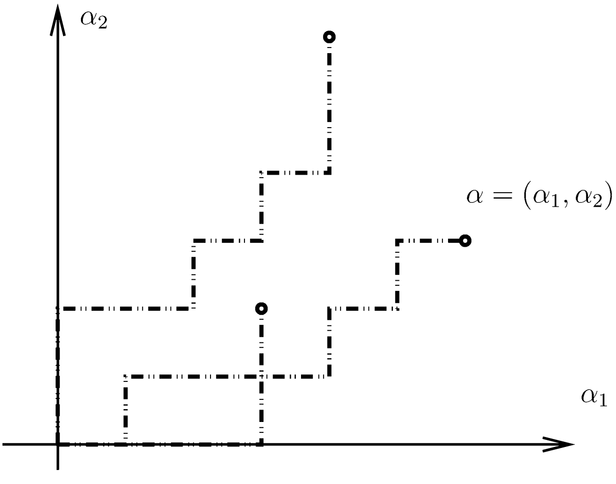


Fig. 7. Equipped vertices.

Let $\mathcal{I}_k = \bigcup_{|\alpha|<2k} \alpha(\Gamma)$ be the union of equipped vertices, $\mathbf{P}_k = \{p_0, \dots, p_{k-1}\} \subset B_\delta^2$ a $k$-tuple. Call $\{(p; p_0, \dots, p_{|\alpha|-1 \pmod k})^{\alpha(\Gamma)}\}_{|\alpha|<2k}$ be the set of Newton monomials *associated to* $\mathcal{I}_k$ *and* $\mathbf{P}_k$.

**Lemma 16.** *This is a basis in the space* $W_{<2k,2}$ *of* 2*-component polynomials in* $(x, y)$ *of degree* $< 2k$.

This lemma, proven below, motivates the following

**Definition 29.** Let $\mathcal{I}_k = \bigcup_{|\alpha|<2k} \alpha(\Gamma)$ be the union of equipped vertices, $\mathbf{P}_k = \{p_0, \dots, p_{k-1}\} \subset B_\delta^2$ be a $k$-tuple, and $\{(p; p_0, \dots, p_{|\alpha(\Gamma)|-1 \pmod k})^{\alpha(\Gamma)}\}_{|\alpha|<2k}$ be the set of Newton monomials associated to $\mathcal{I}_k$ and $\mathbf{P}_k$. We call this basis a dynamical Newton basis if it contains the complete set of dynamically essential Newton monomials associated with $\mathbf{P}_k$.

Now we define the Newton map associated to an equipped multiindex. Implicitly for the standard basis it is defined by (93). It requires taking divided differences of functions of 2 variables (see also [15]) so we need to generalize Definition 26. The definitions below are very much similar to 1-dimensional definitions (80)–(85), but notations are a bit cumbersome. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a $C^\infty$-smooth function of $n$ variables $(x_1, \dots, x_n)$. Our main interest will be the case when $g$ is a polynomial.

**Definition 30.** The first order divided difference of $g$ with respect to $x_i$ is defined as

$$\Delta_{x_i} g(x_1, \dots, x_i', x_i'', \dots, x_n) = \frac{g(x_1, \dots, x_i', \dots, x_n) - g(x_1, \dots, x_i'', \dots, x_n)}{x_i'' - x_i'} \tag{98}$$

for $x'' \neq x'$ and extended by its limit value as $\partial_{x_i} g(x_1, \dots, x_i, \dots, x_n)$ for $x_i'' = x_i' = x_i$.

Clearly, $\Delta_{x_i} g$ is $C^\infty$-smooth function of its arguments, because $g$ is $C^\infty$-smooth. Therefore, iterating this construction is possible. This could lead to somewhat awkward notation, since formally $\Delta_{x_k}\Delta_{x_k}$ makes no sense: one should decide between $\Delta_{x_k'}\Delta_{x_k}$ and $\Delta_{x_k''}\Delta_{x_k}$. Fortunately, the result will be the same, as an easy computation shows [3]. Moreover, it is clear that the operators $\Delta_{x_k}$ and $\Delta_{x_j}$ commute for $k \neq j$, and therefore we use well-defined multiindex notation: for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ and a function $f(x, y)$ of 2 variables $(x, y)$ we denote $\Delta^\alpha f = \Delta_x^{\alpha_1}\Delta_y^{\alpha_2} f$ the mixed divided difference of order $|\alpha| = \alpha_1 + \alpha_2$. Now we need to apply the Definition 30 to incorporate equipped multiindices $\{\alpha(\Gamma)\}$ into this scheme.

Fix an equipped point $\alpha(\Gamma)$, $|\alpha| < 2k$, and a $k$-tuple $\mathbf{P}_k = \{p_0, \dots, p_{k-1}\}$. We define divided differences of the $|\alpha|$th order associated to $\alpha(\Gamma)$ inductively as follows.

- Start with a $C^\infty$-smooth function of 2 variables $h_0(x, y)$. If $\Gamma_1$ is horizontal, take divided difference of $h_0(x, y)$ with respect to $x$. Namely, apply (98) with $n = 2$, $g(x_1, x_2) = h_0(x, y)$, $i = 1$, $x_1 = x$, $x_2 = y$, $x_1' = x_0$, $x_1'' = x$. If $\Gamma_1$ is vertical, put $i = 2$, $x_2' = y_0$, $x_2'' = y$. Now we obtain a function of 3 variables, denote it by $h_1(x_1, x_2, x_3)$.
- Start with a function of 3 variables $h_1(x_1, x_2, x_3)$. For determinacy suppose $\Gamma_1$ is horizontal, then $h_1(x_1, x_2, x_3) = h_1(x_0, x, y) = \Delta_x h_0(x, y)$. If $\Gamma_2$ is horizontal, take divided difference of $h_1(x_0, x, y)$ with respect to $x$. Namely, apply (98) with $n = 3$, $g(x_1, x_2, x_3) = h_1(x_0, x, y)$, $i = 2$, $x_1 = x_0$, $x_2' = x_1$, $x_2'' = x$, $x_3 = y$. If $\Gamma_2$ is vertical, take divided difference with respect to $y$. Namely, apply (98) with $n = 3$, $g(x_1, x_2, x_3) = h_1(x_0, x, y)$, $i = 3$, $x_1 = x_0$, $x_2 = x$, $x_3' = y_1$, $x_3'' = y$. Now we obtain a function of 4 variables, denoted it by $h_2(x_1, x_2, x_3, x_4)$ and so on.

Let $\mathcal{I}_k = \bigcup_{|\alpha|<2k} \alpha(\Gamma)$ be the union of equipped vertices, $\mathbf{P}_k = \{p_0, \dots, p_{k-1}\}$ be a $k$-tuple, and $\{(p; p_0, \dots, p_{|\alpha|-1 \pmod k})^{\alpha(\Gamma)}\}_{|\alpha|<2k}$ be the set of Newton monomials associated to $\mathcal{I}_k$ and $\mathbf{P}_k$. Now we would like to define an associated Newton map implicitly given by

$$\sum_{|\alpha|<2k} \vec{\varepsilon}_\alpha \equiv \sum_{|\alpha|<2k} \vec{u}_a (p; p_0, \dots, p_{|\alpha|-1 \pmod k})^{\alpha(\Gamma)}, \tag{99}$$

where $(p; p_0, \dots, p_{|\alpha|-1 \pmod k})^{\alpha(\Gamma)}$ is defined in (97).

In the case of trivial specification, i.e. $(p; p_0, \dots, p_{|\alpha|-1 \pmod k})^{\alpha(\Gamma)}$ is replaced by $(p; p_0, \dots, p_{|\alpha|-1 \pmod k})^{\alpha}$, given by (92), the standard formula for multivariable Newton map is

$$\vec{u}_\alpha = \vec{\varepsilon}_\alpha + \sum_{\alpha \prec \beta,\, |\beta|<2k} \vec{\varepsilon}_\beta \; p_{\beta_1\alpha_1}(x_0, , \dots, x_{\alpha_1 \pmod k})\, p_{\beta_2\alpha_2}(y_0, \dots, y_{\alpha_2 \pmod k}), \tag{100}$$

where $p_{\beta_1\alpha_1}$ (respectively $p_{\beta_2\alpha_2}$) is the homogeneous polynomial in $x$ (respectively in $y$) of degree $(\beta_1 - \alpha_1)$ (respectively $(\beta_2 - \alpha_2)$) defined by (84) depending on $(\beta_1 - \alpha_1)$ (respectively $(\beta_2 - \alpha_2)$) variables out of $|\beta| - |\alpha|$ (see, e.g., [15,20]). It generalizes the 1-dimensional formula (86) to the case of 2-variables. However, we are not able to provide a formula of type (92) for a general specification. We shall give an indirect definition. Namely, we just prove that there is a linear map of $\{\vec{\varepsilon}_\alpha\}_{|\alpha|<2k}$ into $\{\vec{u}_\alpha\}_{|\alpha|<2k}$ satisfying the identity (99).

Open brackets in the right-hand side of (99). Since points $p_0, \dots, p_{k-1}$ are fixed, we could view both sides as polynomials in $x$ and $y$. Group terms in the right-hand side according to monomials $p^\alpha = x^{\alpha_1} y^{\alpha_2}$. Compare coefficients of both sides. This leads to equalities

$$\vec{\varepsilon}_\alpha = \vec{u}_\alpha + \sum_{\alpha \prec \beta,\, |\beta|<2k} \vec{u}_\beta \mathcal{P}_{\beta,\alpha}(x_0, \dots, x_{\beta_1 - 1}, y_0, \dots, y_{\beta_2 - 1}), \tag{101}$$

where $\mathcal{P}_{\beta,\alpha}$ is a homogeneous polynomial computable using Vieta formulas. This shows that the linear transformation of $\{\vec{\varepsilon}_\alpha\}_{|\alpha|<2k}$ into $\{\vec{u}_\alpha\}_{|\alpha|<2k}$ is given by an upper triangular matrix with units on the diagonal. This leads to the following

**Lemma 17.** *There is an upper triangular matrix $\mathcal{L}^2_{\mathbf{P}_k,\mathcal{I}_k} = \{\mathcal{Q}_{\beta,\alpha}\}_{|\alpha|,|\beta|<2k}$ with units on the diagonal such that for*

$$\vec{u}_{\alpha(\Gamma)} = \vec{\varepsilon}_\alpha + \sum_{\alpha \prec \beta,\, |\beta|<2k} \mathcal{Q}_{\beta,\alpha} \vec{\varepsilon}_\beta \tag{102}$$

*and for any $\{\vec{\varepsilon}_\alpha\}_{|\alpha|<2k}$ and $\{\vec{u}_\alpha\}_{|\alpha|<2k}$ given by this formula* (99) *holds.*

**Proof.** For example, use formula of inversion of a square matrix using complement maximal minors. $\square$

**Definition 31.** We call the linear map $\mathcal{L}^2_{\mathbf{P}_k,\mathcal{I}_k}$, defined by the matrix $\{\mathcal{Q}_{\beta,\alpha}\}_{|\alpha|,|\beta|<2k}$, an associated Newton map to a $k$-tuple $\mathbf{P}_k$ and equipment $\mathcal{I}_k$.

Denote by $W^{u,\mathbf{P}_k,\mathcal{I}_k}_{<2k,2}$ the space of 2-component vector polynomials of degree $< 2k$ in $x$ and $y$ with the Newton basis associated with $\mathcal{I}_k$ and $\mathbf{P}_k$. The map $\mathcal{L}^2_{\mathbf{P}_k,\mathcal{I}_k}$ is called *a dynamical Newton map* if associated set $\{(p; p_0, \ldots, p_{|\alpha|-1})^{\alpha(\Gamma)}\}_{|\alpha|<2k}$ contains the complete set of dynamically essential Newton monomials associated with $\mathbf{P}_k$ and is denoted by

$$\mathcal{L}^{2,\mathrm{dyn}}_{\mathbf{P}_k,\mathcal{I}_k} : W_{<2k,2} \to W^{u,\mathbf{P}_k,\mathcal{I}_k}_{<2k,2}. \tag{103}$$

**Lemma 18.** *The Newton map $\mathcal{L}^2_{\mathbf{P}_k,\mathcal{I}_k} : W_{<2k,2} \to W^{u,\mathbf{P}_k,\mathcal{I}_k}_{<2k,2}$ given by* (102) *preserves all scaled Lebesgue product measures.*[6]

**Proof.** It follows from the fact that the corresponding linear map is given by an upper triangular matrix with units on the diagonal. $\square$

**Proof of Lemma 16.** Notice that for any multiindex $\alpha = (\alpha_1, \alpha_2)$ independently of its equipment $\Gamma_\alpha$ the Newton map (102) is given by an upper triangular matrix with units on the diagonal. Therefore, this linear map is non-degenerate and $\{(p; p_0, \ldots, p_{|\alpha|-1})^{\alpha(\Gamma)}\}_{|\alpha|<2k}$ forms the basis in $W_{<2k,2}$. $\square$

## 11. Discretization Method

In Section 3.1 in Step V we reduced the proof of the results of the paper to the proof of estimate (31). Then in Step VI we mentioned that this estimate will be proven using Newton Interpolation Polynomials. Now we split Step VI into two steps.

**Step VI A** *(Collection). Reduction to estimate of the measure of "bad" parameters associated with a scattered admissible pseudotrajectory of a given type and specified starting points of generalized loops.*

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[6] See Definition 27 of scaled Lebesgue measures.

We need to estimate $\nu_{<2s}\{B_{\mathfrak{Q}}^{\mathrm{adm}}[s, f, \mathcal{N}_k, n, l, m]\}$. Long loops of pseudotrajectories of type $\mathcal{N}_k$ have lengths $n_1^*+1, \dots, n_\tau^*+1$, where $\tau = t(\mathcal{N}_k)$ is the number of generalized loops. Consider testing rectangles $\{\Pi_{n_j^*}\}_{j=1}^{\tau}$ and grids $\{\boldsymbol{\Pi}_{n_j^*}(\mu^{-\alpha_{l,m}n})\}_{j=1}^{\tau}$, defined in Sections 8.1 and 8.2, respectively. Pick a set of $(2s)^{-1}\mu^{-\alpha_{l,m-1}n}$-scattered set of points $\mathcal{R} = \{R_0, \dots, R_{\tau-1}\}$, $R_j \in \boldsymbol{\Pi}_{n_{j+1}^*}(\mu^{-\alpha_{l,m}n})$, consider the set of parameters

$$\begin{aligned} B_{\mathfrak{Q}}^{\mathrm{fixed}}[s, f, \mathcal{N}_k, n, l, m; \mathcal{R}] = \Big\{ & \vec{\varepsilon}_{<2s} \in HB_{<2s}(\zeta) \;\Big|\; f_{\vec{\varepsilon}_{<2s}} \text{ has a } k\text{-loop} \\ & (2s)^{-1}\mu^{-\alpha_{l,m-1}n}\text{-scattered } \mu^{-\alpha_{l,m}n}\text{-admissible pseudotrajectory } \mathfrak{R} \\ & \text{of type } \mathcal{N}_k \text{ having shape } (l,n),\ \ (f_{\vec{\varepsilon}_{<2s}}, \mathfrak{R}) \notin \mathcal{K}\big\{\mathfrak{Q}, l(\mathcal{N}_s), n, (\theta_{l,m}, \xi)\big\}, \\ & \text{and } \mathcal{R} \text{ is the ordered set of starting points of generalized loops}\Big\}. \end{aligned} \tag{104}$$

For each $j$ the number of points in $\boldsymbol{\Pi}_{n_j^*}(\mu^{-\alpha_{l,m}n})$ is bounded by $9\delta^2(\mu^{-2\alpha_{l,m}n})$ (see Lemma 10). Thus, the number of different choices of set $\mathcal{R}$ is bounded by $(9\delta^2\mu^{-2\alpha_{l,m}n})^\tau$. Therefore to prove (31) we need to establish the following estimate:

$$\nu_{<2s}\big\{B_{\mathfrak{Q}}^{\mathrm{fixed}}[s, f, \mathcal{N}_k, n, l, m; \mathcal{R}]\big\} \leqslant \big(9\delta^2\mu^{2\alpha_{l,m}n}\big)^{-\tau} C_s^* \mu^{-h_l n}. \tag{105}$$

**Step VI B.** *Reduction to the cone condition for one generalized loop of a scattered admissible pseudotrajectory of given type and fixed ordered starting points of generalized loops.*

Given $(2s)^{-1}\mu^{-\alpha_{l,m-1}n}$-scattered set $\mathcal{R}$ as above, define the following set of parameters:

$$\begin{aligned} B_{\mathfrak{Q}}^{\mathrm{first}}[s, f, \mathcal{N}_k, n, l, m; \mathcal{R}] = \Big\{ & \vec{\varepsilon}_{<2s} \in HB_{<2s}(\zeta) \;\Big|\; f_{\vec{\varepsilon}_{<2s}} \text{ has a } k\text{-loop} \\ & (2s)^{-1}\mu^{-\alpha_{l,m-1}n}\text{-scattered } \mu^{-\alpha_{l,m}n}\text{-admissible pseudotrajectory } \mathfrak{R} \\ & \text{of type } \mathcal{N}_k \text{ having shape } (l,n),\ \mathcal{R} \text{ is the ordered set of starting points of} \\ & \text{generalized loops, and inclusion } K_{\xi_l n}(R_0) \hookrightarrow_{F^{N_1}_{\mathfrak{R},\vec{\varepsilon}_{<2s}}(R_0)} K_{\theta_{l,m}n}(R_1) \text{ fails}\Big\}, \end{aligned} \tag{106}$$

where $N_1$ is a length of the first generalized loop.

For any ordered set of points $\mathcal{R} = \{R_0, \dots, R_{\tau-1}\}$ define a cyclic permutation $\mathrm{Shift}(\mathcal{R}) = \{R_1, \dots, R_{\tau-1}, R_0\}$. Now we have

$$B_{\mathfrak{Q}}^{\mathrm{fixed}}[s, f, \mathcal{N}_k, n, l, m; \mathcal{R}] \subset \bigcup_{j=0}^{\tau-1} B_{\mathfrak{Q}}^{\mathrm{first}}\big[s, f, \mathcal{N}_k, n, l, m; \mathrm{Shift}^j(\mathcal{R})\big]. \tag{107}$$

Now we reduce the problem to the proof of the following estimate:

$$\nu_{<2s}\big\{B_{\mathfrak{Q}}^{\mathrm{first}}[s, f, \mathcal{N}_k, n, l, m; \mathcal{R}]\big\} \leqslant \frac{1}{s}\big(9\delta^2\mu^{2\alpha_{l,m}n}\big)^{-\tau} C_s^* \mu^{-h_l n}. \tag{108}$$

To obtain this estimate we apply Fubini reduction from $HB_{<2s}(\zeta)$ to $HB_{<2k}(\zeta)$ first and then use Newton Interpolation Polynomials.

*11.1. Fubini reduction from $HB_{<2s}(\zeta)$ to $HB_{<2k}(\zeta)$*

We reduced the problem to estimate (108) of $\nu_{<2s}\{B^{\text{first}}_{\mathfrak{Q}}[s, f, \mathcal{N}_k, n, l, m; \mathcal{R}]\}$. Before we apply Discretization Method we need to reduce this estimate to the estimate of $\nu_{<2k}$-measure of the following set:

$$\begin{aligned} B^{\text{first}}_{\mathfrak{Q}}[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}] = \big\{ & \vec{\varepsilon}_{<2k} \in HB_{<2k}(\zeta) \,\big|\, f_{\vec{\varepsilon}_{<2k}} = f + \vec{\Phi}_{\vec{\varepsilon}_{<2k}} \text{ has a } k\text{-loop} \\ & (2s)^{-1}\mu^{-\alpha_{l,m-1}n}\text{-scattered } \mu^{-\alpha_{l,m}n}\text{-admissible pseudotrajectory } \mathfrak{R} \\ & \text{of type } \mathcal{N}_k \text{ having shape } (l,n),\ \mathcal{R} \text{ is the ordered set of starting points of} \\ & \text{generalized loops, and inclusion } K_{\xi_l n}(R_0) \hookrightarrow_{F^{N_1}_{\mathfrak{R},\vec{\varepsilon}_{<2k}}(R_0)} K_{\theta_{l,m}n}(R_1) \text{ fails}\big\}, \end{aligned} \tag{109}$$

where $N_1$ is a length of the first generalized loop.

One can make this Fubini reduction almost in the same way as in Section 3.2. Namely, consider the following decomposition of the space of parameters, perturbations, and the product measure for $k < s$:

$$\begin{gathered} HB_{<2s}(\zeta) = HB_{<2k}(\zeta) \oplus HB_{\geqslant 2k,<2s}(\zeta), \quad \text{where} \\ HB_{\geqslant 2k,<2s}(\zeta) = \big\{ \varepsilon^q_{ij} \in \mathbb{R} \,\big|\, |\varepsilon^q_{ij}| \leqslant \zeta,\ q = 1,2,\ 0 \leqslant i,j,\ 2k \leqslant i+j < 2s \big\}, \\ \vec{\varepsilon}_{<2s} = (\vec{\varepsilon}_{<2k}, \vec{\varepsilon}_{\geqslant 2k,<2s}) \in HB_{<2k}(\zeta) \oplus HB_{\geqslant 2k,<2s}(\zeta), \\ \Phi^q_{\vec{\varepsilon}_{<2s}}(x,y) = \Phi^q_{\vec{\varepsilon}_{<2k}}(x,y) + \Phi^q_{\vec{\varepsilon}_{\geqslant 2k,<2s}}(x,y) = \sum_{0 \leqslant i,j,\, i+j<2k} \varepsilon^q_{ij} x^i y^j + \sum_{0 \leqslant i,j,\, 2k \leqslant i+j<2s} \varepsilon^q_{ij} x^i y^j, \\ \nu_{<2s} = \nu_{<2k} \times \nu_{\geqslant 2k,<2s}, \quad \text{where} \\ \nu_{<2k} = \bigtimes_{0 \leqslant i,j,\, i+j<2k} \big(\nu^1_{ij} \times \nu^2_{ij}\big), \qquad \nu_{\geqslant 2k,<2s} = \bigtimes_{0 \leqslant i,j,\, 2k \leqslant i+j<2s} \big(\nu^1_{ij} \times \nu^2_{ij}\big). \end{gathered} \tag{110}$$

Suppose we can get the following estimate

$$\nu_{<2k}\big\{B^{\text{first}}_{\mathfrak{Q}}[k, \tilde{f}, \mathcal{N}_k, n, l, m; \mathcal{R}]\big\} \leqslant \tilde{C}_s \big(\mu^{-2\alpha_{l,m}n}\big)^{\tau} \mu^{-h_l n}, \tag{111}$$

where $\tilde{f} = f_{(0,\vec{\varepsilon}_{\geqslant 2k,<2s})} = f + \vec{\Phi}_{\vec{\varepsilon}_{\geqslant 2k,<2s}}$, uniformly over all parameters $\vec{\varepsilon}_{\geqslant 2k,<2s} \in HB_{\geqslant 2k,<2s}$. Since $\delta$ is assumed to be small, Fubini Theorem implies estimate (108) with $C^*_s = s\tilde{C}_s$.

In order to apply the Discretization Method and Newton Interpolation Polynomials we need to partition the cube of parameters $HB_{<2k}(\zeta)$ into bricks of certain size and derive estimates of the type (111) in each of those bricks.

*11.2. From the cube to a brick of at most standard thickness*

**Definition 32.** We call the set of parameters

$$HB^{\text{st}}_{<2k}(\zeta) = \left\{ \varepsilon^q_{ij} \in \mathbb{R} \,\middle|\, |\varepsilon^q_{ij}| \leqslant \frac{\zeta}{(i!j!)^2},\ q = 1,2,\ 0 \leqslant i,j, \text{ and } 2k \leqslant i+j < 2s \right\}$$

the brick of standard thickness with width $\zeta$.[7]

[7] Cf. [22, Sections 3.1 and 4.4] for similar definitions.

Denote by $HB^{\rm st}_{<2k}(\vec\varepsilon^*_{<2k},\zeta)$ the standard thickness with width $\zeta$ centered at $\vec\varepsilon^*_{<2k}$, i.e. conditions $|\varepsilon^q_{ij}|\leqslant\zeta/(i!j!)^2$ are replaced by $|\varepsilon^q_{ij}-(\vec\varepsilon^*)^q_{ij}|\leqslant\zeta/(i!j!)^2$. We shall partition the cube $HB_{<2k}(\zeta)$ into disjoint bricks of standard thickness with width $\zeta$ simply by dividing $ij$-side of the brick into $(i!j!)^2$ equal sides. The probability measure $\nu_{<2k}$ on $HB_{<2k}(\zeta)$, defined in the previous section, induces a measure on each brick of standard thickness with width $\zeta$. Fix one brick $HB^{\rm st}_{<2k}(\vec\varepsilon^*_{<2k},\zeta)$. After normalization $\nu_{<2k}$ induces the probability on this brick denoted $\nu^{\rm st}_{<2k}$. Denote by $B^{\rm first,st}_{\mathfrak{Q}}[k,\tilde f_{\vec\varepsilon^*_{<2k}},\mathcal{N}_k,n,l,m;\mathcal{R}]$ the intersection of the set $B^{\rm first}_{\mathfrak{Q}}[k,\tilde f,\mathcal{N}_k,n,l,m;\mathcal{R}]$ with the brick $HB^{\rm st}_{<2k}(\vec\varepsilon^*_{<2k},\zeta)$. If we can prove that the following estimate holds true

$$\nu^{\rm st}_{<2k}\big\{B^{\rm first,st}_{\mathfrak{Q}}[k,\tilde f_{\vec\varepsilon^*_{<2k}},\mathcal{N}_k,n,l,m;\mathcal{R}]\big\}\leqslant\tilde C_s\big(\mu^{-2\alpha_{l,m}n}\big)^\tau\mu^{-h_ln},\tag{112}$$

where $\tilde f_{\vec\varepsilon^*_{<2k}}=\tilde f+\vec\Phi_{\vec\varepsilon^*_{<2k}}$, uniformly over all parameters $\vec\varepsilon_{<2k}\in HB_{<2k}(\zeta)$, then this implies estimate (111). We shall prove (112) in Collection Lemma (Section 11.6).

### 11.3. Decomposition into pseudotrajectories

Now our goal is to estimate the measure of the "bad" set $B^{\rm first,st}_{\mathfrak{Q}}[k,f,\mathcal{N}_k,n,l,m;\mathcal{R}]$ and prove (112). This set consists of parameters for which there is a $k$-loop scattered (non-recurrent) admissible pseudotrajectory of type $\mathcal{N}_k$ having shape $(l,n)$, given ordered starting points of generalized loops, and associated cone condition of this pseudotrajectory fails after the first generalized loop. We apply Discretization Method similar to the one in [22, Sections 3.1–3.4]. For this purpose we would like to contain the set of "bad" parameters into a finite collection of subsets each of "bad" parameters corresponding to a single $k$-loop $(2s)^{-1}\mu^{-\alpha_{l,m-1}n}$-scattered $\mu^{-\alpha_{l,m}n}$-admissible pseudotrajectory, where $\alpha_{l,m-1}$ and $\alpha_{l,m}$ are defined in Section 7.1. So fix type $\mathcal{N}_k=(n_1,\dots,n_k)$ and its shape $(l,n)$. Consider grids $\boldsymbol{\Pi}_{n_i}(\mu^{-\alpha_{l,m}n})$ and $\tilde{\boldsymbol{\Pi}}_{n_i}(\mu^{-\alpha_{l,m}n})$ in each of testing rectangles $\Pi_{n_i}$ and $\tilde\Pi_{n_i}$ for $i=1,\dots,k$. Denote

$$\boldsymbol{\Pi}_{\mathcal{N}_k}\big(\mu^{-\alpha_{l,m}n}\big)=\bigcup_{1\leqslant i\leqslant k}\boldsymbol{\Pi}_{n_i}\big(\mu^{-\alpha_{l,m}n}\big)\quad\text{and}\quad\tilde{\boldsymbol{\Pi}}_{\mathcal{N}_k}\big(\mu^{-\alpha_{l,m}n}\big)=\bigcup_{1\leqslant i\leqslant k}\tilde{\boldsymbol{\Pi}}_{n_i}\big(\mu^{-\alpha_{l,m}n}\big).$$

Fix starting points of generalized loops $\mathcal{R}=\{R_0,\dots,R_{t-1}\}$ and an admissible $\mathbf{n}$-tuple of points $\mathfrak{R}=\{r_0,\dots,r_{\mathbf{n}-1}\}$, $\mathbf{R}_k=\mathfrak{R}\cap U=\{\mathbf{r}_0,\dots,\mathbf{r}_{k-1}\}$ so that $\mathcal{R}\subseteq\mathbf{R}_k\subset\boldsymbol{\Pi}_{\mathcal{N}_k}(\mu^{-\alpha_{l,m}n})$, $\mathbf{r}_i\in\boldsymbol{\Pi}_{n_{i+1}}(\mu^{-\alpha_{l,m}n})$ for $i=0,\dots,k-1$. Suppose $0=i_0<i_1<\dots<i_{\tau-1}<k$ are indices of starting points of generalized loops in $\mathbf{R}_k$. Since the map $f_{\vec\varepsilon}$ outside $\hat U$ does not depend on $\vec\varepsilon$, a $k$-tuple $\mathbf{R}_k\subset U$ (if admissible) determines a $k$-loop admissible pseudo-orbit $\mathfrak{R}$ uniquely. Therefore, there is a one-to-one correspondence between $\mathbf{n}$-tuple $k$-loop admissible pseudo-orbits $\mathfrak{R}=\{r_0,\dots,r_{\mathbf{n}-1}\}$ and the corresponding intersections $\mathbf{R}_k=\mathfrak{R}\cap U$. Thus, we can consider only $(2s)^{-1}\mu^{-\alpha_{l,m-1}n}$-scattered $\mu^{-\alpha_{l,m}n}$-admissible $k$-tuples $\mathbf{R}_k\subset U$. Let $\mathcal{R}\subseteq\{\mathbf{r}_0,\dots,\mathbf{r}_{k-1}\}$. Define "bad" parameters corresponding to a $\mu^{-\alpha_{l,m}n}$-pseudotrajectory $\mathfrak{R}$ as follows

$$\begin{aligned}B^{\rm first,st}_{\mathfrak{Q}}&\big[k,f,\mathcal{N}_k,n,l,m;\mathcal{R},\{\mathbf{r}_0,\dots,\mathbf{r}_{k-1}\}\big]=\big\{\vec\varepsilon_{<2k}\in HB^{\rm st}_{<2k}(\zeta)\;\big|\;f_{\vec\varepsilon_{<2k}}\text{ has}\\&\text{a }k\text{-loop }(2s)^{-1}\mu^{-\alpha_{l,m-1}n}\text{-scattered }\mu^{-\alpha_{l,m}n}\text{-admissible pseudotrajectory }\mathfrak{R}\\&\text{of type }\mathcal{N}_k\text{ having shape }(l,n),\ \mathbf{R}_k\text{ is the ordered set of starting points of}\\&\text{loops }\mathbf{R}_k=\{\mathbf{r}_0,\dots,\mathbf{r}_{k-1}\}\subset\boldsymbol{\Pi}_{\mathcal{N}_k}(\mu^{-\alpha_{l,m}n}),\ \mathbf{r}_i\in\boldsymbol{\Pi}_{n_{i+1}}(\mu^{-\alpha_{l,m}n})\\&\text{for }i=0,\dots,k-1,\text{ and }\vec\varepsilon_{<2k}\in B^{\rm first}_{\mathfrak{Q}}[k,f,\mathcal{N}_k,n,l,m;\mathcal{R}]\big\}.\end{aligned}\tag{113}$$

To estimate the measure of $B^{\text{first,st}}_{\mathfrak{Q}}[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}]$ we need to define additional sets.[8] Consider the definition of parameters $\vec{\varepsilon}_{<2k}$ for which the map $f_{\vec{\varepsilon}_{<2k}}$ has a prescribed $(2s)^{-1}\mu^{-\alpha_{l,m-1}n}$-scattered $\mu^{-\alpha_{l,m}n}$-admissible pseudo-orbit $\{\mathbf{r}_0, \ldots, \mathbf{r}_{k-1}\} \subset \boldsymbol{\Pi}_{\mathcal{N}_k}(\mu^{-\alpha_{l,m}n})$ that is almost periodic, has appropriate combinatorics, but does not satisfy the cone property after the first generalized loop. Based on it define a set of parameters $\vec{\varepsilon}_{<2k}$ for which only a part of length $\mathbf{n}' < \mathbf{n}$ of the $\mu^{-\alpha_{l,m}n}$-pseudotrajectory $\{r_0, \ldots, r_{\mathbf{n}'-1}\}$ and an ordered set of starting points of generalized loops are prescribed for $f_{\vec{\varepsilon}_{<2k}}$ and $\{r_0, \ldots, r_{\mathbf{n}'-1}\} \cap U = \{\mathbf{r}_0, \ldots, \mathbf{r}_{j-1}\}$:

$$\begin{aligned} &B^{\text{first,st}}_{\mathfrak{Q}}\big[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}, \{\mathbf{r}_0, \ldots, \mathbf{r}_{j-1}\}\big] \\ &\quad = \big\{\vec{\varepsilon}_{<2k} \in HB^{\text{st}}_{<2k}(\zeta) \;\big|\; \text{there is } \{r_{\mathbf{n}'}, \ldots, r_{\mathbf{n}-1}\} \subset \boldsymbol{\Pi}_{\mathcal{N}_k}(\mu^{-\alpha_{l,m}n}) \text{ such that } \{\mathbf{r}_i\}_{i=0}^{k-1} \text{ defines} \\ &\qquad \text{a } k\text{-loop } (2s)^{-1}\mu^{-\alpha_{l,m-1}n}\text{-scattered } \mu^{-\alpha_{l,m}n}\text{-admissible pseudotrajectory associated} \\ &\qquad \text{to } \vec{\varepsilon}_{<2k}, \text{ and } \vec{\varepsilon}_{<2k} \in B^{\text{first,st}}_{\mathfrak{Q}}[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}]\big\}. \end{aligned} \tag{114}$$

Since $\{i_0, \ldots, i_{t-1}\}$ are indices of starting points of generalized loops, for each $j$ not among them we have

$$\begin{aligned} &B^{\text{first,st}}_{\mathfrak{Q}}\big[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}, \{\mathbf{r}_0, \ldots, \mathbf{r}_{j-1}\}\big] \\ &\quad = \bigcup_{\mathbf{r}_j \in \boldsymbol{\Pi}_{n_{j+1}}(\mu^{-\alpha_{l,m}n})} B^{\text{first,st}}_{\mathfrak{Q}}\big[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}, \{\mathbf{r}_0, \ldots, \mathbf{r}_j\}\big] \end{aligned} \tag{115}$$

and for each $j \in \{i_0, \ldots, i_{t-1}\}$ we have the corresponding point of pseudotrajectory fixed so

$$\begin{aligned} &B^{\text{first,st}}_{\mathfrak{Q}}\big[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}, \{\mathbf{r}_0, \ldots, \mathbf{r}_{j-1}\}\big] \\ &\quad = B^{\text{first,st}}_{\mathfrak{Q}}\big[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}, \{\mathbf{r}_0, \ldots, \mathbf{r}_j\}\big]. \end{aligned} \tag{116}$$

Let $j$ be the smallest index such that $\mathbf{r}_j$ is not a starting point of a generalized loop. If such $j$ does not exist, i.e. all the loops are long, the sets (109) and (113) coincide. Otherwise inductive application of these formulas gives

$$\begin{aligned} &B^{\text{first,st}}_{\mathfrak{Q}}[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}], \\ &\bigcup_{\mathbf{r}_j \in \boldsymbol{\Pi}_{n_{j+1}}(\mu^{-\alpha_{l,m}n})} B^{\text{first,st}}_{\mathfrak{Q}}\big[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}, \{\mathbf{r}_0, \ldots, \mathbf{r}_j\}\big]. \end{aligned} \tag{117}$$

The first step is to estimate the measure of $B^{\text{first,st}}_{\mathfrak{Q}}[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}, \{\mathbf{r}_0, \ldots, \mathbf{r}_{k-1}\}]$ and then "collect" over admissible $k$-tuples $\{\mathbf{r}_0, \ldots, \mathbf{r}_{k-1}\}$ and get an estimate of the measure of $B^{\text{first,st}}_{\mathfrak{Q}}[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}]$.[9]

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[8] See [22, Section 3.2] for a similar construction in a simpler case.

[9] We shall follow the same strategy as in [22, Sections 3.3–3.4].

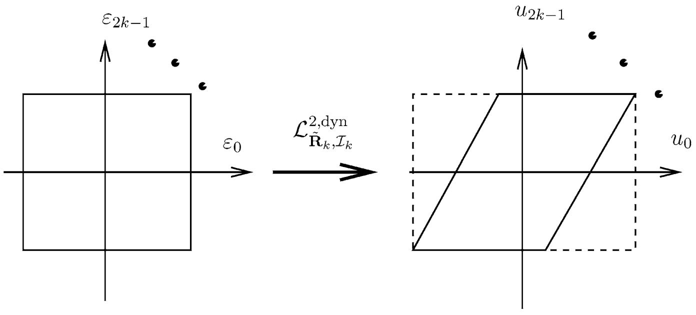


Fig. 8. Distortion by the Newton map.

### 11.4. Distortion Lemma

In this section we formulate the Distortion Lemma for a Newton map $\mathcal{L}^{2,\mathrm{dyn}}_{\tilde{\mathbf{R}}_k,\mathcal{I}_k}$ defined in (100) and in the next four sections complete the proof of the key estimate (108) by collecting all possible scattered admissible "bad" pseudotrajectories defined above (see the Collection Lemma in Section 11.6).

Consider an ordered $k$-tuple of points $\tilde{\mathbf{R}}_k = \{\tilde{\mathbf{r}}_j\}_{j=0}^{k-1} \subset \tilde{U}$. Select the complete set of dynamically essential monomials associated with $\tilde{\mathbf{R}}_k$ (see Definition 28) and the dynamical Newton map $\mathcal{L}^{2,\mathrm{dyn}}_{\tilde{\mathbf{R}}_k,\mathcal{I}_k} : W_{<2k,2} \to W^{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}_{<2k,2}$, defined by (103). By definition the dynamically essential monomials form a subset of basis vectors in $W^{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}_{<2k,2}$. We now estimate the distortion of the Newton map $\mathcal{L}^{2,\mathrm{dyn}}_{\tilde{\mathbf{R}}_k,\mathcal{I}_k}$ as the map from the standard basis $\{\vec{e}_\alpha\}_{|\alpha|<2k}$ in $W_{<2k,2}$ to the dynamical Newton basis $\{\vec{u}_{\alpha,\mathcal{I}_k}\}_{|\alpha|<2k}$ in $W^{u,\mathbf{R}_k,\mathcal{I}_k}_{<2k,2}$. It helps to have in mind the following picture characterizing the distortion of the Newton map (see Fig. 8).

Recall that in Section 11.2 we restrict ourself to the brick of standard thickness with width $\zeta$ as space of parameters

$$HB^{\mathrm{st}}_{<2k}(\zeta) = \left\{\vec{\varepsilon}_{ij} = \left(\varepsilon^1_{ij}, \varepsilon^2_{ij}\right) \in \mathbb{R}^2 \;\middle|\; \left|\varepsilon^q_{ij}\right| \leqslant \frac{\zeta}{(i!j!)^2},\ q = 1, 2,\ 0 \leqslant i, j,\ i + j < 2k\right\},$$

where $\zeta$ is small enough to guarantee that $G_{\vec{\varepsilon}_{<2k}}(\tilde{U}) \supset U$ for all $\vec{\varepsilon}_{<2k} \in HB^{\mathrm{st}}_{<2k} \subset HB_{<2k}(\zeta)$ and the family of perturbations is:

$$G_{\vec{\varepsilon}_{<2k}}(\tilde{x}, \tilde{y}) = G(\tilde{x}, \tilde{y}) + \vec{\Phi}_{\vec{\varepsilon}_{<2k}}(\tilde{x}, \tilde{y}),$$

$$\vec{\Phi}_{\vec{\varepsilon}_{<2k}}(\tilde{x}, \tilde{y}) = \begin{pmatrix} \Phi^1_{\vec{\varepsilon}_{<2k}}(\tilde{x}, \tilde{y}) \\ \Phi^2_{\vec{\varepsilon}_{<2k}}(\tilde{x}, \tilde{y}) \end{pmatrix}, \qquad \Phi^q_{\vec{\varepsilon}_{<2k}}(\tilde{x}, \tilde{y}) = \sum_{0 \leqslant i, j, i+j<2k} \varepsilon^q_{ij} \tilde{x}^i \tilde{y}^j, \quad q = 1, 2. \tag{118}$$

**Distortion Lemma.** *Let* $\tilde{\mathbf{R}}_k = \{\tilde{\mathbf{r}}_0, \ldots, \tilde{\mathbf{r}}_{k-1}\} \subset B^2_\delta$ *be an ordered* $k$*-tuple of points in the* $\delta$*-ball* $B^2_\delta$*, and* $\mathcal{L}^2_{\tilde{\mathbf{R}}_k,\mathcal{I}_k} : W_{<2k,2} \to W^{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}_{<2k,2}$ *be the Newton map, defined by* (100)*. Then the image of the* *brick of standard thickness $HB^{\mathrm{st}}_{<2k}(\zeta)$ with edge $\zeta$ is contained in the brick of standard thickness $HB^{\mathrm{st}}_{<2k}(\zeta/(1-4\delta))$ with edge $\zeta/(1-4\delta)$:*

$$\mathcal{L}^2_{\tilde{\mathbf{R}}_k,\mathcal{I}_k}\big(HB^{\mathrm{st}}_{<2k}(\zeta)\big) \subset HB^{\mathrm{st}}_{<2k}\left(\frac{\zeta}{1-4\delta}\right) \subset W^{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}_{<2k,2}. \tag{119}$$

*In other words, independently of the choice of an $k$-tuple $\{\tilde{\mathbf{r}}_0,\dots,\tilde{\mathbf{r}}_{k-1}\} \subset B^2_\delta$ for any $0 \leqslant i < k$, the coefficients $u^i_{\alpha,\mathcal{I}_k}$ have the range of values bounded by $|u^i_{\alpha,\mathcal{I}_k}| \leqslant \zeta/(1-4\delta)$ in the image $\mathcal{L}^2_{\mathbf{R}_k,\mathcal{I}_k}(HB^{\mathrm{st}}_{<2k}(\zeta))$.*

**Proof.** Recall that for $\{\vec{\varepsilon}_\alpha\}_{|\alpha|<2k} \in HB^{\mathrm{st}}_{<2k}(\zeta)$, therefore, for each of these $\alpha$ and $q = 1,2$ we have that $|\varepsilon^q_\alpha| \leqslant \zeta(\alpha!)^{-2}$. We cannot use arguments of the proof of Distortion Lemma from Section 3.4 [22], because the Newton map is given by an implicit formula (see Lemma 17 and (102)). Note that for the standard Newton basis we do have explicit formulas of Newton map (see (86) and (100) in 1- and 2-dimensional cases, respectively). We shall prove it by backward induction in $|\alpha|$. By Lemma 17 we know that for $|\alpha| = 2k-1$ we have $u^q_\alpha = \varepsilon^q_\alpha$, $q = 1,2$. Therefore, $|u^q_\alpha| \leqslant \zeta(\alpha!)^{-2}$.

We proceed by inductively decreasing $|\alpha|$. Fix $\alpha$ such that $|\alpha| = 2k-2$. Consider identity (99) as polynomial equality. Differentiate it $\alpha_1$ times with respect to $x$ and $\alpha_2$ times with respect to $y$. We get the following identity:

$$\begin{aligned}
&\vec{\varepsilon}_\alpha \alpha_1!\alpha_2! + \vec{\varepsilon}_{\alpha+(1,0)}(\alpha_1+1)!\alpha_2!x + \vec{\varepsilon}_{\alpha+(0,1)}\alpha_1!(\alpha_2+1)!y \\
&\quad \equiv \vec{u}_\alpha \alpha_1!\alpha_2! + \vec{u}_{\alpha+(1,0)}\alpha_1!\alpha_2!\left((\alpha_1+1)x - \sum_{j\in J^x_\alpha} x_j\right) \\
&\qquad + \vec{u}_{\alpha+(0,1)}\alpha_1!\alpha_2!\left((\alpha_2+1)y - \sum_{j\in J^y_\alpha} y_j\right),
\end{aligned} \tag{120}$$

where $J^x_\alpha$ and $J^y_\alpha$ are two sets of indices of $(\alpha_1+1)$ and $(\alpha_2+1)$ elements. We shall not use precise form of these sets of indices. Since $\vec{u}_{\alpha+(1,0)} = \vec{\varepsilon}_{\alpha+(1,0)}$ and $\vec{u}_{\alpha+(0,1)} = \vec{\varepsilon}_{\alpha+(0,1)}$, we could cancel the corresponding terms. All $x_j$'s and $y_j$'s are from the $\delta$-ball. It implies that

$$\begin{aligned}
|u^q_\alpha| &\leqslant |\varepsilon^q_\alpha| + (\alpha_1+1)\delta|u^q_{\alpha+(0,1)}| + (\alpha_2+1)\delta|u^q_{\alpha+(1,0)}| \\
&= |\varepsilon^q_\alpha| + \binom{\alpha_1+1}{1}\delta|u^q_{\alpha+(0,1)}| + \binom{\alpha_2+1}{1}\delta|u^q_{\alpha+(1,0)}|.
\end{aligned}$$

This implies that

$$|u^q_\alpha| \leqslant \frac{\zeta}{(1-4\delta)(\alpha!)^2}, \tag{121}$$

where we use $(1+2\delta) < (1-4\delta)^{-1}$.

Suppose we prove the above estimate for all $\alpha$ such that $|\alpha| > m$. Fix $\alpha$ such that $|\alpha| = m$. Consider identity (99) as the polynomial equality. Differentiate it $\alpha_1$ times with respect to $x$ and

$\alpha_2$ times with respect to $y$. In the identity obtained this way plug in $x = y = 0$ and replace each $x_j$'s and $y_j$'s by its upper bound $\delta$. For $q = 1, 2$ we get the following inequality:

$$\left|u_\alpha^q\right|\alpha_1!\alpha_2! \leqslant \left|\varepsilon_\alpha^q\right|\alpha_1!\alpha_2! + \sum_{\alpha\prec\beta,\,|\beta|<2k} \delta^{|\beta|-|\alpha|}\left|u_\beta^q\right|\binom{\beta_1}{\alpha_1}\alpha_1!\binom{\beta_1}{\alpha_1}\alpha_2!.$$

Applying upper bounds (121) on $u_\beta^q$ we get

$$\left|u_\alpha^q\right| \leqslant \frac{\zeta}{(\alpha!)^2}\left(1 + \sum_{\alpha\prec\beta,\,|\beta|<2k} \frac{\delta^{|\beta|-|\alpha|}}{1-4\delta}\,\frac{\alpha_1!}{\beta_1!(\beta_1-\alpha_1)!}\,\frac{\alpha_2!}{\beta_2!(\beta_2-\alpha_2)!}\right).$$

Write the right-hand side as the power series of $\delta$ and estimates coefficients. Direct verification shows that the right-hand side is bounded by (121). Indeed, for $\alpha \prec \beta$, $|\beta - \alpha| = 1$ the coefficient next to $\delta$ is bounded by

$$\frac{1}{\alpha_1+1} + \frac{1}{\alpha_2+1} \leqslant 4.$$

For $\alpha \prec \beta$, $|\beta - \alpha| = 2$ the coefficient next to $\delta^2$ is bounded by

$$4\left[\frac{1}{\alpha_1+1} + \frac{1}{\alpha_2+1}\right] + \frac{1}{(\alpha_1+1)(\alpha_1+2)} + \frac{1}{(\alpha_1+1)(\alpha_2+1)} + \frac{1}{(\alpha_2+1)(\alpha_2+2)} \leqslant 16$$

and so on. This completes the proof of the lemma. □

For a given $k$-tuple $\tilde{\mathbf{R}}_k = \{\tilde{\mathbf{r}}_j\}_{j=0}^{k-1} \subset B_\delta^2$, the parallelepiped

$$\mathcal{P}^{\mathrm{st}}_{<2k,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\zeta) = \mathcal{L}^{2,\mathrm{dyn}}_{\tilde{\mathbf{R}}_k,\mathcal{I}_k}\left(HB^{\mathrm{st}}_{<2k}(\zeta)\right) \subset W^{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}_{<2k,2} \tag{122}$$

is the set of parameters $\{\vec{u}_{\alpha,\mathcal{I}_k}\}_{|\alpha|<2k}$ that correspond to parameters $\{\vec{\varepsilon}_\alpha\}_{|\alpha|<2k}$ from $HB^{\mathrm{st}}_{<2k}(\zeta)$. In other words, these are the Newton parameters *allowed by the family* (118) for the $k$-tuple $\tilde{\mathbf{R}}_k$. We already knew by Lemma 18 that $\mathcal{P}^{\mathrm{st}}_{<2k,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\zeta)$ has the same volume as $HB^{\mathrm{st}}_{<2k}(\zeta)$, but the Distortion Lemma tells us in addition that the projection of $\mathcal{P}^{\mathrm{st}}_{<2k,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\zeta)$ onto any coordinate axis is at most a factor of $(1-4\delta)^{-1}$ longer than the projection of $HB^{\mathrm{st}}_{<2k}(\zeta)$.

Let $\tilde{\mathbf{R}}_j = \{\tilde{\mathbf{r}}_i\}_{i=0}^{j-1}$ be the $j$-tuple of first $j$ points of the $k$-tuple $\tilde{\mathbf{R}}_k$. We now consider which Newton parameters are allowed by the family (118) when $\tilde{\mathbf{R}}_j$ is fixed but $\tilde{\mathbf{r}}_j, \ldots, \tilde{\mathbf{r}}_{k-1}$ are such that $\tilde{\mathbf{r}}_j \in \boldsymbol{\Pi}_{n_{j+1}}(\mu^{-\alpha_{l,m}n})$. Since we will only be using the definitions below for admissible discretized $k$-tuples $\tilde{\mathbf{R}}_k \subset \tilde{\boldsymbol{\Pi}}_{\mathcal{N}_k}(\mu^{-\alpha_{l,m}n})$, we consider only the (finite number of) possibilities $\tilde{\mathbf{r}}_j, \ldots, \tilde{\mathbf{r}}_{k-1} \subset \tilde{\boldsymbol{\Pi}}_{\mathcal{N}_k}(\mu^{-\alpha_{l,m}n})$. Let

$$\pi^{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}_{<2k\leqslant j} : W^{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}_{<2k,2} \to W^{u,\tilde{\mathbf{R}}_j,\mathcal{I}_j}_{\leqslant j,2} \quad \text{and} \quad \pi^{u,\tilde{\mathbf{R}}_j,\mathcal{I}_j}_{<2k,j} : W^{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}_{<2k,2} \to W^{u,\tilde{\mathbf{R}}_j,\mathcal{I}_j}_{j,2}$$

be the natural projections onto the space $W^{u,\tilde{\mathbf{R}}_j,\mathcal{I}_j}_{\leqslant j,2}$ of 2-component polynomials of degree $j$ and the space $W^{u,\tilde{\mathbf{R}}_j,\mathcal{I}_j}_{j,2}$ of 2-component homogeneous polynomials of degree $j$ respectively. Denote the unions over all $\tilde{\mathbf{r}}_j,\dots,\tilde{\mathbf{r}}_{k-1}\subset\tilde{\boldsymbol{\Pi}}_{\mathcal{N}_k}(\mu^{-\alpha_{l,m}n})$ of the images of $\mathcal{P}^{\mathrm{st}}_{<2k,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\zeta)$ under the respective projections $\pi^{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}_{<2k,\leqslant j}$ and $\pi^{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}_{<2k,j}$ by

$$\mathcal{P}^{\mathrm{st}}_{<2k,\leqslant j,\tilde{\mathbf{R}}_j,\mathcal{I}_j}(\zeta)=\bigcup_{\tilde{\mathbf{r}}_j,\dots,\tilde{\mathbf{r}}_{k-1}\subset\tilde{\boldsymbol{\Pi}}_{\mathcal{N}_k}(\mu^{-\alpha_{l,m}n})}\pi^{u,\tilde{\mathbf{R}}_k}_{<2k,\leqslant j}\big(\mathcal{P}^{\mathrm{st}}_{<2k,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\zeta)\big)\subset W^{u,\tilde{\mathbf{R}}_j,\mathcal{I}_j}_{\leqslant j,2},$$

$$\mathcal{P}^{\mathrm{st}}_{<2k,j,\tilde{\mathbf{R}}_j,\mathcal{I}_j}(\zeta)=\bigcup_{\tilde{\mathbf{r}}_j,\dots,\tilde{\mathbf{r}}_{k-1}\subset\tilde{\boldsymbol{\Pi}}_{\mathcal{N}_k}(\mu^{-\alpha_{l,m}n})}\pi^{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}_{<2k,j}\big(\mathcal{P}^{\mathrm{st}}_{<2k,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\zeta)\big)\subset W^{u,\tilde{\mathbf{R}}_j,\mathcal{I}_j}_{j,2}. \tag{123}$$

### *11.5. Probability estimates of an elementary event*

Consider a set of starting points of loops $\mathbf{R}_k=\{\mathbf{r}_0,\dots,\mathbf{r}_{k-1}\}\subset U$ of type $\mathcal{N}_k=(n_1,\dots,n_k)$, shape $(l,n)$, and scale number $m$, where $\mathbf{r}_i\in\boldsymbol{\Pi}_{n_{i+1}}(\mu^{-\alpha_{l,m}n})$. Let $\tilde{\mathbf{R}}_k=\{\tilde{\mathbf{r}}_0,\dots,\tilde{\mathbf{r}}_{k-1}\}\subset\tilde{U}$ be a set of ending points of the corresponding loops, $\mathcal{I}_k=\{\alpha(\Gamma)\}_{|\alpha|<2k}$ be a set of equipped multiindices. It defines the family of perturbations by Newton polynomials associated to the $k$-tuple $\mathbf{R}_k$ and equipped multiindices $\mathcal{I}_k$

$$\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(x)=\tilde{f}(x)+\sum_{|\alpha|<2k}\vec{u}^{\mathrm{dyn}}_{\alpha,\mathcal{I}_k}(p;p_0,\dots,p_{|\alpha|-1\ (\mathrm{mod}\ k)})^{\alpha(\Gamma)}, \tag{124}$$

where the Newton monomial $(p;p_0,\dots,p_{|\alpha|-1\ (\mathrm{mod}\ k)})^{\alpha(\Gamma)}$ is defined in (97) and

$$\vec{u}^{\mathrm{dyn}}_{\mathcal{I}_k}=\big\{\vec{u}^{\mathrm{dyn}}_{\alpha,\mathcal{I}_k}\big\}_{|\alpha|<2k}\in\mathcal{P}^{\mathrm{st}}_{<2k,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\zeta).$$

We omit sub-subindex $\mathcal{I}_k$ for brevity. We choose equipments of multiindices so that all dynamically essential monomials (95)–(96) are present. It is possible by Lemma 17.

Among the dynamical Newton monomials we have a set of dynamically essential monomials $\{Q^{\mathrm{dyn},q}(p,\tilde{\mathbf{R}}_j),\ Q^{\mathrm{dyn},q}_m(p,\tilde{\mathbf{R}}_j,\tilde{\mathbf{R}}'_j)\}$, $j=0,\dots,k-1$, $m=1,2$, and $q=1,2$ is an index of vector component. Dynamically essential monomials have the following degrees:

$$\deg\big(Q^{\mathrm{dyn},q}(p,\tilde{\mathbf{R}}_j)\big)=j,\qquad\deg\big(Q^{\mathrm{dyn},q}_m(p,\tilde{\mathbf{R}}_j,\tilde{\mathbf{R}}'_j)\big)=k+j,$$

and satisfy the following inequalities:

$$\big|Q^{\mathrm{dyn},q}(\tilde{\mathbf{r}}_j,\tilde{\mathbf{R}}_j)\big|\geqslant 2^{-j/2}\prod_{i=0}^{j-1}|\tilde{\mathbf{r}}_j-\tilde{\mathbf{r}}_i|,$$

$$\left|\frac{\partial}{\partial x}Q^{\mathrm{dyn},q}_1(\tilde{\mathbf{r}}_j,\tilde{\mathbf{R}}_j,\tilde{\mathbf{R}}'_j)\right|\geqslant 2^{-(k+j-1)/2}\prod_{i=0}^{j-1}|\tilde{\mathbf{r}}_j-\tilde{\mathbf{r}}_i|^2\prod_{i=j+1}^{k-1}|\tilde{\mathbf{r}}_j-\tilde{\mathbf{r}}_i|,$$

$$\left|\frac{\partial}{\partial y} Q_2^{\mathrm{dyn},q}(\tilde{\mathbf{r}}_j, \tilde{\mathbf{R}}_j, \tilde{\mathbf{R}}'_j)\right| \geqslant 2^{-(k+j-1)/2} \prod_{i=0}^{j-1} |\tilde{\mathbf{r}}_j - \tilde{\mathbf{r}}_i|^2 \prod_{i=j+1}^{k-1} |\tilde{\mathbf{r}}_j - \tilde{\mathbf{r}}_i|. \tag{125}$$

Recall that by Distortion Lemma the set of Newton parameters $\mathcal{P}^{\mathrm{st}}_{<2k,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\zeta)$ allowed by the family (118) is contained in the cube $HB^{\mathrm{st}}_{<2k}(\zeta/(1-4\delta))$ (see (122) for definition). Therefore, it suffices to consider $|u_\alpha^{\mathrm{dyn},q}| < \zeta/(1-4\delta)$ for each $q = 1, 2$ and $|\alpha| < 2k$. We denote $\vec{u}^{\mathrm{dyn}}_{j,\mathcal{I}_k} \in HB^{\mathrm{st}}_q(\zeta/(1-4\delta))$ if $|u^{\mathrm{dyn},q}_{\alpha,\mathcal{I}_k}| < \zeta/(1-4\delta)$ for all $|\alpha| = j$ and $q = 1, 2$, where $j = 0, \ldots, 2k-1$. Define $\nu^{\mathrm{st}}_j = \bigtimes_{|\alpha|=j} (\nu^{\mathrm{st},1}_\alpha \times \nu^{\mathrm{st},2}_\alpha)$.

Due to the choice of Newton polynomials the image $\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_0)$ is independent of $\vec{u}^{\mathrm{dyn}}_{\alpha,\mathcal{I}_k} = (u^{\mathrm{dyn},1}_{\alpha,\mathcal{I}_k}, u^{\mathrm{dyn},2}_{\alpha,\mathcal{I}_k})$ for all $|\alpha| > 0$. Therefore the position of $\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_0)$ *depends only on* $\vec{u}^{\mathrm{dyn}}_{0,\mathcal{I}_k} = (u^{\mathrm{dyn},1}_{0,\mathcal{I}_k}, u^{\mathrm{dyn},2}_{0,\mathcal{I}_k})$. Recall that $\nu^1_{00}$ and $\nu^2_{00}$ are 1-dimensional Lebesgue measures scaled by $1/(2\zeta)$. This gives

$$\begin{aligned}
&\nu^{\mathrm{st}}_0 \left\{ \vec{u}^{\mathrm{dyn}}_{0,\mathcal{I}_k} \in HB^{\mathrm{st}}_0\left(\frac{\zeta}{1-4\delta}\right) \,\Big|\, \left|\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_0) - \mathbf{r}_1\right| \leqslant \gamma \right\} \\
&\quad \leqslant \left(\frac{(0!)^2}{2\zeta} 2\gamma\right)^2 = \left(\frac{(0!)^2 \gamma}{\zeta}\right)^2.
\end{aligned} \tag{126}$$

Denote the right-hand side by $\mathrm{Prob}_0(\gamma, \zeta)$. To fit notations below we put $\mathrm{Prob}_0(\gamma, \zeta) = \mathrm{Prob}_0(\tilde{\mathbf{R}}_1, \gamma, \zeta, \delta)$.

Fix $\vec{u}^{\mathrm{dyn}}_{0,\mathcal{I}_k} = (u^{\mathrm{dyn},1}_{0,\mathcal{I}_k}, u^{\mathrm{dyn},2}_{0,\mathcal{I}_k})$. Similarly, the position of $\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_1)$ *depends only on* $\vec{u}^{\mathrm{dyn}}_{\alpha,\mathcal{I}_k} = (u^{\mathrm{dyn},1}_{\alpha,\mathcal{I}_k}, u^{\mathrm{dyn},2}_{\alpha,\mathcal{I}_k})$ *for* $|\alpha| = 1$, i.e. $\alpha$ is either $(0, 1)$ or $(1, 0)$. Call $\vec{u}^{\mathrm{dyn}}_{1,\mathcal{I}_k} = \{\vec{u}^{\mathrm{dyn}}_{\alpha,\mathcal{I}_k}\}_{|\alpha|=1}$.

Thus, using the dynamically essential monomials and Distortion Lemma, we get

$$\begin{aligned}
&\nu^{\mathrm{st}}_1 \left\{ \vec{u}^{\mathrm{dyn}}_{1,\mathcal{I}_k} \in HB^{\mathrm{st}}_1\left(\frac{\zeta}{1-4\delta}\right) \,\Big|\, \left|\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_1) - \mathbf{r}_2\right| \leqslant \gamma \right\} \\
&\quad \leqslant \frac{(2 \cdot (1!)^2 \gamma)^2}{(2\zeta(1-4\delta))^2 |Q^{\mathrm{dyn}}_1(\tilde{\mathbf{r}}_1, \tilde{\mathbf{R}}_1) \cdot Q^{\mathrm{dyn}}_2(\tilde{\mathbf{r}}_1, \tilde{\mathbf{R}}_1)|} \leqslant 2\left(\frac{(1!)^2\gamma}{\zeta(1-4\delta)}\right)^2 |\tilde{\mathbf{r}}_1 - \tilde{\mathbf{r}}_0|^{-2}.
\end{aligned} \tag{127}$$

Denote the right-hand side by $\mathrm{Prob}_1(\tilde{\mathbf{R}}_2, \gamma, \zeta, \delta)$.

Inductively for $j = 2, \ldots, k-1$, fix $\vec{u}_{\alpha,\mathcal{I}_k} = (u^{\mathrm{dyn},1}_{\alpha,\mathcal{I}_k}, u^{\mathrm{dyn},2}_{\alpha,\mathcal{I}_k})$ for $|\alpha| < j$. Then the position of $\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_j)$ *depends only on* $\vec{u}^{\mathrm{dyn}}_j = \{\vec{u}^{\mathrm{dyn}}_{\alpha,\mathcal{I}_k}\}_{|\alpha|=j}$. Moreover, for $j = 2, \ldots, k-1$ we have

$$\begin{aligned}
&\nu^{\mathrm{st}}_j \left\{ \vec{u}^{\mathrm{dyn}}_{j,\mathcal{I}_k} \in HB^{\mathrm{st}}_j\left(\frac{\zeta}{1-4\delta}\right) \,\Big|\, \left|\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_j) - \mathbf{r}_{j+1 \ (\mathrm{mod}\ k)}\right| \leqslant \gamma \right\} \\
&\quad \leqslant \frac{(j!)^4 \gamma^2}{\zeta^2 (1-4\delta)^{2j}} \left| Q^{\mathrm{dyn},1}(\tilde{\mathbf{r}}_j, \tilde{\mathbf{R}}_j) \cdot Q^{\mathrm{dyn},2}(\tilde{\mathbf{r}}_j, \tilde{\mathbf{R}}_j)\right|^{-1} \\
&\quad \leqslant \frac{2^j (j!)^4 \gamma^2}{\zeta^2 (1-4\delta)^{2j}} \prod_{i=0}^{j-1} |\tilde{\mathbf{r}}_j - \tilde{\mathbf{r}}_i|^{-2}.
\end{aligned} \tag{128}$$

Denote the right-hand side by $\mathrm{Prob}_j(\tilde{\mathbf{R}}_{j+1}, \gamma, \zeta, \delta)$.

Now consider a cone $K = K(\tilde{\mathbf{r}}_{i_1-1})$, $\angle K < \pi/2$. Recall that $\mathbf{r}_{i_1} = R_1$ is the starting point of the second generalized loop. We want to estimate the measure of parameters, for which the image of this cone under $D\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_{i_1-1})$ does not intersect the cone $K^*_{\theta_{l,m}n}(\mathbf{r}_{i_1}) = \overline{\mathbb{R}^2 \setminus K_{\theta_{l,m}n}(\mathbf{r}_{i_1})}$. As before we identify tangent spaces at points $\mathbf{r}_{i_1}$ and $\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_{i_1-1})$ with $\mathbb{R}^2$. Note that $\vec{u}^{\mathrm{dyn}}_{\alpha,\mathcal{I}_k}$ with $|\alpha| > k+i_1-1$ does not affect to the image of the cone $K$. Fix parameters $\{\vec{u}^{\mathrm{dyn}}_{\alpha,\mathcal{I}_k}\}_{|\alpha|<k+i_1-1}$. Consider $\vec{u}^{\mathrm{dyn}}_{k+i_1-1,\mathcal{I}_k} = \{\vec{u}^{\mathrm{dyn}}_{\alpha,\mathcal{I}_k}\}_{|\alpha|=k+i_1-1}$. We shall prove in Section 11.8 that

$$\begin{aligned}
&\nu^{\mathrm{st}}_{k+i_1-1}\left\{\vec{u}^{\mathrm{dyn}}_{k+i_1-1,\mathcal{I}_k} \in HB^{\mathrm{st}}_{k+i_1-1}\left(\frac{\zeta}{1-4\delta}\right) \,\middle|\, D\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k} K(\tilde{\mathbf{r}}_{i_1-1}) \cap K^*_{\theta_{l,m}n}(\mathbf{r}_{i_1 \pmod k}) \neq \{0\}\right\} \\
&\quad\leqslant \frac{(1-4\delta)^{-2(k+i_1-1)}}{\zeta}\big((k+i_1-1)!\big)^4 2^{(k+i_1-1)/2}\big(2M_1^3\angle K + M_1\mu^{-\theta_{l,m}n}\big) \\
&\qquad\times \prod_{i=0}^{i_1-2} |\tilde{\mathbf{r}}_{i_1-1} - \tilde{\mathbf{r}}_i|^{-2} \prod_{i=i_1}^{k-1} |\tilde{\mathbf{r}}_{i_1-1} - \tilde{\mathbf{r}}_i|^{-1}.
\end{aligned} \tag{129}$$

Denote the right-hand side by Prob-cone$(\tilde{\mathbf{R}}_k, i_1, n, M_1, \zeta, \delta, \theta_{l,m}, \angle K)$.

Combining estimates (126)–(129) along with Distortion Lemma[10] we get

$$\begin{aligned}
&\nu^{\mathrm{st}}_{<2k}\big\{B^{\mathrm{first,st}}_{\Omega}\big[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}, \{\mathbf{r}_0, \dots, \mathbf{r}_{k-1}\}\big]\big\} \\
&\quad\leqslant (1-4\delta)^{-3k^2}\ \text{Prob-cone}\big(\tilde{\mathbf{R}}_k, i_1, n, M_1, \zeta, \delta, \theta_{l,m}, \angle K_{\max}\big) \\
&\qquad\times \prod_{j=0}^{k-1} \mathrm{Prob}_j\big(\tilde{\mathbf{R}}_{j+1}, (3+2M_1)\mu^{-\alpha_{l,m}n}, \zeta, \delta\big).
\end{aligned} \tag{130}$$

We derive an upper estimate on $\angle K_{\max}$ in the case under consideration later. One could just keep in mind that it is of order $\mu^{-(\theta_{l,m}+h_l)n}$ and is exponentially small in $n$ (see (156)).

### *11.6. Collection Lemma*

For each $j < k$, the sets $\mathcal{P}^{\mathrm{st}}_{<2k,\leqslant j,\tilde{\mathbf{R}}_j,\mathcal{I}_j}(\zeta)$ and $\mathcal{P}^{\mathrm{st}}_{<2k,j,\tilde{\mathbf{R}}_j,\mathcal{I}_j}(\zeta)$ are polyhedrons. Both depend only on the $j$-tuple $\tilde{\mathbf{R}}_j$, the set $\mathcal{I}_j$ of equipped multiindices $|\alpha| \leqslant j$, and width $\zeta$ of the cube $HB^{\mathrm{st}}_{<2k}(\zeta)$. The set $\mathcal{P}^{\mathrm{st}}_{<2k,\leqslant j,\tilde{\mathbf{R}}_j}(\zeta)$ consists of all Newton parameters $\{\vec{u}_{\alpha,\mathcal{I}_k}\}_{|\alpha|\leqslant j} \in W^{u,\tilde{\mathbf{R}}_j,\mathcal{I}_j}_{j,2}$ that are allowed by the family (118) for the $j$-tuple $\mathbf{R}_j$.

For each $j < k$, we introduce the family of diffeomorphisms

$$\tilde{f}_{\vec{u}^{\mathrm{dyn}}_{\mathcal{I}_k}(j),\tilde{\mathbf{R}}_j,\mathcal{I}_j}(x) = \tilde{f}(x) + \sum_{|\alpha|\leqslant j} \vec{u}^{\mathrm{dyn}}_{\alpha,\mathcal{I}_j}(p; p_0, \dots, p_{|\alpha|-1})^{\alpha(\Gamma)}, \tag{131}$$

[10] Cf. with [22, Section 3.3].

where the Newton monomial $(p; p_0, \dots, p_{|\alpha|-1})^{\alpha(\Gamma)}$ is defined in (97), equipment $\Gamma$ of each multiindex $\alpha$ is given by the set $\mathcal{I}_j$, and $\vec{u}^{\,\mathrm{dyn}}_{\mathcal{I}_k}(j) = \{u^{\mathrm{dyn}}_{\alpha,\mathcal{I}_j}\}_{|\alpha|\leqslant j} \in \mathcal{P}^{\mathrm{st}}_{<2k,\leqslant j,\tilde{\mathbf{R}}_j,\mathcal{I}_j}(\zeta)$ (cf. (124)). For each possible continuation $\tilde{\mathbf{R}}_k$ of $\tilde{\mathbf{R}}_j$, the family $\tilde{f}_{\vec{u}^{\mathrm{dyn}}(j),\tilde{\mathbf{R}}_j,\mathcal{I}_j}$ includes the subfamily of $\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}$ (with $\vec{u}^{\,\mathrm{dyn}}_{\mathcal{I}_k} \in \mathcal{P}^{\mathrm{st}}_{<2k,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\zeta)$) corresponding to $\vec{u}^{\,\mathrm{dyn}}_{\alpha,\mathcal{I}_k} = 0$ for all $|\alpha| > j$. However, the action of $\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}$ on $\tilde{\mathbf{r}}_0, \dots, \tilde{\mathbf{r}}_{j-1}$ does not depend on $\tilde{\mathbf{r}}_j, \dots, \tilde{\mathbf{r}}_k$, so for these points the family $\tilde{f}_{\vec{u}^{\mathrm{dyn}}(j),\mathbf{R}_j,\mathcal{I}_j}$ is representative of the entire family $\tilde{f}_{\vec{u}^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}$. This motivates the definition

$$
\begin{aligned}
&T^{2,\gamma}_{<2k,\leqslant j}(\tilde{f}; \mathbf{r}_0, \dots, \mathbf{r}_{j-1}, \mathbf{r}_j, \mathbf{r}_{j+1}) \\
&\quad = \Big\{ \vec{u}^{\,\mathrm{dyn}}_{\mathcal{I}_k}(j) \in \mathcal{P}_{<2k,\leqslant j,\tilde{\mathbf{R}}_j,\mathcal{I}_j}(\zeta) \subset W^{u,\tilde{\mathbf{R}}_j,\mathcal{I}_j}_{\leqslant j,2} \ \Big|\ \big| \tilde{f}_{\vec{u}^{\mathrm{dyn}}(j),\tilde{\mathbf{R}}_j,\mathcal{I}_j}(\tilde{\mathbf{r}}_{i-1}) - \mathbf{r}_i \big| \leqslant \gamma \\
&\qquad\qquad \text{for } i = 1, \dots, j+1 \Big\}.
\end{aligned}
\tag{132}
$$

In a view of the family of perturbations (118) and the form of the map (7) notice that knowing type $\mathcal{N}_k$ of a $k$-tuple $\tilde{\mathbf{R}}_k$ one can reconstruct the corresponding $k$-tuple $\mathbf{R}_k$ by applying inverse of the linear map $L$ the prescribed number of times. It provides natural identification of $\mathbf{R}_k$ and $\tilde{\mathbf{R}}_k$.

$T^{2,\gamma}_{<2k,\leqslant j}(\tilde{f}; \mathbf{r}_0, \dots, \mathbf{r}_j, \mathbf{r}_{j+1})$ represents the set of Newton parameters $\vec{u}^{\,\mathrm{dyn}}_{\mathcal{I}_k}(j) = \{\vec{u}^{\,\mathrm{dyn}}_{\alpha,\mathcal{I}_k}\}_{|\alpha|\leqslant j}$ allowed by the family (118) for which $\mathbf{r}_0, \dots, \mathbf{r}_j$ is a $\gamma$-pseudotrajectory of $\tilde{f}_{\vec{u}^{\mathrm{dyn}}(j),\tilde{\mathbf{R}}_j,\mathcal{I}_j}$ (and hence of $\tilde{f}_{\vec{u}^{\mathrm{dyn}}(j),\tilde{\mathbf{R}}_j,\mathcal{I}_j}$ for all valid extensions $\vec{u}^{\,\mathrm{dyn}}_{\mathcal{I}_k}$ and $\tilde{\mathbf{R}}_k$ of $\vec{u}^{\,\mathrm{dyn}}_{\mathcal{I}_k}(j)$ and $\tilde{\mathbf{R}}_j$).

In the following lemma, we collect all possible $(2s)^{-1}\mu^{-\alpha_{l,m-1}n}$-scattered $\mu^{-\alpha_{l,m}n}$-pseudotrajectories and estimates of "bad" measure corresponding to those $\mu^{-\alpha_{l,m}n}$-pseudotrajectories. Denote $\mathcal{R}_j = (R_0, \dots, R_{j-1})$ the first $j$ starting points of generalized loops and define indices $\{i_0, \dots, i_{\tau-1}\}$ as follows $R_0 = \mathbf{r}_{i_0}, \dots, R_{t-1} = \mathbf{r}_{i_{\tau-1}}$. According to our notations $\mathcal{R}_\tau = \mathcal{R}$.

**Collection Lemma.** *With the notations above, for all $\tilde{\mathbf{r}}_0 \in \Pi_{n_1}(\mu^{-\alpha_{l,m}n})$ the measure of the "bad" parameters satisfies*

$$
\begin{aligned}
&\nu^{\mathrm{st}}_{<2k}\big\{B^{\mathrm{first,st}}_{\mathfrak{Q}}[k, f, \mathcal{N}_k, n, l, m, \mu^{-\alpha_{l,m}n}; \mathcal{R}]\big\} \\
&\quad \leqslant (1-4\delta)^{-2k(2k-1)+2\sum_{j=1}^{t} i_j} \,\text{Prob-cone}\big(\tilde{\mathbf{R}}_k, i_1, n, M_1, \zeta, \delta, \theta_{l,m}, 4M_1^{2(s-1)}\mu^{-(\theta_{l,m}+h_l)n}\big) \\
&\qquad \times \prod_{j=1}^{\tau} \mathrm{Prob}_{i_j-1}\big(\tilde{\mathbf{R}}_{i_j}, (3+2M_1)\mu^{-\alpha_{l,m}n}, \zeta, \delta\big) \\
&\quad \leqslant \big(\mu^{-2\alpha_{l,m}n}\big)^{\tau} \tilde{C}_s \mu^{-h_l n},
\end{aligned}
\tag{133}
$$

*where $\tilde{C}_s$ is some explicitly commutable constant* (144) *and constants $\alpha_{l,m}, \theta_{l,m}, h_l$ are defined in* (51).

**Proof.** Denote $\mu^{-\alpha_{l,m}n}$ by $\gamma_n$ for brevity in the proof below. We prove by backward induction on $j$ that for $\mathbf{r}_0 \in \boldsymbol{\Pi}_{n_1}(\mu^{-\alpha_{l,m}n}), \dots, \mathbf{r}_j \in \boldsymbol{\Pi}_{n_{j+1}}(\mu^{-\alpha_{l,m}n})$ such that $\mathbf{r}_j$ is not a starting point of a generalized loop,

$$\nu^{\mathrm{st}}_{<2k}\big\{B^{\mathrm{first,st}}_{\mathfrak{Q}}\big[k,f,\mathcal{N}_k,n,l,m;\mathcal{R}\subseteq\{\mathbf{r}_0,\dots,\mathbf{r}_j\}\cup\mathcal{R}\big]\big\}$$
$$\leqslant(1-4\delta)^{-(2k-j)(2k+j+1)+\sum_{l=1}^{t_j}i_l+k+i_1}\ \text{Prob-cone}\big(\tilde{\mathbf{R}}_k,i_1,n,M_1,\zeta,\delta,\theta_{l,m},\angle K\big)$$
$$\times\prod_{l=t_j+1}^{\tau}\text{Prob}_{i_l-1}\big(\tilde{\mathbf{R}}_{i_l},(3+2M_1)\gamma_n,\zeta,\delta\big)\ \nu_{<j}\big\{T^{2,\gamma_n}_{<2k,<j}(\tilde f;\mathbf{r}_0,\dots,\mathbf{r}_j)\big\}\mu^{-2\alpha_{l,m}n},\qquad(134)$$

where $t_j$ is the number of starting points of generalized loops among first $j$ points in $\mathbf{R}_j$. In the case $\mathbf{r}_j$ is a starting point of a generalized loop we have equality (116). Even though the measure stays unchanged the above formula changes as indices in terms depending on $j$ change. This estimate results in the first inequality in (133) for $j=0$.

Consider the case $j=k-1$. Fix a $k$-loop $(2s)^{-1}\mu^{-\alpha_{l,m-1}n}$-scattered (non-recurrent) $\gamma_n$-admissible pseudotrajectory $\{r_0,\dots,r_{\mathbf{n}-1}\}$ of type $\mathcal{N}_k$ having shape $(l,n)$ whose starting points of generalized loops $\mathcal{R}=\{R_0,\dots,R_{\tau-1}\}\subset\boldsymbol{\Pi}_{\mathcal{N}_k}(\gamma_n)\cap\{r_0,\dots,r_{\mathbf{n}-1}\}$ are fixed. Recall that we denote by $\mathbf{R}_k=\{\tilde{\mathbf{r}}_0,\dots,\tilde{\mathbf{r}}_{k-1}\}$ intersection of the pseudotrajectory with $\tilde U$ and by $\tilde{\mathcal{R}}=\{\tilde R_1,\dots,\tilde R_{\tau-1}\}$ the preimages of the above starting points. Notice that $\tilde{\mathbf{R}}_k$ is not uniquely determine by the fixed $\mathbf{R}_k$ and depends on $\vec\varepsilon_{<2k}$. Using formulas (128) and (129), we have

$$\nu^{\mathrm{st,dyn}}_{k-1}\left\{\vec u^{\,\mathrm{dyn}}_{k-1,\mathcal{I}_k}\in HB^{\mathrm{st}}_{k-1}\left(\frac{\zeta}{1-4\delta}\right)\ \Big|\ \big|\tilde f_{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_{k-1})-\mathbf{r}_0\big|\leqslant(3+2M_1)\gamma_n\right\}$$
$$\leqslant\text{Prob}_{k-1}\big(\tilde{\mathbf{R}}_k,(3+2M_1)\gamma_n,\zeta,\delta\big)\qquad(135)$$

and

$$\nu^{\mathrm{st,dyn}}_{k+i_1-1}\left\{\vec u^{\,\mathrm{st,dyn}}_{k+i_1-1,\mathcal{I}_k}\in HB^{\mathrm{st}}_{k+i_1-1}\left(\frac{\zeta}{1-4\delta}\right)\ \Big|\ D\tilde f_{\vec u^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}K(\tilde{\mathbf{r}}_{i_1-1})\cap K^*_{\theta_{l,m}n}(\mathbf{r}_{i_1\ (\mathrm{mod}\ k)})\neq\{0\}\right\}$$
$$\leqslant\text{Prob-cone}(\tilde{\mathbf{R}}_k,i_1,n,M_1,\zeta,\delta,\theta_{l,m},\angle K_{\max}).\qquad(136)$$

We omit $HB^{\mathrm{st}}_{j\ (\text{respectively }k+i_1-1)}(\zeta/(1-4\delta))$ in corresponding estimates for brevity. The Fubini Theorem, Lemma 18, and definition (16) of the product measure $\nu^{\mathrm{st}}_{<2k}$ imply that

$$\nu^{\mathrm{st,dyn}}_{<2k}\big\{B^{\mathrm{first,st}}_{\mathfrak{Q}}\big[k,f,\mathcal{N}_k,n,l,m;\mathcal{R}\subseteq\{\mathbf{r}_0,\dots,\mathbf{r}_{k-1}\}\cup\mathcal{R}\big]\big\}$$
$$\leqslant\nu^{\mathrm{st,dyn}}_{k-1}\big\{\vec u^{\,\mathrm{st,dyn}}_{k-1,\mathcal{I}_k}\ \big|\ \big|\tilde f_{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_{k-1})-\mathbf{r}_0\big|\leqslant(3+2M_1)\gamma_n\big\}$$
$$\times\prod_{k\leqslant j\leqslant 2k-1,\ j\neq k+i_1-1}\nu^{\mathrm{st}}_j\big\{\mathcal{P}_{<2k,j,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\zeta)\big\}\nu^{\mathrm{st,dyn}}_{<k-1}\left\{T^{2,(3+2M_1)\gamma_n}_{<2k,<k-1}(\tilde f;\mathbf{r}_0,\dots,\mathbf{r}_{k-1})\right\}$$
$$\times\nu^{\mathrm{st,dyn}}_{k+i_1-1}\left\{\vec u^{\,\mathrm{dyn}}_{k+i_1-1,\mathcal{I}_k}\ \big|\ D\tilde f_{\vec u^{\mathrm{dyn}},\tilde{\mathbf{R}}_k,\mathcal{I}_k}K(\tilde{\mathbf{r}}_{i_1-1})\cap K^*_{\theta_{l,m}n}(\mathbf{r}_{i_1\ (\mathrm{mod}\ k)})\neq\{0\}\right\}$$
$$\leqslant\nu^{\mathrm{st,dyn}}_{<k-1}\left\{T^{2,(3+2M_1)\gamma_n}_{<2k,<k-1}(\tilde f;\mathbf{r}_0,\dots,\mathbf{r}_{k-1})\right\}\text{Prob}_{k-1}\big(\tilde{\mathbf{R}}_k,(3+2M_1)\gamma_n,\zeta,\delta\big)$$
$$\times(1-4\delta)^{-(3k+1)k+k+i_1}\text{Prob-cone}(\tilde{\mathbf{R}}_k,i_1,n,M_1,\zeta,\delta,\theta_{l,m},\angle K_{\max}),\qquad(137)$$

where $\angle K_{\max}$ is the maximal possible angle of $F^{N_1-1}_{\mathcal{R},\vec u_{<2k}}(K_{\xi_l n}(R_0))$ estimated in (156) from above by $4M_1^{2(s-1)}\mu^{-(\theta_{l,m}+h_l)n}$.

The last inequality follows from the Distortion Lemma, which says that for each $\alpha$ such that $|\alpha| \leqslant 2k-2$ we have

$$\nu_j^{\mathrm{st,dyn}}\big\{\mathcal{P}^{\mathrm{st}}_{<2k,j,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\zeta)\big\} \leqslant (1-4\delta)^{-2(j+1)}. \tag{138}$$

This yields the required estimate (134) for $j = k-1$.

Suppose now that for $j+1$, (134) is true and we would like to prove it for $j$. There are two different cases: either $\mathbf{r}_{j+1}$ is not a starting point of a generalized loop or it is. Estimates in both cases follow the same strategy of implicit collection. Consider the first case. Denote by $G^{2,(3+2M_1)\gamma_n}_{<2k,j}(\tilde{f}, \vec{u}^{\mathrm{dyn}}_{\mathcal{I}_k}(j-1); \mathbf{r}_0, \dots, \mathbf{r}_j) \subset \boldsymbol{\Pi}_{n_{j+2}}(\gamma_n)$ the set of points $\mathbf{r}_{j+1}$ of the grid $\boldsymbol{\Pi}_{n_{j+2}}(\gamma_n)$ such that the $(j+2)$-tuple $\mathbf{r}_0, \dots, \mathbf{r}_{j+1}$ is a $\gamma_n$-admissible pseudotrajectory associated to some extension $\vec{u}^{\mathrm{dyn}}_{\mathcal{I}_k}(j) \in \mathcal{P}^{\mathrm{st}}_{<2k,\leqslant j,\tilde{\mathbf{R}}_j}(\zeta)$ of $\vec{u}^{\mathrm{dyn}}_{\mathcal{I}_k}(j-1)$. In other words, $G^{2,(3+2M_1)\gamma_n}_{<2k,j}(\tilde{f}, \vec{u}^{\mathrm{dyn}}_{\mathcal{I}_k}(j-1); \mathbf{r}_0, \dots, \mathbf{r}_j)$ is the set of all possible continuations of the $\gamma_n$-admissible pseudotrajectory $\mathbf{r}_0, \dots, \mathbf{r}_j$ using all possible Newton parameters $\vec{u}^{\mathrm{dyn}}_{\mathcal{I}_k}(j)$ allowed by the family (118).

Now let $\mathbf{r}_0, \dots, \mathbf{r}_j$ be a $\gamma_n$-admissible pseudotrajectory associated to $\vec{u}^{\mathrm{dyn}}_{\mathcal{I}_k}(j) = \{\vec{u}^{\mathrm{dyn}}_{\alpha,\mathcal{I}_k}\}_{|\alpha|\leqslant j}$, then at most 4 points $\mathbf{r}_{j+1} \in \boldsymbol{\Pi}_{n_{j+2}}(\gamma_n)$ are within $\gamma_n$ of $\tilde{f}_{\vec{u}^{\mathrm{dyn}}(j),\tilde{\mathbf{R}}_j,\mathcal{I}_j}(\tilde{\mathbf{r}}_j)$. Therefore, for fixed $\vec{u}^{\mathrm{dyn}}_{\mathcal{I}_k}(j) = \{\vec{u}^{\mathrm{dyn}}_{\alpha,\mathcal{I}_k}\}_{|\alpha|\leqslant j} \in \mathcal{P}^{\mathrm{st}}_{<2k,\leqslant j,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\zeta)$, each value of $\vec{u}^{\mathrm{dyn}}_{j,\mathcal{I}_k} \in \mathcal{P}^{\mathrm{st}}_{<2k,j,\tilde{\mathbf{R}}_k}(\zeta)$ corresponds to at most 1 points in $G^{2,(3+2M_1)\gamma_n}_{<2n,j}(\tilde{f}, \vec{u}^{\mathrm{dyn}}_{\mathcal{I}_k}(j-1); \mathbf{r}_0, \dots, \mathbf{r}_j)$. In the case when a point is exactly in the middle of 2 or 4 grid points we associate it to the right or the right-top neighbor respectively. It follows that

$$\sum_{\mathbf{r}_{j+1}\in G^{2,(3+2M_1)\gamma_n}_{<2k,j}(\tilde{f},\vec{u}^{\mathrm{dyn}}(j-1);\mathbf{r}_0,\dots,\mathbf{r}_j)} \nu^{\mathrm{st}}_{\leqslant j}\big\{T^{2,(3+2M_1)\gamma_n}_{<2k,<j+1}(\tilde{f}; \mathbf{r}_0, \dots, \mathbf{r}_{j+1})\big\}$$
$$\leqslant \nu_j^{\mathrm{st}}\big\{\mathcal{P}^{\mathrm{st}}_{<2k,j,\tilde{\mathbf{R}}_j,\mathcal{I}_j}(\zeta)\big\}\nu^{\mathrm{st}}_{\leqslant j-1}\big\{T^{2,(3+2M_1)\gamma_n}_{<2k,<j}(\tilde{f}; \mathbf{r}_0, \dots, \mathbf{r}_j)\big\}. \tag{139}$$

The Distortion Lemma then implies that

$$\sum_{\mathbf{r}_{j+1}\in G^{2,(3+2M_1)\gamma_n}_{<2k,j}(\tilde{f},\vec{u}^{\mathrm{dyn}}(j-1);\mathbf{r}_0,\dots,\mathbf{r}_j)} \nu^{\mathrm{st}}_{\leqslant j}\big\{T^{2,(3+2M_1)\gamma_n}_{<2k,<j+1}(\tilde{f}; \mathbf{r}_0, \dots, \mathbf{r}_{j+1})\big\}$$
$$\leqslant (1-4\delta)^{-2j}\nu^{\mathrm{st}}_{\leqslant j-1}\big\{T^{2,(3+2M_1)\gamma_n}_{<2k,<j}(\tilde{f}; \mathbf{r}_0, \dots, \mathbf{r}_j)\big\}. \tag{140}$$

Consider the second case when $\mathbf{r}_{j+1}$ is a starting point of a generalized loop. According to our notations it means $j+1 = i_q$ and $R_q = \mathbf{r}_{j+1}$ for some $1 \leqslant q \leqslant \tau$. In this case the Distortion Lemma and estimate (128) imply

$$\nu^{\mathrm{st}}_{\leqslant j}\big\{T^{2,(3+2M_1)\gamma_n}_{<2k,<j+1}(\tilde{f}; \mathbf{r}_0, \dots, \mathbf{r}_{j+1})\big\}$$
$$\leqslant \mathrm{Prob}_j\big(\tilde{\mathbf{R}}_{j+1}, (3+2M_1)\gamma_n, \zeta, \delta\big)\nu^{\mathrm{st}}_{\leqslant j-1}\big\{T^{2,(3+2M_1)\gamma_n}_{<2k,<j}(\tilde{f}; \mathbf{r}_0, \dots, \mathbf{r}_j)\big\}. \tag{141}$$

Inductive application of this formula proves (134). In the case $j = 0$ we get

$$
\begin{aligned}
&\nu^{\mathrm{st}}_{<2k}\bigl\{B^{\mathrm{first,st}}_{\mathfrak{Q}}[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}]\bigr\} \\
&\quad\leqslant (1-4\delta)^{-2k(2k-1)+2\sum_{j=1}^{\tau} i_j} \text{Prob-cone}(\tilde{\mathbf{R}}_k, i_1, n, M_1, \zeta, \delta, \theta_{l,m}, \angle K) \\
&\quad\quad\times \prod_{j=1}^{\tau} \text{Prob}_{i_j-1}\bigl(\tilde{\mathbf{R}}_{i_j}, (3+2M_1)\gamma_n, \zeta, \delta\bigr). \qquad (142)
\end{aligned}
$$

Apply to the right-hand side Lemma 21. Direct calculation (see (156)) shows that angle of the cone after the first generalized loop (in our notations of length $N_1$) satisfies

$$\angle K_{\max} = \angle F^{N_1-1}_{\mathfrak{R},\vec{u}_{<2k}}\bigl(K_{\xi_l n}(R_0)\bigr) < 4M_1^{2(s-1)}\mu^{-(\theta_{l,m}+h_l)n}. \qquad (143)$$

Put

$$\tilde{C}_s = \prod_{j=0}^{s-1}(j!)^4\bigl((2s-1)!\bigr)^4\left[\frac{2^{4s^2+s}(3+2M_1)^{2s}M_1^{3s^2}s^{s^2+s}}{(1-4\delta)^{2s^2+4s}\zeta^{s+1}}\right]\bigl(8M_1^{2s+1}+M_1\bigr). \qquad (144)$$

Recall that $\tau \leqslant s$. Combining (148), (149) with the above estimates we get

$$\nu^{\mathrm{st}}\bigl\{B^{\mathrm{first,st}}_{\mathfrak{Q}}[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}]\bigr\} \leqslant \tilde{C}_s\mu^{-\theta_{l,m}n+(s^2-1)(\alpha_{l,m-1}+2d_{l+1})n}\bigl(\mu^{-2\alpha_{l,m}n}\bigr)^{\tau}.$$

By Lemma 4 we have

$$2\bigl(s^2-1\bigr)d_{l+1} + \bigl(s^2-1\bigr)\alpha_{l,m-1} - \theta_{l,m} < -h_l.$$

For large enough $n$ and $(2s)^{-1}\mu^{-\alpha_{l,m}n}$-scattered set $\mathcal{R}$ this leads to the estimate

$$\nu^{\mathrm{st}}\bigl\{B^{\mathrm{first,st}}_{\mathfrak{Q}}[k, f, \mathcal{N}_k, n, l, m; \mathcal{R}]\bigr\} \leqslant \bigl(\mu^{2\alpha_{l,m}n}\bigr)^{-\tau}\tilde{C}_s\mu^{-h_l n}. \qquad (145)$$

This proves the Collection Lemma. □

### *11.7. Auxiliary estimates*

We are interested only in scattered admissible pseudotrajectories. Recall that $\tilde{\mathcal{R}} \subset \tilde{U}$ is a set of ending points of generalized loops. To get lower estimates of pairwise distances between an ending point of a generalized loop and another point from the set $\tilde{\mathbf{R}}_k = \{\tilde{\mathbf{r}}_0, \ldots, \tilde{\mathbf{r}}_{k-1}\} \subset \tilde{U}$ we have applied two following lemmas in the section above.

**Lemma 19.** *For a $k$-loop $\gamma'$-admissible pseudotrajectory of type $\mathcal{N}_k$, shape $(l, n)$, and for any pair of points $\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_j \in \tilde{\mathbf{R}}_k$ such that $\tilde{\mathbf{r}}_i \in \tilde{\mathcal{R}}$ and $\tilde{\mathbf{r}}_j \notin \tilde{\mathcal{R}}$ the following inequality holds*:

$$|\tilde{\mathbf{r}}_j - \tilde{\mathbf{r}}_i| \geqslant M_1^{-1}\bigl((1-\delta) - (1+\delta)\mu^{-(d_l-d_{l+1})n} - 2(3+2M_1)\gamma'\mu^{d_{l+1}n}\bigr)\mu^{-d_{l+1}n}. \qquad (146)$$

**Remark 15.** In the case under consideration we have $\mu^{-\alpha_{l,m}n}$-admissible pseudotrajectories. It means that $\gamma' = \mu^{-\alpha_{l,m}n}$. According to the choice of combinatorial constants $d_{l+1} \ll \alpha_{l,m}$

(see Sections 5.1, 7.1, 8.3). Therefore, the inequality (146) implies that for $(1-\delta)-(1+\delta)\times\mu^{-(d_l-d_{l+1})n}-2(3+2M_1)\gamma'\mu^{d_{l+1}n}>1/2$ (see Section 11.9) we have

$$|\tilde{\mathbf{r}}_j-\tilde{\mathbf{r}}_i|\geqslant\frac{1}{2}M_1^{-1}\mu^{-d_{l+1}n}.$$

**Lemma 20.** *For a $k$-loop $\gamma''$-scattered $\gamma'$-admissible pseudotrajectory of type $\mathcal{N}_k$ and shape $(l,n)$ for any pair $\tilde{\mathbf{r}}_i,\tilde{\mathbf{r}}_j\in\tilde{\mathcal{R}}$ the following inequality holds*:

$$|\tilde{\mathbf{r}}_j-\tilde{\mathbf{r}}_i|\geqslant M_1^{-1}\big(1-2(3+2M_1)\gamma'(\gamma'')^{-1}\big)\gamma''. \tag{147}$$

**Remark 16.** In the case under consideration we have only $\mu^{-\alpha_{l,m}n}$-admissible $\frac{1}{2s}\mu^{-\alpha_{l,m-1}n}$-scattered pseudotrajectories, where $\alpha_{l,m-1}\ll\alpha_{l,m}$ (see Sections 5.1, 7.1, 8.3). Therefore, the inequality (147) implies that for $2(3+2M_1)\mu^{(-\alpha_{l,m}+\alpha_{l,m-1})n}<1/2$ (see Section 11.9) we have

$$|\tilde{\mathbf{r}}_j-\tilde{\mathbf{r}}_i|\geqslant(4sM_1)^{-1}\mu^{-\alpha_{l,m-1}n}.$$

**Lemma 21.** *With the notations of Lemmas* 19 *and* 20 *above, we have for periodicity*

$$\begin{aligned}
&\prod_{j=1}^{\tau}\mathrm{Prob}_{i_j-1}\big(\tilde{\mathbf{R}}_{i_j},(3+2M_1)\mu^{-2\alpha_{l,m}n},\zeta,\delta\big)\\
&\quad\leqslant 2^{\sum_{j=1}^{\tau}(i_j-1)}(1-4\delta)^{-2\sum_{j=1}^{\tau}i_j}\prod_{j=0}^{s-1}(j!)^4\zeta^{-2\tau}(3+2M_1)^{2\tau}(4sM_1)^{\tau(\tau-1)}\\
&\qquad\times(2M_1)^{2\tau(\tau-1)}\big(\mu^{-2\tau\alpha_{l,m}n}\big)\mu^{\tau(\tau-1)\alpha_{l,m-1}n+2\tau(\tau-1)d_{l+1}n},
\end{aligned}\tag{148}$$

*where $\{i_j\}_{j=0}^{\tau-1}$ are indices of starting points of generalized loops. For cone property we have*

$$\begin{aligned}
&\text{Prob-cone}(\tilde{\mathbf{R}}_k,i_1,n,M_1,\zeta,\delta,\theta_{l,m},\angle K)\\
&\quad\leqslant(1-4\delta)^{-4k}\big((2s-1)!\big)^4\zeta^{-1}\big[2^{5(s-1)}M_1^{3s-4}s^{s-2}\big]\big(2M_1^3\angle K+M_1\mu^{-\theta_{l,m}n}\big)\\
&\qquad\times\mu^{2(s-1)d_{l+1}n+(\tau-1)\alpha_{l,m-1}n}.
\end{aligned}\tag{149}$$

**Proof.** We start by proving (148). Using definitions of the numbers $\mathrm{Prob}_{i-1}(\tilde{\mathbf{R}}_i,(3+2M_1)\mu^{-2\alpha_{l,m}n},\zeta,\delta)$, $\{i_j\}_j$, and Remarks 15 and 16 we have that

$$\begin{aligned}
&\prod_{j=1}^{\tau}\mathrm{Prob}_{i_j-1}\big(\tilde{\mathbf{R}}_{i_j},(3+2M_1)\mu^{-2\alpha_{l,m}n},\zeta,\delta\big)\\
&\quad\leqslant\prod_{j=1}^{\tau}\big((i_j-1)!\big)^4\prod_{j=1}^{\tau}\big(2^{i_j-1}(1-4\delta)^{-2(i_j-1)}(3+2M_1)^2\zeta^{-2}\big)\\
&\qquad\times\mu^{-2\tau\alpha_{l,m}n}\big(4sM_1\mu^{\alpha_{l,m-1}n}\big)^{2+4+6+\cdots+2(\tau-1)}\big(2M_1\mu^{d_{l+1}n}\big)^{2(\tau-1)\tau}.
\end{aligned}$$

To obtain the second estimate application of (129) shows that we need to estimate the last two products of distances there. Consider the former of them. Recall that $\tilde{R}_{j-1} = \tilde{\mathbf{r}}_{i_j-1}$. This implies that among points $\{\tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_{i_j-2}\}$ there are $(j-1)$ points from $\tilde{\mathcal{R}}$ and $(i_j - j)$ from the complement. By Remarks 15 and 16 for large enough $n$ we have an estimate

$$\prod_{q=0}^{i_j-2} |\tilde{\mathbf{r}}_{i_j-1} - \tilde{\mathbf{r}}_q|^{-2} \leqslant \left(2sM_1\mu^{\alpha_{l,m-1}n}\right)^{2(j-1)}\left(2M_1\mu^{d_{l+1}n}\right)^{2(i_j-j)}$$
$$\leqslant \left(4sM_1\mu^{\alpha_{l,m-1}n}\right)^{2(j-1)}\left(2M_1\mu^{d_{l+1}n}\right)^{2(k-\tau)} \leqslant \left(4sM_1\mu^{\alpha_{l,m-1}n}\right)^{2(j-1)}\left(2M_1\mu^{d_{l+1}n}\right)^{2(s-1)}.$$

Recall that $\tilde{R}_0 = \tilde{\mathbf{r}}_{i_1-1}$. Using the above estimates we estimate the following product:

$$\prod_{j=0}^{i_1-2} |\tilde{\mathbf{r}}_{i_1-1} - \tilde{\mathbf{r}}_j|^{-2} \prod_{j=i_1}^{k-1} |\tilde{\mathbf{r}}_{i_1-1} - \tilde{\mathbf{r}}_j|^{-1}$$
$$\leqslant \left(2M_1\mu^{d_{l+1}n}\right)^{2(i_1-1)}\left(4sM_1\mu^{\alpha_{l,m-1}n}\right)^{\tau-1}\left(2M_1\mu^{d_{l+1}n}\right)^{k-i_1-\tau+1}$$
$$\leqslant \left[2^{4s-4}M_1^{3s-4}s^{s-2}\right]\mu^{2(s-1)d_{l+1}n+(\tau-1)\alpha_{l,m-1}n}.$$

This completes the proof of the lemma. □

We shall prove Lemmas 19, 20, and estimate (129) in the next Section 11.8.

### *11.8. The proof of auxiliary estimate*

This section is devoted to the proof of estimate (129) and Lemmas 19, 20 from Section 21.

We start with the proof of (129). Consider an image

$$D\tilde{f}_{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_{i_1-1})K(\tilde{\mathbf{r}}_{i_1-1}) = K^{\{\vec{u}_\alpha\}_{|\alpha|=k+i_1-1}}(\mathbf{r}_{i_1}).$$

Recall that $M_1$ is an upper bound on $C^1$-norm of $\tilde{f}_{\vec{u},\tilde{\mathbf{R}}_k,\mathcal{I}_k}$ and its inverse, where $\vec{u}$ is allowed by the family (9). For any set of parameters $\{\vec{u}_\alpha\}_{|\alpha|=k+i_1-1}$ by Lemma 12 we have

$$\angle K^{\{\vec{u}_\alpha\}_{|\alpha|=k+i_1-1}}(\mathbf{r}_{i_1}) \leqslant 2M_1^2\angle K(\tilde{\mathbf{r}}_{i_1-1}).$$

Consider a cone $\hat{K}(\mathbf{r}_{i_1})$ which is wider than $K^*_{\theta_{l,m}n}(\mathbf{r}_{i_1})$ by $2M_1^2\angle K(\tilde{\mathbf{r}}_{i_1-1})$ on both sides, in particular,

$$\angle\hat{K}(\mathbf{r}_{i_1}) = \angle K^*_{\theta_{l,m}n}(\mathbf{r}_{i_1}) + 4M_1^2\angle K(\tilde{\mathbf{r}}_{i_1-1}),$$

and

$$\sin\frac{1}{2}\angle\hat{K}(\mathbf{r}_{i_1}) = \sin\left(\frac{1}{2}K^*_{\theta_{l,m}n}(\mathbf{r}_{i_1}) + 2M_1^2\angle K(\tilde{\mathbf{r}}_{i_1-1})\right)$$
$$< \sin\left(\frac{1}{2}K^*_{\theta_{l,m}n}(\mathbf{r}_{i_1})\right) + \sin\left(2M_1^2\angle K(\tilde{\mathbf{r}}_{i_1-1})\right)$$
$$< 2M_1^2\angle K(\tilde{\mathbf{r}}_{i_1-1}) + \mu^{-\theta_{l,m}n}. \tag{150}$$

Take any unit vector $(a,b) \in K(\tilde{\mathbf{r}}_{i_1-1})$, $a^2+b^2=1$. If $D\tilde{f}_{\vec{u},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_q)\binom{a}{b} \notin \hat{K}(\mathbf{r}_{i_1})$ then

$$D\tilde{f}_{\vec{u},\tilde{\mathbf{R}}_k,\mathcal{I}_k} K(\tilde{\mathbf{r}}_{i_1-1}) \cap K^*_{\theta_{l,m}n}(\mathbf{r}_{i_1}) = \{0\}. \tag{151}$$

Denote

$$D\tilde{f}_{\vec{u},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_{i_1-1})\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}.$$

It is enough to check that $|\tilde{b}|/\sqrt{\tilde{a}^2+\tilde{b}^2} > \sin\frac{1}{2}\angle\hat{K}(\mathbf{r}_{i_1})$ to claim that (151) holds. We have

$$\sqrt{\tilde{a}^2+\tilde{b}^2} = \left| D\tilde{f}_{\vec{u},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_q)\begin{pmatrix} a \\ b \end{pmatrix}\right| \leqslant M_1,$$

therefore it is enough to estimate the measure of parameters for which

$$|\tilde{b}| > M_1\mu^{-\theta_{l,m}n} + 2M_1^3\angle K(\tilde{\mathbf{r}}_{i_1-1}) > M_1 \sin\frac{1}{2}\angle\hat{K}(\mathbf{r}_{i_1}).$$

We have

$$\begin{aligned} &D\tilde{f}_{\vec{u},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_{i_1-1}) \\ &\quad = \begin{pmatrix} A + u^1_{\alpha_1}\frac{\partial}{\partial x}Q_1^{\mathrm{dyn}}(\tilde{\mathbf{r}}_{i_1-1}, \tilde{\mathbf{R}}_{i_1-1}, \tilde{\mathbf{R}}'_{i_1-1}) & B + u^1_{\alpha_2}\frac{\partial}{\partial y}Q_2^{\mathrm{dyn}}(\tilde{\mathbf{r}}_{i_1-1}, \tilde{\mathbf{R}}_{i_1-1}, \tilde{\mathbf{R}}'_{i_1-1}) \\ C + u^2_{\alpha_1}\frac{\partial}{\partial x}Q_1^{\mathrm{dyn}}(\tilde{\mathbf{r}}_{i_1-1}, \tilde{\mathbf{R}}_{i_1-1}\tilde{\mathbf{R}}'_{i_1-1}) & D + u^2_{\alpha_2}\frac{\partial}{\partial y}Q_2^{\mathrm{dyn}}(\tilde{\mathbf{r}}_{i_1-1}, \tilde{\mathbf{R}}_{i_1-1}, \tilde{\mathbf{R}}'_{i_1-1}) \end{pmatrix}, \end{aligned} \tag{152}$$

where $A, B, C, D$ do not depend on $\{\vec{u}_{\alpha_m}\}_{m=1,2}$. Notice that by construction of dynamical Newton monomials we have

$$\begin{aligned} \frac{\partial}{\partial x}Q_1^{\mathrm{dyn}}(\tilde{\mathbf{r}}_{i_1-1}, \tilde{\mathbf{R}}_{i_1-1}, \tilde{\mathbf{R}}'_{i_1-1}) &= \frac{\partial}{\partial y}Q_2^{\mathrm{dyn}}(\tilde{\mathbf{r}}_{i_1-1}, \tilde{\mathbf{R}}_{i_1-1}, \tilde{\mathbf{R}}'_{i_1-1}) \\ &= \prod_{i=0}^{i_1-1-2}|\tilde{\mathbf{r}}_{i_1-1}-\tilde{\mathbf{r}}_i|^{-2}\prod_{j=i_1}^{2k-1}|\tilde{\mathbf{r}}_{i_1-1}-\tilde{\mathbf{r}}_i|^{-1}. \end{aligned} \tag{153}$$

Now we have

$$\begin{aligned} &D\tilde{f}_{\vec{u},\tilde{\mathbf{R}}_k,\mathcal{I}_k}(\tilde{\mathbf{r}}_{i_1-1})\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} \\ &\quad = \begin{pmatrix} Aa + Bb + a\,u^1_{\alpha_1}\frac{\partial}{\partial x}Q_1^{\mathrm{dyn}}(\tilde{\mathbf{r}}_{i_1-1}, \tilde{\mathbf{R}}_{i_1-1}, \tilde{\mathbf{R}}'_{i_1-1}) + bu^1_{\alpha_2}\frac{\partial}{\partial y}Q_2^{\mathrm{dyn}}(\tilde{\mathbf{r}}_{i_1-1}, \tilde{\mathbf{R}}_{i_1-1}, \tilde{\mathbf{R}}'_{i_1-1}) \\ Ca + Db + au^2_{\alpha_1}\frac{\partial}{\partial x}Q_1^{\mathrm{dyn}}(\tilde{\mathbf{r}}_{i_1-1}, \tilde{\mathbf{R}}_{i_1-1}, \tilde{\mathbf{R}}'_{i_1-1}) + bu^2_{\alpha_2}\frac{\partial}{\partial y}Q_2^{\mathrm{dyn}}(\tilde{\mathbf{r}}_{i_1-1}, \tilde{\mathbf{R}}_{i_1-1}, \tilde{\mathbf{R}}'_{i_1-1}) \end{pmatrix}. \end{aligned}$$

Since $a^2+b^2=1$, either $|a| \geqslant 1/\sqrt{2}$, or $|b| \geqslant 1/\sqrt{2}$. Assume that $|a| \geqslant 1/\sqrt{2}$ (opposite case is similar). In this case we get an estimate

$$\nu^{\mathrm{st}}_{k+i_1-1}\big\{\vec{u}^{\mathrm{dyn}}_{k+i_1-1} \in \mathcal{P}^{\mathrm{st}}_{<2k,k+i_1-1,\tilde{\mathbf{R}}_k}(\zeta) \bigm| D\tilde{f}_{u,\tilde{\mathbf{R}}_k,\mathcal{I}_k} K(\tilde{\mathbf{r}}_{i_1-1}) \cap K^*_{\theta_{l,m}n}(\mathbf{r}_{i_1 \pmod k}) \neq \{0\}\big\}$$
$$\leqslant \frac{(1-4\delta)^{-2(k+i_1-1)}}{\zeta}\sqrt{2}\frac{2M_1^3\angle K + M_1\mu^{-\theta_{l,m}n}}{|\frac{\partial}{\partial x}Q_1^{\mathrm{dyn}}(\tilde{\mathbf{r}}_{i_1-1}, \tilde{\mathbf{R}}_{i_1-1}, \tilde{\mathbf{R}}'_{i_1-1})|}$$
$$\leqslant (1-4\delta)^{-2(k+i_1-1)}\big((k+i_1-1)!\big)^4\zeta^{-1}2^{(k+i_1-1)/2}\big(2M_1^3\angle K + M_1\mu^{-\theta_{l,m}n}\big)$$
$$\times \prod_{i=0}^{i_1-2}|\tilde{\mathbf{r}}_{i_1-1} - \tilde{\mathbf{r}}_i|^{-2}\prod_{i=i_1}^{k-1}|\tilde{\mathbf{r}}_{i_1-1} - \tilde{\mathbf{r}}_{i_1-1}|^{-1}. \tag{154}$$

This proves (129).

Now we derive an upper estimate on $\angle K$. Take the cone $K_{\xi_l n}(R_0)$. We want to consider the cone $F^{N_1-1}_{\mathfrak{R},\vec{\varepsilon}_{<2k}}(K_{\xi_l n}(R_0))$ as a cone $K$ in a statement above. Let us estimate its size. The image of the cone $K_{\xi_l n}(R_0)$ under $L^{n_1}$ is a cone $K_{\xi_l n-(1+\Im)n_1}$, and we have

$$\sin\angle K_{\xi_l n-(1+\Im)n_1} < 2\mu^{\xi_l n-(1+\Im)n_1} < 2\mu^{\xi_l n-(1+\Im)d_l n}.$$

After application of the map along short loops we have

$$\sin\angle F^{N_1-1}_{\mathfrak{R},\vec{\varepsilon}_{<2k}}\big(K_{\xi_l n}(R_0)\big) \leqslant M_1^{2h_1}\left(\frac{\mu}{\lambda}\right)^{n_2+\dots+n_{h_1+1}}\sin\angle K_{\xi_l n-(1+\Im)n_1}$$
$$< 2M_1^{2(k-t)}\mu^{(k-t)d_{l+1}n(1+\Im)}\mu^{\xi n-(1+\Im)d_l n}$$
$$\leqslant 2M_1^{2(s-1)}\mu^{((s-1)(1+\Im)d_{l+1}+\xi_l-(1+\Im)d_l)n}, \tag{155}$$

and finally applying Lemma 4 we have

$$\angle F^{N_1-1}_{\mathfrak{R},\vec{\varepsilon}_{<2k}}\big(K_{\xi_l n}(R_0)\big) < 4M_1^{2(s-1)}\mu^{((s-1)(1+\Im)d_{l+1}+\xi_l-(1+\Im)d_l)n}$$
$$< 4M_1^{2(s-1)}\mu^{-(\theta_{l,m}+h_l)n}. \tag{156}$$

This proves estimate (143).

**Proof of Lemma 19.** By definition of admissible pseudotrajectory

$$\big|G_{\vec{\varepsilon}}(\tilde{\mathbf{r}}_i) - \mathbf{r}_{i+1 \pmod k}\big| \leqslant (3+2M_1)\gamma' \quad \text{and} \quad \big|G_{\vec{\varepsilon}}(\tilde{\mathbf{r}}_j) - \mathbf{r}_{j+1 \pmod k}\big| \leqslant (3+2M_1)\gamma'.$$

For any point (in particular, for $\mathbf{r}_{i+1 \pmod k}$) from testing rectangle $\Pi_{n_{i+1 \pmod k)+1}}$ its $y$-coordinate $\leqslant (1+\delta)\mu^{-n_{i+1 \pmod k)+1}}$.

For any point (in particular, for $\mathbf{r}_{j+1 \pmod k}$) from testing rectangle $\Pi_{n_{j+1 \pmod k)+1}}$ its $y$-coordinate $\geqslant (1-\delta)\mu^{-n_{j+1 \pmod k)+1}}$. Therefore

$$|\mathbf{r}_{i+1 \pmod k} - \mathbf{r}_{j+1 \pmod k}| \geqslant (1-\delta)\mu^{-n_{j+1 \pmod k)+1}} - (1+\delta)\mu^{-n_{i+1 \pmod k)+1}}$$
$$\geqslant (1-\delta)\mu^{-d_{l+1}n} - (1+\delta)\mu^{-d_l n}$$
$$= \big((1-\delta) - (1+\delta)\mu^{-(d_l-d_{l+1})n}\big)\mu^{-d_{l+1}n}. \tag{157}$$

This gives the following:

$$\begin{aligned}
&\big|G_{\vec\varepsilon}(\tilde{\mathbf{r}}_i) - G_{\vec\varepsilon}(\tilde{\mathbf{r}}_j)\big| \\
&\quad \geqslant |\mathbf{r}_{i+1 \pmod k} - \mathbf{r}_{j+1 \pmod k}| - \big|G_{\vec\varepsilon}(\tilde{\mathbf{r}}_i) - \mathbf{r}_{i+1 \pmod k}\big| - \big|G_{\vec\varepsilon}(\tilde{\mathbf{r}}_j) - \mathbf{r}_{j+1 \pmod k}\big| \\
&\quad \geqslant \big((1-\delta) - (1+\delta)\mu^{-(d_l - d_{l+1})n} - 2(3+2M_1)\gamma' \mu^{d_{l+1}n}\big)\mu^{-d_{l+1}n}. \qquad (158)
\end{aligned}$$

Finally we get

$$\begin{aligned}
|\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j| &\geqslant M_1^{-1}\big|G_{\vec\varepsilon}(\tilde{\mathbf{r}}_i) - G_{\vec\varepsilon}(\tilde{\mathbf{r}}_j)\big| \\
&\geqslant M_1^{-1}\big((1-\delta) - (1+\delta)\mu^{-(d_l - d_{l+1})n} - 2(3+2M_1)\gamma' \mu^{d_{l+1}n}\big)\mu^{-d_{l+1}n}. \qquad (159)
\end{aligned}$$

Lemma 19 is proved. □

**Proof of Lemma 20.** By definition of admissible pseudotrajectory

$$\big|G_{\vec\varepsilon}(\tilde{\mathbf{r}}_i) - \mathbf{r}_{i+1 \pmod k}\big| \leqslant (3+2M_1)\gamma' \quad \text{and} \quad \big|G_{\vec\varepsilon}(\tilde{\mathbf{r}}_j) - \mathbf{r}_{j+1 \pmod k}\big| \leqslant (3+2M_1)\gamma'.$$

By definition of a scattered pseudotrajectory we have

$$|\mathbf{r}_{i+1 \pmod k} - \mathbf{r}_{j+1 \pmod k}| \geqslant \gamma''.$$

This implies

$$\begin{aligned}
&\big|G_{\vec\varepsilon}(\tilde{\mathbf{r}}_i) - G_{\vec\varepsilon}(\tilde{\mathbf{r}}_j)\big| \\
&\quad \geqslant |\mathbf{r}_{i+1 \pmod k} - \mathbf{r}_{j+1 \pmod k}| - \big|G_{\vec\varepsilon}(\tilde{\mathbf{r}}_i) - \mathbf{r}_{i+1 \pmod k}\big| - \big|G_{\vec\varepsilon}(\tilde{\mathbf{r}}_j) - \mathbf{r}_{j+1 \pmod k}\big| \\
&\quad \geqslant \gamma'' - 2(3+2M_1)\gamma'. \qquad (160)
\end{aligned}$$

Finally we have

$$|\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j| \geqslant M_1^{-1}\big(1 - 2(3+2M_1)\gamma'(\gamma'')^{-1}\big)\gamma''$$

Lemma 20 is proven. □

### *11.9. Estimates of constants in Auxiliary Theorem I*

The last estimate in (133) gives Auxiliary Theorem I, as explained in Section 3.1. To finish the proof we just need to estimate the constants $C_s$, $\mathbf{h}_s(\aleph)$ and $\mathbf{N}^*(s,\aleph)$.

Estimate $\mathbf{h}_s(\aleph)$ first. We can take $\mathbf{h}_s(\aleph) = \min_{l=1,\dots,s} h_l$. By our choice (Definition 19) $h_l = \beta^{(2s+2)l+1}$, therefore $\min_l h_l = h_s = \beta^{(2s+2)s+1}$. So we can take

$$\mathbf{h}_s = \beta^{2s^2+2s+1} = \big(5s^2\aleph^{-1}(1+\Im)\big)^{-(2s^2+2s+1)}.$$

To estimate $C_s$ note that we can take $C_s \geqslant (s!s2^s)C_s^*$ (to ensure that (28) implies (26)), $C_s^* = s\tilde{C}_s$ (to ensure that (108) implies (105)), and $\tilde{C}_s$ given by (144). This implies that we can take $C_s = \exp(s^2(A + 9\ln s))$, where $A$ depends on $\{M_1, \delta, \zeta\}$ only.

To estimate how large $\mathbf{N}^*(s,\aleph)$ should be taken, let us recall that we made an assumption that period $n$ is large enough in the proof of Lemma 3, in the proof of Proposition 4, assuming that inequality (70) holds, and in Remarks 15 and 16. In all these cases the assumption could be written in the form $A_1 A_2^{-s}\mu^{-A_3 n} < 1$, where constants $A_1, A_2 > 0$ depend on parameters $\mathfrak{W}$ only, and $A_3 \geqslant \mathbf{h}_s = (5s^2\aleph^{-1}(1+\Im))^{-(2s^2+2s+1)}$. Therefore all these assumptions follows from the condition $n > \mathbf{N}^*(s,\aleph) > (\ln A_1 - s\ln A_2)/(A_3 \ln\mu)$. In particular, we can take

$$\mathbf{N}^*(s,\aleph) = Bs\bigl(5s^2\aleph^{-1}(1+\Im)\bigr)^{2s^2+2s+1},$$

where $B$ depends on parameters $\mathfrak{W}$ only.

## 12. Prevalence of hyperbolicity of localized periodic orbits

In this section we prove Auxiliary Theorem II stated in Section 4.3.

### *12.1. Hyperbolicity of linear operators*

Recall that a linear operator $A:\mathbb{R}^N \to \mathbb{R}^N$ is *hyperbolic* if it has no eigenvalues on the unit circle $\{|z|=1\} \subset \mathbb{C}$. Denote by $|\cdot|$ the Euclidean norm in $\mathbb{C}^N$. Define the *hyperbolicity* of a linear operator $L$ by

$$\operatorname{Hyp}(A) = \inf_{\phi\in[0,1)} \inf_{|v|=1} \bigl|Av - \exp(2\pi i\phi)v\bigr|. \tag{161}$$

It is clear that a linear operator $A$ is hyperbolic if and only if $\operatorname{Hyp}(A) = 0$.

This notion of hyperbolicity (in some cases more appropriate than the minimum distance of the eigenvalues of $A$ from the unit circle in $\mathbb{C}$) was introduced and studied in [22]. In particular, we need the following statements proved in Appendix D of [22].

**Lemma 22.** *For any pair of linear operators $L$ and $\Delta$ of $\mathbb{R}^N$ into itself, hyperbolicity satisfies the estimate*

$$\operatorname{Hyp}(L+\Delta) \geqslant \operatorname{Hyp}(L) - \|\Delta\|. \tag{162}$$

**Proof.** By the definition of hyperbolicity,

$$\operatorname{Hyp}(L+\Delta) = \inf_{\phi\in[0,1)} \inf_{\|v\|=1} \bigl|(L+\Delta)v - \exp(2\pi i\phi)v\bigr|. \tag{163}$$

By triangle inequality, for all $v \in \mathbb{R}^N$,

$$\bigl|(L+\Delta)v - \exp(2\pi i\phi)v\bigr| \geqslant \bigl|Lv - \exp(2\pi i\phi)v\bigr| - |\Delta v|. \tag{164}$$

This implies the statement of Lemma 22. □

**Proposition 5.** [22, Proposition A.5] *Let $r \leqslant 1 \leqslant K$ be positive numbers and $A, B$ be linear operators of $\mathbb{R}^N$ into itself given by $N \times N$ matrices from $M_N(\mathbb{R})$ with real entries. Consider an*

*$N^2$-parameter family $\{A_U = A + UB\}_{U\in C^{N^2}(r)}$, where $C^{N^2}(r)$ is the cube in $M_N(\mathbb{R})$ whose entries are bounded in absolute value by $r$. Suppose that $\|B\|$, $\|B^{-1}\| \leqslant K$. Then for the Lebesgue product probability measure $\mu_{r,N^2}$ on the cube $C^{N^2}(r)$ and all $0 < \gamma \leqslant \min(r, 1)$, we have*

$$\mu_{r,N^2}\left\{U \in C^{N^2}(r) \mid \mathrm{Hyp}(A_U) \leqslant \gamma\right\} \leqslant \frac{C(N)K^{2N^2}\gamma}{r^2}, \tag{165}$$

*where the constant $C(N)$ depends only on $N$.*

*12.2. Completely scattered periodic orbits of given type*

Set initial parameters of the problem $\mathfrak{W} = \{\mu, \lambda, M_1, M_2, V, \delta, \zeta\}$. Introduce the following sets in space of parameters $HB_{<2s}(\zeta)$:

$$X^{\text{non-hyp}}_{\mathfrak{W}}[f, s, s'] = \left\{\vec{\varepsilon} \in HB_{<2s}(\zeta) \;\middle|\; f_{\vec{\varepsilon}} \text{ has a non-hyperbolic } (\mathcal{V}, s')\text{-localized periodic orbit}\right\}. \tag{166}$$

Recall that $\nu_{<2s}$ is the Lebesgue product probability measure on $HB_{<2s}(\zeta)$, as defined by (12). Auxiliary Theorem II is equivalent to the following equality:

$$\nu_{<2s}\left\{\bigcup_{s'=1}^{s} X^{\text{non-hyp}}_{\mathfrak{W}}[f, s, s']\right\} = 0.$$

Therefore we need to show that for each $s' = 1, \dots, s$

$$\nu_{<2s}\left\{X^{\text{non-hyp}}_{\mathfrak{W}}[f, s, s']\right\} = 0. \tag{167}$$

To show that perturbations of $f$ do not have non-hyperbolic $s'$-loop periodic orbits for $\nu_{<2s'}$-almost every perturbation, our method requires to consider polynomial perturbations of degree $2s' - 1$. Therefore we can reduce the space of parameters for $s' < s$ in the following way. Introduce the sets:

$$X^{\text{red}}_{\mathfrak{W}}[f, s'] = \left\{\vec{\varepsilon} \in HB_{<2s'}(\zeta) \;\middle|\; f_{\vec{\varepsilon}} \text{ has a non-hyperbolic } (\mathcal{V}, s')\text{-localized periodic orbit}\right\}. \tag{168}$$

Using Fubini reduction from Section 3.2 one can show that if $\tilde{f}(\tilde{x}, \tilde{y}) = f(\tilde{x}, \tilde{y}) + \vec{\Phi}_{\vec{\varepsilon}_{\geqslant 2s', <2s}}(\tilde{x}, \tilde{y})$ and

$$\nu_{<2s'}\left\{X^{\text{red}}_{\mathfrak{W}}[\tilde{f}, s']\right\} = 0 \tag{169}$$

for all $\vec{\varepsilon}_{\geqslant 2s', <2s}$, then (167) holds. So we need to prove (169) in order to prove Auxiliary Theorem II.

**Definition 33.** An $(\mathcal{V}, s')$-localized periodic orbit is called $\rho$-completely scattered if the distance between starting points of any two different loops is at least $\rho$.

Define

$$X^{\mathrm{scatt}}_{\mathfrak{W}}[\tilde f, \mathcal{N}_{s'}, \rho] = \bigl\{\vec\varepsilon_{<2s'} \in HB_{<2s'}(\zeta) \bigm| f_{\vec\varepsilon} \text{ has a non-hyperbolic } (\mathcal{V}, s)\text{-localized } \rho\text{-completely scattered periodic orbit of type } \mathcal{N}_{s'}\bigr\}. \tag{170}$$

For any $s'$-loop periodic orbit it is $\rho$-scattered for some $\rho > 0$. Therefore $X^{\mathrm{red}}_{\mathfrak{W}}[\tilde f, s']$ can be decomposed into the countable union

$$X^{\mathrm{reg}}_{\mathfrak{W}}[\tilde f, s'] = \bigcup_{m\in\mathbb{N}} \bigcup_{\mathcal{N}_{s'}} X^{\mathrm{scatt}}_{\mathfrak{W}}\left[\tilde f, \mathcal{N}_{s'}, \frac{1}{m}\right].$$

Therefore we reduce (169) and, therefore, Auxiliary Theorem II to the following statement:

$$\text{For each type } \mathcal{N}_{s'} \text{ and each } \rho > 0 \text{ we have } \nu_{<2s'}\bigl\{X^{\mathrm{scatt}}_{\mathfrak{W}}[\tilde f, \mathcal{N}_{s'}, \rho]\bigr\} = 0. \tag{171}$$

### *12.3. Admissible periodic orbits of small hyperbolicity*

To prove (171) we fix type $\mathcal{N}_{s'} = (n_1, \dots, n_{s'})$, $\rho > 0$, and consider grids of a small size $\gamma \ll \rho$ (see Sections 8.1–8.3). By Proposition 3 for any $\rho$-completely scattered $s'$-loop periodic orbit $\mathfrak{P} = \{p_0, \dots, p_{n-1}\}$ of a map $f_{\vec\varepsilon_{<2s'}}$ there exists a $\gamma$-admissible (see Definition 23) periodic pseudo-orbit $\mathfrak{R} = \{r_0, \dots, r_{n-1}\}$ of the same type which is $(\rho - 4\gamma)$-completely scattered and $\operatorname{dist}(p_i, r_i) \leqslant 2\gamma$, $i = 0, \dots, n-1$.

Calculations similar to (60) show that

$$\bigl\| Df^n_{\vec\varepsilon_{<2s'}}(p_0) - F^n_{\mathfrak{R},\vec\varepsilon_{<2s'}}(r_0)\bigr\| \leqslant \bigl[2s' M_1^{s'-1} M_2 \mu^{n-s'}\bigr]\gamma,$$

so (by Lemma 22) if $\operatorname{Hyp}(Df^n_{\vec\varepsilon}(p_0)) = 0$ then

$$\operatorname{Hyp}\bigl(F^n_{\mathfrak{R},\vec\varepsilon_{<2s'}}(r_0)\bigr) \leqslant \bigl[2s' M_1^{s'-1} M_2 \mu^{n-s'}\bigr]\gamma.$$

Define the following sets

$$X^{\mathrm{adm}}_{\mathfrak{W}}[\tilde f, \mathcal{N}_{s'}, \rho, \gamma] = \bigl\{\vec\varepsilon \in HB_{<2s'}(\zeta) \bigm| f_{\vec\varepsilon_{<2s'}} \text{ has a } \gamma\text{-admissible } (\rho - 4\gamma)\text{-completely scattered } (\mathcal{V}, s')\text{-localized periodic orbit } \mathfrak{R} \text{ of type } \mathcal{N}_{s'} \text{ such that } \operatorname{Hyp}\bigl(F^n_{\mathfrak{R},\vec\varepsilon_{<2s'}}(r_0)\bigr) \leqslant \bigl[2s' M_1^{s'-1} M_2 \mu^{n-s'}\bigr]\gamma\bigr\}. \tag{172}$$

We have for all $\gamma < \rho/10$

$$X^{\mathrm{adm}}_{\mathfrak{W}}[\tilde f, \mathcal{N}_{s'}, \rho, \gamma] \supset X^{\mathrm{scatt}}_{\mathfrak{W}}[\tilde f, \mathcal{N}_{s'}, \rho].$$

A set $X^{\mathrm{scatt}}_{\mathfrak{W}}[\tilde f, \mathcal{N}_{s'}, \rho]$ does not depend on $\gamma$, and

$$\nu_{<2s'}\bigl\{X^{\mathrm{scatt}}_{\mathfrak{W}}[\tilde f, \mathcal{N}_{s'}, \rho]\bigr\} \leqslant \nu_{<2s'}\bigl(X^{\mathrm{adm}}_{\mathfrak{W}}[\tilde f, \mathcal{N}_{s'}, \rho, \gamma]\bigr).$$

Therefore to prove that $\nu_{<2s'}\{X^{\text{scatt}}_{\mathfrak{W}}[\tilde f,\mathcal{N}_{s'},\rho]\}=0$ it is enough to show that

$$\nu_{<2s'}\big\{X^{\text{adm}}_{\mathfrak{W}}[\tilde f,\mathcal{N}_{s'},\rho,\gamma]\big\}\leqslant C(\tilde f,\mathfrak{W},\mathcal{N}_{s'},\rho)\gamma. \tag{173}$$

Take any $\rho$-completely scattered $\gamma$-admissible pseudo-orbit $\mathfrak{R}=\{r_0,\dots,r_{n-1}\}$ of type $\mathcal{N}_{s'}$. Recall that we denote the intersection of $\mathfrak{R}$ with $U$ (with $\tilde U$) by $\mathbf{R}_{s'}=\{\mathbf{r}_0,\dots,\mathbf{r}_{s'-1}\}$ (by $\tilde{\mathbf{R}}_{s'}=\{\tilde{\mathbf{r}}_0,\dots,\tilde{\mathbf{r}}_{s'-1}\}$ respectively). Note that the entire pseudo-orbit $\mathfrak{R}$ is uniquely defined by the choice of set $\mathbf{R}_{s'}=\{\mathbf{r}_0,\dots,\mathbf{r}_{s-1}\}$, where $\mathbf{r}_i\in\boldsymbol{\Pi}_{n_{i+1}}(\gamma)$. By Lemma 10 there exist at most $(9\delta^2\gamma^{-2})^{s'}$ different sets $\mathbf{R}_{s'}$ with this property. For each $\mathbf{R}_{s'}=\{\mathbf{r}_0,\dots,\mathbf{r}_{s-1}\}$, $\mathbf{r}_i\in\boldsymbol{\Pi}_{n_{i+1}}(\gamma)$, which is $(\rho-4\gamma)$-scattered (that is, $\operatorname{dist}(r_i,r_j)\geqslant\rho-4\gamma$ for each pair $i\neq j$), define a set of parameters

$$\begin{aligned}X^{\text{fixed}}_{\mathfrak{W}}[\tilde f,\gamma,\mathbf{R}_{s'}]=\big\{\vec\varepsilon\in HB_{<2s'}(\zeta)\ \big|\ &f_{\vec\varepsilon}\text{ has a }\gamma\text{-admissible }(\mathcal{V},s')\text{-localized periodic}\\ &\text{orbit }\mathfrak{R}\text{ such that }\mathbf{R}_{s'}\text{ is the ordered set of starting points of its loops and}\\ &\operatorname{Hyp}\big(F^n_{\mathfrak{R},\vec\varepsilon_{<2s'}}(\mathbf{r}_0)\big)\leqslant\big[2s'M_1^{s'-1}M_2\mu^{n-s'}\big]\gamma\big\}.\end{aligned} \tag{174}$$

We will prove that for any $(\rho-4\gamma)$-completely scattered set $\mathbf{R}_{s'}$ and $\gamma\ll\rho$ we have

$$\nu_{<2s'}\big\{X^{\text{fixed}}_{\mathfrak{W}}[\tilde f,\gamma,\mathbf{R}_{s'}]\big\}\leqslant C^*(\tilde f,\mathfrak{W},\mathcal{N}_{s'},\rho)\gamma^{2s'+1}. \tag{175}$$

This will imply that

$$\begin{aligned}&\nu_{<2s'}\big\{X^{\text{adm}}_{\mathfrak{W}}[\tilde f,\mathcal{N}_{s'},\rho,\gamma]\big\}\\ &\quad\leqslant\nu_{<2s'}\Bigg(\bigcup_{\{\mathbf{R}_{s'}|\mathbf{R}_{s'}\text{ is }(\rho-4\gamma)\text{-completely scattered}\}}X^{\text{fixed}}_{\mathfrak{W}}[\tilde f,\gamma,\mathbf{R}_{s'}]\Bigg)\\ &\quad\leqslant\big(9\delta^2\gamma^{-2}\big)^{s'}\cdot C^*(\tilde f,\mathfrak{W},\mathcal{N}_{s'},\rho)\gamma^{2s'+1}=C(\tilde f,\mathfrak{W},\mathcal{N}_{s'},\rho)\gamma,\end{aligned} \tag{176}$$

which is exactly the required estimate (173). Therefore we reduced the proof of Auxiliary Theorem II to the estimate (175).

### *12.4. Newton Interpolation Polynomials and estimates of the measure of "bad" parameters*

To prove (175) we follow exactly the same strategy as to prove (108). The key element is application of Discretization Method from Section 11. Actually we need just estimates from Section 11.5. Combining estimates (126)–(128) for $j=0,\dots,s'-1$ we get an estimate of probability of $\mathbf{R}_{s'}$ being $\gamma$-admissible pseudo-orbit. To estimate probability of non-hyperbolicity we apply Proposition 5 with matrix $N=2$, the unperturbed $2\times2$ matrix $A=F^n_{\mathfrak{R},0}(\mathbf{r}_0)$, matrix $B=\text{Const}\cdot F^{n-1}_{\mathfrak{R},0}(\mathbf{r}_0)$, and $U=\{u^q_{\alpha_m}\}_{q,m=1,2}$. Indeed, calculation from Section 11.8 and, in particular, expression (152) shows that the corresponding composition of linearizations $F^n_{\mathfrak{R},\vec\varepsilon_{<2s'}}(\mathbf{r}_0)$ written in dynamical Newton basis has the form (152) and by (153) we have that

$$\text{Const}=\prod_{j=0}^{s'-1}|\tilde{\mathbf{r}}_{s'}-\tilde{\mathbf{r}}_j|^{-2}.$$

Now we substitute (129), the family (152), and the proof of it by (5), the family

$$\left\{F^n_{\mathfrak{R},\vec{\varepsilon}_{<2s'}}(\mathbf{r}_0) = F^n_{\mathfrak{R},0}(\mathbf{r}_0) + \mathrm{Const}\, U \cdot F^{n-1}_{\mathfrak{R},0}(\mathbf{r}_0)\right\}_{U\in C^4(\zeta/(1-4\delta))},$$

and Proposition 5. We get

$$\begin{aligned}\nu_{2s'-1}&\left\{\vec{u}^{\,\mathrm{dyn}}_{2s'-1,\mathcal{I}_k} \in HB_{2s'-1}\left(\frac{\zeta}{1-4\delta}\right) \,\Big|\, \mathrm{Hyp}\left(F^n_{\mathfrak{R},\vec{\varepsilon}_{<2s'}}(\mathbf{r}_0)\right) \leqslant \left[2s' M_1^{s'-1} M_2 \mu^{n-s'}\right]\gamma\right\}\\ &\leqslant \frac{(1-4\delta)^{-2s'}}{\zeta^2} 2^{s'}\left[2s' M_1^{s'-1} M_2\mu^{n-s'}\right] M_1^{8n} \prod_{j=0}^{s'-1} |\tilde{\mathbf{r}}_{s'} - \tilde{\mathbf{r}}_j|^{-2}\gamma. \end{aligned} \tag{177}$$

Denote the right-hand side by Prob-non-hyp$(\mathbf{R}_{s'}, n, M_1, \zeta, \delta, \gamma)$.

Combining (126–128) for $q = 0, \dots, s'-1$ we derive an analog of (130)

$$\begin{aligned}\nu_{<2s'}\left\{X^{\mathrm{fixed}}_{\mathfrak{W}}[\tilde{f}, \gamma, \mathbf{R}_{s'}]\right\} &\leqslant (1-4\delta)^{-2s'(2s'+1)} \text{Prob-non-hyp}(\tilde{\mathbf{R}}_{s'}, n, M_1, \zeta, \delta, \gamma)\\ &\quad \times \prod_{j=0}^{s'-1} \mathrm{Prob}_j(\tilde{\mathbf{R}}_{j+1}, \gamma, \zeta, \delta). \end{aligned} \tag{178}$$

Products of distances in $\{\mathrm{Prob}_j\}_j$ and in Prob-non-hyp are bounded from below by powers of $(\rho - 4\gamma)$. Provided that $\gamma \ll \rho$ we could replace $(\rho - 4\gamma)$ by $\rho/2$. All the other constants depend on parameters of the problem $\mathfrak{W}$ and are uniform. This proves (175) and, therefore, complete the proof of Auxiliary Theorem II.

## 13. Nonlinear results

### *13.1. The proof of the main result for a non-resonant saddle fixed point*

Consider a $C^r$ smooth diffeomorphism $f : M \to M$ of a smooth compact surface $M$ with $r \geqslant 2$. Suppose $p' = f(p')$ is a non-resonant saddle fixed point, i.e. eigenvalues $|\lambda| < 1 < |\mu|$ of $Df(p')$ have no integer relations $\lambda^{n_1}\mu^{n_2} = 1$. Suppose it has a homoclinic tangency at some point $q'$. Let $\tilde{q}' = f^{-1}(q')$. Since $p'$ is non-resonant, by Sternberg's linearization theorem, in a small neighborhood of $p'$ there is a $C^r$ smooth normal coordinate system $(x, y)$ such that $f(x, y) = (\lambda x, \mu y)$ in it. Extend the coordinate neighborhood by iterating forward and backward until the first time it contains $\tilde{q}$ and $q$ respectively. Denote such a neighborhood by $V''$ and call *a normal neighborhood*. Similarly extend the coordinate neighborhood by iterating forward and backward until the first time it contains $\tilde{q}$ and $f(q)$ respectively. Denote it by $V'$. By definition $V'$ does not contain $q$ (see Fig. 9). Consider a neighborhood $U'$ (respectively $\tilde{U}' \subset \hat{U}'$) of $q'$ (respectively $\tilde{q}'$) such that $f(U') \cap U' = \emptyset$ (respectively $f^{-1}(\hat{U}') \cap \hat{U}' = \emptyset$), $f(\tilde{U}') \supset U'$, and $f(\hat{U}) \cap V = \emptyset$. Consider the normal chart $T : V'' \to \mathbb{R}^2$ as a subset of $\mathbb{R}^2$ and denote it by $T(V') = V$. Denote $T(\tilde{q}') = \tilde{q}$, $T(q') = q$, and $T(p') = p$ respectively. As the result we get $C^r$ smooth transition maps from $\tilde{U}' \subset \hat{U}'$ (respectively $U'$) into $\tilde{U} \subset \hat{U}$ (respectively $U$), both are restrictions of $T$.

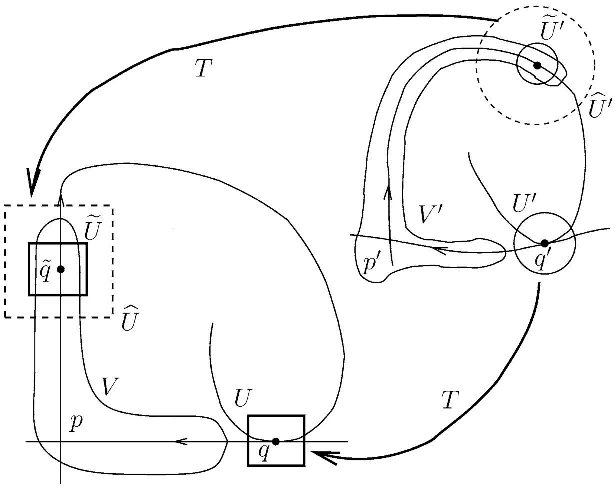


Fig. 9. A non-resonant saddle fixed point.

The family of perturbations (9) becomes

$$f_{\vec{\varepsilon}}(x', y') = \begin{cases} f_0(x', y'), & \text{if } (x', y') \in V' \setminus \hat{U}', \\ T^{-1} \circ G_{\vec{\varepsilon}} \circ T(x', y'), & \text{if } (x', y') \in \tilde{U}'. \end{cases} \tag{179}$$

In the linearized coordinates this family has the following form:

$$f_{\vec{\varepsilon}}(x, y) = \begin{cases} L(x, y), & \text{if } (x, y) \in V \setminus \hat{U}, \\ G_{\vec{\varepsilon}}(x, y), & \text{if } (x, y) \in \tilde{U}. \end{cases} \tag{180}$$

Then Theorems A, B, A′, and B′ holds true for this family.

**Remark 17.** Probably it would be more natural to consider the following family of perturbations

$$f_{\vec{\varepsilon}}(x', y') = \begin{cases} f_0(x', y'), & \text{if } (x', y') \in V' \setminus U', \\ G'_{\vec{\varepsilon}}(x', y'), & \text{if } (x', y') \in \tilde{U}'. \end{cases} \tag{181}$$

It seems that our method still works for this family. However, this leads to variety of technical complications in the proof so we omit detailed explanation.

*13.2. Statement of the main result for saddle periodic points with homoclinic tangency*

Consider a $C^r$ smooth diffeomorphism $f: M \to M$ of a smooth compact surface $M$ with $r \geqslant 2$. Suppose $p' = f^k(p')$ is a non-resonant saddle periodic point for some integer $k$, i.e. eigenvalues $|\lambda| < 1 < |\mu|$ of $Df^k(p')$ have no integer relations $\lambda^{n_1}\mu^{n_2} = 1$. Denote by $W^s(p')$ and $W^u(p')$ stable and unstable manifolds of $p'$. Suppose $p'$ has a homoclinic tangency at some point $q'$, i.e. $W^s(p')$ and $W^u(p')$ contain and do not transverse at $q'$. Let $\tilde{q}' = f^{-k}(q')$. Since $p'$ is non-resonant, by Sternberg's linearization theorem, in a small neighborhood of $p'$ there is a

$C^r$ smooth normal coordinate system $(x, y) \subset \tilde{V}'$ such that $f^k(x, y) = (\lambda x, \mu y)$. Extend the coordinate neighborhood by iterating forward and backward until the first time it contains $\tilde{q}$ and $q$ respectively. Denote such a neighborhood by $V_0''(p')$. Similarly extend the coordinate neighborhood by iterating forward and backward until the first time it contains $\tilde{q}$ and $f(q)$ respectively. Denote it by $V_0'(p')$ and call *a normal neighborhood*. By definition $V_0'(p')$ does not contain $q$ (see Fig. 9). Consider images of $V_0'(p')$, denoted $V_j'(p') = f^j(V'(p'))$ for $j = 1, \dots, k-1$. Even by decreasing $V_0'(p')$ we cannot not claim that $V_j'(p')$'s are pairwise disjoint.[11] Consider a neighborhood $U'$ (respectively $\tilde{U}' \subset \hat{U}'$) of $q'$ (respectively $\tilde{q}'$) such that $f(U') \cap U' = \emptyset$ (respectively $f^{-1}(\hat{U}') \cap \hat{U}' = \emptyset$), $f(\tilde{U}') \supset U'$, $f(\hat{U}') \cap V' = \emptyset$, and both $U'$ and $\tilde{U}'$ are disjoint from $\bigcup_{j=0}^{k-1} V_j'(p')$. This is always possible to achieve by decreasing corresponding $V$'s and $U$'s, because $q'$ and $\tilde{q}'$ belongs to $W^s(p') \cap W^u(p')$ and therefore, cannot belong to $W^s(f^j(p'))$ or $W^u(f^j(p'))$ for $j = 1, \dots, k-1$. Indeed, under forward (respectively backward) $f^k$-iterates it should converge only to $p$. After such a choice of $U'$ and $\tilde{U}'$ we reduce the case of a saddle periodic point with a homoclinic tangency to the case of a saddle fixed point with a homoclinic tangency. Namely, we consider only $\mathcal{V}$-localized sets for $f^k$ (not $f$!) defined as in Definition 1 with $f$ replaced by $f^k$.

Consider the normal neighborhood $T : V_0''(p') \to \mathbb{R}^2$ as a subset of $\mathbb{R}^2$ and denote it by $T(V'(p')) = V$. Denote $T(\tilde{q}') = \tilde{q}$, $T(q') = q$, and $T(p') = p$ respectively. As the result we get $C^r$ smooth transition map from $\tilde{U}' \subset \hat{U}'$ and $U'$ into $\tilde{U} \subset \hat{U}$ and $U$ respectively, which we also denote by $T$. Consider the images $f^{k-1}(\tilde{U}') = \tilde{U}'_{k-1} \subset f^{k-1}(\hat{U}') = \hat{U}'_{k-1}$. The maps $T \circ f^{1-k}$ induces charts on $\tilde{U}'_{k-1}$ and $\hat{U}'_{k-1}$. Notice that because of the way charts are defined the map $f^{k-1}$ restricted to $\hat{U}'$ is the identity map. With respect to these charts we consider the family of maps of perturbations of the form (179) with $T$ replaced by $T \circ f^{1-k}$ and the family $\{f_{\vec{\varepsilon}}\}$ replaced by the family $\{f^k_{\vec{\varepsilon}}\}$. The rest of the proof is the same as in Section 13.1.

## 14. Auxiliary computations

### *14.1. Proof of Addendum 2.1*

**Proof of Addendum 2.1.** We need to choose a sequence $\{\mathbf{N}_s(\aleph)\}_s$ in such a way that the series (42) is convergent. For any period $n$ and cyclicity $s$ there exists at most $n^s$ different types $\mathcal{N}_s$, $|\mathcal{N}_s| = n$, so it is enough to require the convergence of the following series:

$$\sum_{s \in \mathbb{N}} \sum_{n \geqslant \mathbf{N}_s(\aleph)} n^s C_s \mu^{-\mathbf{h}_s(\aleph) n}. \tag{182}$$

It is enough to choose $\mathbf{N}_s(\aleph)$ in such a way that for some $s_0 \in \mathbb{N}$ the following inequality holds for each $s \geqslant s_0$:

$$\sum_{n \geqslant \mathbf{N}_s(\aleph)} n^s C_s \mu^{-\mathbf{h}_s(\aleph) n} < e^{-s}. \tag{183}$$

This is equivalent to the following:

[11] Moreover, there are examples when this property have to fail.

$$\sum_{m=0}^{\infty} \big(m+\mathbf{N}_s(\aleph)\big)^s C_s \mu^{-\mathbf{h}_s(\aleph)(m+\mathbf{N}_s(\aleph))} < e^{-s}, \quad \text{or} \tag{184}$$

$$\sum_{m=0}^{\infty} \exp\big\{s^2(A+9\ln s) + s\ln\big(m+\mathbf{N}_s(\aleph)\big) + s - \beta^{2s^2+2s+1}(\ln\mu)\big(m+\mathbf{N}_s(\aleph)\big)\big\} < 1,$$

where $\beta = \aleph/(5s^2(1+\Im))$. Since $\ln(m+\mathbf{N}_s(\aleph)) \leqslant \ln(\mathbf{N}_s(\aleph)) + m/\mathbf{N}_s(\aleph)$, we have

$$\begin{aligned}
&\sum_{m=0}^{\infty} \exp\big\{s^2(A+9\ln s) + s\ln\big(m+\mathbf{N}_s(\aleph)\big) + s - \beta^{2s^2+2s+1}(\ln\mu)\big(m+\mathbf{N}_s(\aleph)\big)\big\} \\
&\quad \leqslant \exp\big\{s^2(A+9\ln s) + s\ln\big(\mathbf{N}_s(\aleph)\big) + s - \beta^{2s^2+2s+1}(\ln\mu)\mathbf{N}_s(\aleph)\big\} \\
&\quad\quad \times \sum_{m=0}^{\infty} \exp\bigg[\bigg(\frac{s}{\mathbf{N}_s(\aleph)} - \beta^{2s^2+2s+1}(\ln\mu)\bigg)m\bigg].
\end{aligned} \tag{185}$$

We can set the following condition in advance:

$$\mathbf{N}_s(\aleph) > \frac{2s}{\beta^{2s^2+2s+1}\ln\mu}. \tag{186}$$

In this case

$$\begin{aligned}
\sum_{m=0}^{\infty} \exp\bigg[\bigg(\frac{s}{\mathbf{N}_s(\aleph)} - \beta^{2s^2+2s+1}\ln\mu\bigg)m\bigg] &\leqslant \sum_{m=0}^{\infty} \bigg(\exp\bigg(-\frac{1}{2}\beta^{2s^2+2s+1}\ln\mu\bigg)m\bigg) \\
&= \frac{1}{1-\exp(-\frac{1}{2}\beta^{2s^2+2s+1}\ln\mu)}.
\end{aligned} \tag{187}$$

Take $s_0$ such that for $s \geqslant s_0$ the inequality holds $\beta^{2s^2+2s+1}\ln\mu < 1$. For any $y \in (0,1)$ the inequality holds $1/(1-e^{-y}) < 2y^{-1}$. This implies that the value (185) for $s \geqslant s_0$ can be estimated from above by

$$\frac{4}{\beta^{2s^2+2s+1}\ln\mu} \exp\big\{s^2(A+9\ln s) + s + s\ln\big(\mathbf{N}_s(\aleph)\big) - \beta^{2s^2+2s+1}(\ln\mu)\mathbf{N}_s(\aleph)\big\}. \tag{188}$$

To estimate (188) we use the following lemma.

**Lemma 23.** *Given $s \in \mathbb{N}$, for any $X_1, X_2, X_3 > 0$ such that*

$$X_2 + \ln X_1 - s\ln X_3 > s\ln(3s), \tag{189}$$

*and for any $N \geqslant 3sX_3^{-1}(X_2 + \ln X_1 - s\ln X_3)$ the inequality holds*

$$X_1 \exp(X_2 + s\ln N - X_3 N) < 1.$$

**Proof.** Substitution $\tilde{N} = X_3 N$ reduce the statement of Lemma 23 to the following one.

For any $X_1, X_2, X_3 > 0$ such that (189) holds and for any $\tilde{N} \geqslant 3s(X_2 + \ln X_1 - s \ln X_3)$ the inequality holds

$$X_1 \exp(X_2 - s \ln X_3 + s \ln \tilde{N} - \tilde{N}) < 1. \tag{190}$$

Inequality (190) is equivalent to the inequality

$$X_2 + \ln X_1 - s \ln X_3 + s \ln \tilde{N} - \tilde{N} < 0.$$

Denote $Y = X_2 + \ln X_1 - s \ln X_3$. Note that we reduced the proof of Lemma 23 to the following one. □

**Lemma 24.** *Given $s \in \mathbb{N}$, for any $Y > s \ln(3s)$ and $\tilde{N} \geqslant 3sY$ the inequality holds*

$$Y + s \ln \tilde{N} - \tilde{N} < 0. \tag{191}$$

**Proof.** Set $\tilde{N} = 3sY\Delta$, where $\Delta \geqslant 1$. Inequality (191) is equivalent to the inequality

$$Y + s \ln(3s) + s \ln Y + s \ln \Delta - 3sY\Delta < 0. \tag{192}$$

We will prove that

$$Y + s \ln(3s) + s \ln Y - 3sY < 0 \quad \text{and} \tag{193}$$

$$s \ln \Delta - 3sY(\Delta - 1) \leqslant 0. \tag{194}$$

Sum of these two inequalities gives (192) and, therefore, this will prove Lemma 24.

Let us show that (193) holds.

$$\begin{aligned} &-Y + s \ln(3s) < 0 \\ &\quad \Longrightarrow \quad Y(1 - 2s) + s \ln(3s) < 0 \\ &\quad \Longrightarrow \quad Y + s \ln(3s) + sY - 3sY < 0 \\ &\quad \Longrightarrow \quad Y + s \ln(3s) + s \ln Y - 3sY < 0. \end{aligned} \tag{195}$$

Now let us show that (194) holds.

$$\begin{aligned} &\forall s \in \mathbb{N} \quad 1 < s \ln(3s) < Y \\ &\qquad \Longrightarrow \quad 1 - 3Y < 0 \\ &\qquad \Longrightarrow \quad s - 3sY < 0 \\ &\qquad \Longrightarrow \quad s(\Delta - 1) - 3sY(\Delta - 1) \leqslant 0 \\ &\qquad \Longrightarrow \quad s \ln \Delta - 3sY(\Delta - 1) \leqslant 0. \end{aligned} \tag{196}$$

Lemma 24 (and, therefore, Lemma 23) is proved. □

Apply Lemma 23 to

$$X_1 = \frac{4}{\beta^{2s^2+2s+1}\ln\mu}, \qquad X_2 = s^2(A + 9\ln s) + s, \qquad X_3 = \beta^{2s^2+2s+1}\ln\mu. \tag{197}$$

Check the condition (189) for $s \geqslant s_0$:

$$X_1 \geqslant 2, \quad X_3 \leqslant \frac{1}{2} \quad \Longrightarrow \quad X_2 + \ln X_1 - s\ln X_3 > X_2 > As^2 > s\ln(3s)$$

for any $A > 3$. Therefore the quantity (188) is less than 1 (and, hence, the inequality (183) holds) for any

$$\begin{aligned} \mathbf{N}_s(\aleph) \geqslant 3s\frac{(5s^2\aleph^{-1}(1+\Im))^{2s^2+2s+1}}{\ln\mu}\bigl(s^2(A+9\ln s) + s + \ln 4 - (s+1)\ln\ln\mu \\ - \bigl(2s^2+2s+1\bigr)(s+1)\bigl(\ln\aleph - \ln\bigl(5s^2\bigr) - \ln(1+\Im)\bigr)\bigr). \end{aligned} \tag{198}$$

In particular, we can take

$$\mathbf{N}_s(\aleph) = 3s\bigl(\mathcal{B}s^4 - \bigl(2s^2+2s+1\bigr)(s+1)\ln\aleph\bigr)\frac{(5s^2\aleph^{-1}(1+\Im))^{2s^2+2s+1}}{\ln\mu},$$

where $\mathcal{B} = \mathcal{B}(\mathfrak{W})$ depends on parameters of the problem. It is clear that in this case condition (186) holds and we can satisfy the requirement $\mathbf{N}_s(\aleph) > \mathbf{N}^*(s,\aleph)$ (increasing $\mathcal{B}$, if necessary). Addendum 2.1 is proved. □

### *14.2. Proof of Lemma 7*

The proof is by induction on the number of edges. Lemma 7 holds for a graph loop, i.e. a graph with one edge and one vertex.

Now take a connected oriented pseudograph such that at each vertex the number of ingoing edges is equal to the number of outgoing edges. Take any vertex and let us construct an oriented path without repeating edges. After each edge added to the path check whether any two vertices of the path coincide. If no two vertices coincide, we can continue to construct the path. Indeed, at each vertex the number of ingoing edges is equal to the number of outgoing edges. Therefore at last vertex of the path at least one edge is outgoing. The number of edges of the initial pseudograph is finite, so at some moment two vertices of the path coincide. Hence the part of the path which begins and ends at this vertex is a cycle, and by construction this cycle is properly oriented. Remove the edges of this cycle from the initial pseudograph. For any vertex or no edges were removed either one ingoing and one outgoing edge were removed. Therefore for each connected component of the rest (if the rest does contain any edges) the claim of lemma can be applied by induction. This completes the proof of Lemma 7.

## 15. Notations

- $L$ – linear part of the map;
- $0 < \lambda < 1 < \mu$ – eigenvalues of $L$, $\Im = -(\ln\lambda)/(\ln\mu)$;

- $f$ – initial map;
- $q, \tilde{q} = f^{-1}(q)$ – homoclinic points;
- $\{f_{\vec{\varepsilon}}\}$ – family of maps under consideration;
- $G$ – parabolic part of the initial map;
- $\{G_{\vec{\varepsilon}}\}$ – corresponding family;
- $U, \tilde{U}$ – neighborhoods of $q$ and $\tilde{q}$;
- $\delta$ – size of $U$ and $\tilde{U}$;
- $V$ – a neighborhood of a homoclinic contour;
- $\Lambda$ – maximal invariant set of $f$ in $V$;
- $\mathfrak{P} = \{p_0, \ldots, p_{n-1}\}$ – a periodic point of $f_{\vec{\varepsilon}}$;
- $n$ – period of a periodic orbit;
- $s$ and $s' \leqslant s$ – number of loops of periodic orbits;
- $k \leqslant s$ – number of loops of a pseudotrajectories under consideration;
- $(\Phi^1_{\vec{\varepsilon}}, \Phi^2_{\vec{\varepsilon}})$ – family of analytic perturbation;
- $(\Phi^1_{\vec{\varepsilon}_{<2s}}, \Phi^2_{\vec{\varepsilon}_{<2s}})$ – family of perturbations by polynomials of degree $2s-1$;
- $\vec{\varepsilon} = \{\varepsilon^k_{ij}\}$ – family of parameters;
- $\vec{\varepsilon}_{<2s} = \{\varepsilon^k_{ij}\}_{i+j<2s}$ – family of parameters of degree $2s-1$;
- $HB(\zeta)$ – the space of parameters for unbounded cyclicity;
- $HB_{<2s}(\zeta)$ – the space of parameters for cyclicity $s$;
- $\zeta$ – size of the cube of parameters;
- $M_1, M_2$ – $C^1$ and $C^2$-norms of the family, respectively;
- $\nu^k_{ij}$ – probability measure on the interval $[-\zeta, \zeta] \ni \varepsilon^k_{ij}$;
- $\nu$ – probability measure on $HB_{<2s}(\zeta)$;
- $\mathbf{N}$ – lower bound of periods of periodic orbits under consideration;
- $\aleph, C$ – constants in a definition of $(C, \mu, \aleph)$-trace hyperbolic point;
- $\mathcal{N}_s = (n_1, \ldots, n_s)$ – type of an $s$-loop periodic orbit of type $\mathcal{N}_s$;
- $|\mathcal{N}_s| = n_1 + \cdots + n_s + s$ – period of an $s$-loop periodic orbit of type $\mathcal{N}_s$;
- $\mathbf{P}_s = \{\mathbf{p}_0, \ldots, \mathbf{p}_{s-1}\}$ – points of an intersection of a periodic $s$-loop orbit $\mathfrak{P}$ with a neighborhood $U$;
- $\tilde{\mathbf{P}}_s = \{\tilde{\mathbf{p}}_0, \ldots, \tilde{\mathbf{p}}_{s-1}\}$ – points of an intersection of a periodic $s$-loop orbit $\mathfrak{P}$ with a neighborhood $\tilde{U}$;
- $\mathbf{h}_s, C_s, C^*_s, C'_s$ – constants in estimates of the measure of "bad" parameters for a given type of periodic orbits;
- $\beta, \{d_i\}, \{b_i\}$ – constants in a definition of short and long loops;
- $l = l(\mathcal{N}_s)$ – shape of a periodic orbit of type $\mathcal{N}_s$;
- $\mathcal{N}_k \subseteq_l \mathcal{N}_s$ – an $l$-subtype of a type $\mathcal{N}_s$;
- $t$ – number of generalized loops;
- $\tau = t(\mathcal{N}_k)$ – number of generalized loops of scattered pseudo-orbit;
- $N_j$ – length of $j$th generalized loop;
- $n^*_j + 1$ – length of the $j$th long loop;
- $\mathcal{P} = \{P_0, \ldots, P_{t-1}\}$ – starting points of generalized loops;
- $\tilde{P}_j = f^{-1}(P_{j+1})$ – ending point of the $j$th generalized loop;
- $h_j$ – the number of short loops in $j$th generalized loop;
- $K_A(P)$ – vertical cone at point $P$;
- $\angle K_A(P)$ – width (an angle) of $K_A(P)$;
- $(\{\theta_{l,m}\}^s_{l=1}, \xi)$ – sizes of cones;

- $\Pi_n$ – testing rectangle;
- $\tilde{\Pi}_n$ – image of a testing rectangle $\Pi_n$ under $L^n$;
- $\boldsymbol{\Pi}_n(\gamma), \tilde{\boldsymbol{\Pi}}_n(\gamma)$ – grids of size $\gamma \times \gamma\mu^{-n}$ in $\Pi_n$ and of size $\gamma\lambda^n \times \gamma$ in $\tilde{\Pi}_n$, respectively;
- $\{\alpha_{l,i}\}_{l=1}^{s}$ – exponents of sizes of grids;
- $\{\gamma_i\}_{i=0}^{t-1}$ – scales, $\gamma_{t-i} = \mu^{-\alpha_{l,i}n}$, where $(l, n)$ is type;
- $m$ – scale number;
- $k'$ – number of clouds in Cloud decomposition;
- $\mathcal{Z} = \{z_0, \dots, z_{\mathbf{n}-1}\}$ – $k$-loop pseudotrajectory;
- $\mathbf{n}$ – period of a pseudotrajectory;
- $\mathbf{Z}_k = \{\mathbf{z}_0, \dots, \mathbf{z}_{k-1}\}$ – intersection of $\mathcal{Z}$ with $U$;
- $\tilde{\mathbf{Z}}_k = \{\tilde{\mathbf{z}}_0, \dots, \tilde{\mathbf{z}}_{k-1}\}$ – intersection of $\mathcal{Z}$ with $\tilde{U}$;
- $\mathfrak{R} = \{r_0, \dots, r_{\mathbf{n}-1}\}$ – admissible pseudotrajectory of period $\mathbf{n}$;
- $\mathcal{R} = \{R_0, \dots, R_{t-1}\}$ – starting points of generalized loops of an admissible pseudotrajectory;
- $\mathbf{R}_k = \{\mathbf{r}_0, \dots, \mathbf{r}_{k-1}\}$ – intersection of $\mathfrak{R}$ with $U$;
- $\tilde{\mathbf{R}}_k = \{\tilde{\mathbf{r}}_0, \dots, \tilde{\mathbf{r}}_{k-1}\}$ – intersection of $\mathfrak{R}$ with $\tilde{U}$;
- $F^N_{\mathcal{Z},\vec{\varepsilon}}(p)$ – composition of differentials of $f_{\vec{\varepsilon}}$ in $N$ subsequent points of $\mathcal{Z}$ starting at $p \in \mathcal{Z}$;
- $i_0, i_1, \dots, i_{\tau-1}$ – indices of starting points of generalized loops of admissible pseudo-orbit;
- $\tau_j$ – the number of starting points of generalized loops among first $j$ points $\mathbf{R}_j$ of admissible pseudo-orbit;
- $W_{k,2}$ – the space of 2-component homogeneous polynomials of degree $k$ in $(x, y)$;
- $W^{u,\mathbf{R}_k}_{k,2}$ – the space of 2-component homogeneous polynomials of degree $k$ in $(x, y)$ vanishing at all the points $\mathbf{R}_k$ with the standard basis;
- $\mathcal{L}^2_{\mathbf{R}_k}$ – the Newton map of $W_{\leqslant 2s,2}$ to $W^{u,\mathbf{R}_k}_{\leqslant 2s,2}$;
- $m(v)$ – the index of maximal component of vector $v$;
- $(p; p_0, \dots, p_{k-1})^{\alpha}$ – the standard Newton monomial, where $|\alpha| = k$;
- $\alpha(\Gamma)$ – a symbol which stands for a multiindex $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_+$ and an oriented right-up path $\Gamma_\alpha$ connecting $\alpha$ with the origin;
- $\mathcal{I}_s$ – the union of symbols $\alpha(\Gamma)$ over all $|\alpha| \leqslant s$;
- $(p; p_0, \dots, p_{k-2})^{\alpha(\Gamma)}$ – the Newton monomial associated with the corresponding oriented path $\Gamma_\alpha$;
- $Q^{\mathrm{dyn}}_0(r, \mathbf{R}_k)$ – the dynamically essential Newton monomial vanishing at $\mathbf{R}_k$;
- $\mathbf{Q}^{\mathrm{dyn}}_{\mathbf{R}_k} = \{Q^{\mathrm{dyn}}(p, \tilde{\mathbf{R}}_1), Q^{\mathrm{dyn}}_m(p, \tilde{\mathbf{R}}_1, \tilde{\mathbf{R}}'_1), \dots, Q^{\mathrm{dyn}}(p, \tilde{\mathbf{R}}_k), \ Q^{\mathrm{dyn}}_m(p, \tilde{\mathbf{R}}_k, \tilde{\mathbf{R}}'_k)\}$, where $m = 1, 2$ – complete set of dynamically essential monomials;
- $W^{\mathrm{dyn},\mathbf{R}_k}_{k,2}$ – the space of 2-component homogeneous polynomials of degree $k$ in $(x, y)$ vanishing at all the points $\mathbf{R}_k$ with the basis containing $\mathbf{Q}^{\mathrm{dyn}}_{\mathbf{R}_k}$;
- $\mathcal{L}^{2,\mathrm{dyn}}_{\mathbf{R}_k}$ – the dynamical Newton map of $W_{\leqslant 2s,2}$ to $W^{\mathrm{dyn},\mathbf{R}_k}_{\leqslant 2s,2}$;
- $p_{k,m}(x_0, \dots, x_m)$ – homogeneous polynomial of all monomials of degree $k - m$ in $x_0, \dots, x_m$ with unit coefficients;
- $\mathcal{DD}^{2,s}(B^2, \mathbb{R}^2)$ – the space of divided differences;
- $\mathbb{N}$ – the set of positive integer numbers;
- $\mathbb{Z}_+$ – the set of nonnegative integer numbers.

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## Appendix A. Infinite number of localized sinks of bounded cyclicity and Kupka–Smale property

**Definition 34.** A diffeomorphism $f : M \to M$ of a closed manifold $M$ is called a Kupka–Smale diffeomorphism, if

(1) all periodic points of $f$ are hyperbolic;
(2) for any two (not necessarily different) periodic points the stable manifold of one is transversal to the unstable manifold of the other.

Suppose that $M$ is a closed manifold of dimension $d \geqslant 2$, $f : M \to M$ is a diffeomorphism with hyperbolic periodic point $p$ of period $k$, i.e. $f^k(p) = p$. Denote $\dim W^u(p) = d^u$, $\dim W^s(p) = d^s$, then $d^u + d^s = d$. Let $q \in W^u(p) \cap W^s(p) \setminus p$ be a homoclinic point. For small $\rho$ we define a neighborhood $U_\rho(q)$ of a homoclinic orbit $\{f^i(q)\}_{i\in\mathbb{Z}}$ in the following way. Set $\tilde{q} = f^{-k}(q)$. Let $V'_\rho(p)$ be a $\rho$-neighborhood of $p$. Since $p$ is a hyperbolic periodic point and $\rho$ is small, one can assume that $f^k|_{V'_{\rho'}(p)}$ is topologically conjugated to $Df^k(p)$. There are integers $n_1$ and $n_2$ such that $f^{kn_1}(q) \in V'_\rho(p)$ and $f^{-kn_2}(\tilde{q}) \in V'_\rho(p)$. Choose a small neighborhood $U(q)$ of $q$ such that $f^{kn_1}(U(q)) \subset V'_\rho(p)$. Choose a small neighborhood $\tilde{U}(\tilde{q})$ of $\tilde{q}$ such that $f^{-kn_2}(\tilde{U}(\tilde{q})) \subset V'_\rho(p)$ and $f^k(\tilde{U}(\tilde{q})) \subset U(q)$. Finally, set

$$\mathcal{V}_\rho(p) = V'_\rho(p) \bigcup_{i=0}^{kn_1} f^i\big(U(q)\big) \bigcup_{i=-kn_2}^{k-1} f^i\big(\tilde{U}_\rho(q)\big).$$

Definition of $(U(q), s)$-localized periodic orbits is the same as in Definition 3 with $V = \mathcal{V}_\rho(p)$ and $U = U(q)$.

**Proposition 6.** *Suppose a $C^1$-diffeomorphism $f : M \to M$ has a hyperbolic periodic point $p \in M$ with a homoclinic point $q$. Consider a neighborhood $\mathcal{V}_\rho(p)$, associated with this homoclinic point defined above. For any $s \in \mathbb{N}$ existence of an infinite number of $(U(q), s)$-localized sinks implies that $f$ is not a Kupka–Smale diffeomorphism.*

**Proof.** Consider a sequence of $(U(q), s)$-localized sinks. Numerate them. Each sink intersects a neighborhood $U(q)$ at $s$ points, denoted by $\{\mathbf{p}_1^m, \mathbf{p}_2^m, \ldots, \mathbf{p}_s^m\}$ for $m$th sink. Set $\mathbf{p}_{s+1}^m = \mathbf{p}_1^m$.

Denote $\tilde{\mathbf{p}}_l^i = f^{-k}(\mathbf{p}_{l+1}^i) \in \tilde{U}(q)$, $l = 1, \dots, s$. For each $i \in \mathbb{N}$ and $l = 1, \dots, s$ there exists $N_l^i \in \mathbb{N}$ such that

$$f^{kN_l^i}\left(\mathbf{p}_l^i\right) = \tilde{\mathbf{p}}_l^i.$$

One can choose a subsequence in such a way that the corresponding sequence of finite sets has a limit:

$$\left\{\mathbf{p}_1^i, \mathbf{p}_2^i, \dots, \mathbf{p}_s^i\right\} \to \left\{\mathbf{p}_1^*, \mathbf{p}_2^*, \dots, \mathbf{p}_s^*\right\} \quad \text{as } i \to +\infty.$$

It is clear that the following lemma holds.

**Lemma 25.** *For each* $1 \leqslant l \leqslant s$ *there exists a limit* $\lim_{i\to\infty} N_l^i = N_l^* \in \mathbb{N} \cup \{\infty\}$, *and*

$$\mathbf{p}_l^* \in W^s(p) \quad \Leftrightarrow \quad \tilde{\mathbf{p}}_l^* \in W^u(p) \quad \Leftrightarrow \quad N_l^* = \infty.$$

Since a limit of a sequence of periodic points of the same period has to be a non-hyperbolic periodic point we have: if all the limit points $\{\mathbf{p}_1^*, \mathbf{p}_2^*, \dots, \mathbf{p}_s^*\}$ does not belong to $W^s(p)$, then the map $f$ is not a Kupka–Smale diffeomorphism.

Assume that $t$ points out of $s$ points $\{\mathbf{p}_1^*, \mathbf{p}_2^*, \dots, \mathbf{p}_s^*\}$ belong to $W^s(p)$ for some $t \geqslant 1$. Denote them by $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_t\} \subseteq \{\mathbf{p}_1^*, \mathbf{p}_2^*, \dots, \mathbf{p}_s^*\}$. This implies that $t$ out of $s$ points $\{\tilde{\mathbf{p}}_1^*, \tilde{\mathbf{p}}_2^*, \dots, \tilde{\mathbf{p}}_s^*\}$ belong to $W^u(p)$. Denote them by $\{\tilde{\mathbf{q}}_1, \tilde{\mathbf{q}}_2, \dots, \tilde{\mathbf{q}}_t\} \subseteq \{\tilde{\mathbf{p}}_1^*, \tilde{\mathbf{p}}_2^*, \dots, \tilde{\mathbf{p}}_s^*\}$. Set $\mathbf{q}_{t+1} = \mathbf{q}_1$, $\tilde{\mathbf{q}}_{t+1} = \tilde{\mathbf{q}}_1$, $N_{\max}^{\#} = (s-t) + \sum_{N_l^* \neq \infty} N_l^*$. For each $j = 1, \dots, t$ there exists $N_j^{\#} \leqslant N_{\max}^{\#}$ such that

$$f^{kN_j^{\#}}(\tilde{\mathbf{q}}_j) = \mathbf{q}_{j+1}.$$

In particular, all $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_t\}$ and $\{\tilde{\mathbf{q}}_1, \tilde{\mathbf{q}}_2, \dots, \tilde{\mathbf{q}}_t\}$ are homoclinic points.

To show absence of sinks we shall construct an invariant cone field for $f$ in an open subset of $\mathcal{V}_\rho(p)$ so that trajectories $\{\mathbf{p}_j^i\}_{1 \leqslant j \leqslant m}$ visit it for large $i$. Below we construct both: an invariant cone field and a subset.

There exists a continuous invariant (under $f^k$) splitting of a tangent bundle of $\mathcal{V}_\rho(p)$, $T_xM = E_x^s \oplus E_x^u$ for all $x \in \mathcal{V}_\rho(p)$, $\dim E_x^s = d^s$ and $\dim E_x^u = d^u$. By iterations of $f^k$ (respectively $f^{-k}$) we can extend this splitting to $U(q)$ (respectively $\tilde{U}(\tilde{q})$). Assume that stable and unstable manifolds of the points $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_t$ intersect transversally (otherwise $f$ is not Kupka–Smale), i.e. $Df^{kN_j^{\#}}(E_{\tilde{\mathbf{q}}_j}^u) \oplus E_{\mathbf{q}_{j+1}}^s = T_{\mathbf{q}_{j+1}}M$. Consider a cone field in $\tilde{U}(\tilde{q})$ (respectively $U(q)$) of the following form:

$$\forall x \in \tilde{U}(\tilde{q}) \quad \tilde{K}_x^u(A) = \left\{v \in T_xM \;\middle|\; v = v^u + v^s,\ v^u \in E_x^u,\ v^s \in E_x^s,\ \left|v^u\right| \geqslant A\left|v^s\right|\right\},$$
$$\forall y \in U(q) \quad K_y^u(A) = \left\{v \in T_yM \;\middle|\; v = v^u + v^s,\ v^u \in E_y^u,\ v^s \in E_y^s,\ \left|v^u\right| \geqslant A^{-1}\left|v^s\right|\right\}.$$

Since $Df^{kN_j^{\#}}(E_{\tilde{\mathbf{q}}_j}^u) \oplus E_{\mathbf{q}_{j+1}}^s = T_{\mathbf{q}_{j+1}}M$, for $A$ large enough and each $j = 1, \dots, t$ we have

$$Df^{kN_j^{\#}}\left(\tilde{K}_{\mathbf{q}_j}^u(A)\right) \subset K_{\mathbf{q}_{j+1}}^u(A).$$

Take small $\delta > 0$ and consider $\delta$-neighborhoods $U_\delta(\tilde{\mathbf{q}}_j)$, $j = 1, \dots, t$. Due to the continuity of stable and unstable subbundles, if $A$ is large and $\delta$ is small enough, for all $x \in \tilde{U}_\delta(\tilde{\mathbf{q}}_j)$ we have

$$Df^{kN_j^\#}\big(\tilde{K}_x^u(A)\big) \subset K^u_{f^{kN_j^\#}(x)}(A).$$

Decreasing $\delta$, if necessary, we can also get the following property. For any point $y \in U_\delta(\tilde{\mathbf{q}}_j)$ such that for some positive integer $N_y$ the finite orbit $\{y, f^k(y), \dots, f^{N_y k}(y)\}$ forms a one loop, i.e. belongs to a set

$$\mathcal{V}_\rho(p) \bigcup_{i=1}^{n_1} f^{ik}\big(U(q)\big) \bigcup_{i=-n_2}^{0} f^{ik}\big(\tilde{U}(\tilde{q})\big),$$

and with $f^{ik}(y) \in U(q)$ only for $i = N_y$ the following inclusion holds

$$Df^{N_y k}\big(K_y^u(A)\big) \subset \tilde{K}^u_{f^{N_y k}(y)}(A).$$

Moreover,

$$\forall v \in K_y^u(A) \quad \big|Df^{N_y k}(v)\big| \geqslant 2M_1^{N^\#_{\max}}|v|,$$

where $M_1 = \max\{\|f\|_{C^1}, \|f^{-1}\|_{C^1}\}$.

Therefore for any $s$-loop periodic orbit $\mathfrak{P}$ from the initial subsequence whose intersection with $U(q)$ is close enough to the set $\{\mathbf{p}_1^*, \mathbf{p}_2^*, \dots, \mathbf{p}_s^*\}$ we have the following combinatorics. By construction return $\mathfrak{P} \cap U$ are either close to $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_t\}$ and belong to their $\delta$-neighborhood or do not belong there and length of the corresponding loops following after them is bounded. More exactly, the sum of lengths of all such loops is bounded by $N^\#_{\max}$. This implies distortion of vectors is bounded and, therefore, for any $y \in U_\delta(\mathbf{q}_j)$ that is a starting point of a loop,

$$\forall v \in K_y^u(A) \quad \big|Df^{kN_y + kN_j^\#}(v)\big| \geqslant 2|v| \quad \text{and}$$
$$Df^{kN_y + kN_j^\#}(v) \in K^u_{f^{kN_y + kN_j^\#}(y)}(A).$$

Therefore, some vectors in its tangent space are expanding after each "generalized" loop, i.e. a loop starting at $U_\delta(\mathbf{q}_i)$ and ending at $U_\delta(\mathbf{q}_j)$ for some $1 \leqslant i, j \leqslant t$. This is impossible for a sink. This contradiction proves Proposition 6. □

## Appendix B. Estimate of measure of "non-hyperbolic" parameters

Here we reproduce the proof of Proposition 5 (Proposition A.5. from [22]).

**Proof of Proposition 5.** For $0 < \gamma \leqslant 1$ and $\phi \in [0, 1)$, define the sets of non-$\gamma$-hyperbolic matrices by

$$NH_N^{\gamma}(\mathbb{R}) = \{L \in M_N(\mathbb{R}) \mid \mathrm{Hyp}(L) \leqslant \gamma\},$$
$$NH_N^{\gamma,\phi}(\mathbb{R}) = \Big\{L \in M_N(\mathbb{R}) \;\Big|\; \inf_{|v|=1} \big|\big(L - \exp(2\pi i\phi)\big)v\big| \leqslant \gamma\Big\}. \tag{198}$$

Then

$$NH_N^{\gamma}(\mathbb{R}) = \bigcup_{\phi\in[0,1)} NH_N^{\gamma,\phi}(\mathbb{R}). \tag{199}$$

We claim that

$$NH_N^{\gamma}(\mathbb{R}) \subset \bigcup_{j=0,\dots,[5/\gamma]-1} NH_N^{2\gamma,j/[5/\gamma]}(\mathbb{R}). \tag{200}$$

Indeed, suppose that $L \in NH_N^{\gamma}(\mathbb{R})$. Then for some number $\phi \in [0,1)$ and vector $v \in \mathbb{R}^N$ with $|v| = 1$, we have $|(L - \exp(2\pi i\phi))v| \leqslant \gamma$. Let $j$ be the nearest integer to $[5/\gamma]\phi$ and let $\phi_\gamma = j/[5/\gamma]$; then $\phi - \phi_\gamma \leqslant 1/(2(5/\gamma - 1)) < \gamma/(2\pi)$. Thus

$$\big|\big(L - \exp(2\pi i\phi_\gamma)\big)v\big| \leqslant \big|\big(L - \exp(2\pi i\phi)\big)v\big| + \big|\exp(2\pi i\phi) - \exp(2\pi i\phi_\gamma)\big| \leqslant 2\gamma, \tag{201}$$

and $L \in NH_N^{2\gamma,j/[\pi/\gamma+1]}(\mathbb{R})$ as claimed.

Next, we claim that every matrix in $NH_N^{2\gamma,j/[5/\gamma]}(\mathbb{R})$ lies within $2\gamma$ of a matrix in $NH_N^{0,j/[5/\gamma]}(\mathbb{R})$, where we use the Euclidean ($\mathbb{R}^{N^2}$) norm on $M_N(\mathbb{R})$ (not the matrix norm). Consider $L \in NH_N^{2\gamma,j/[5/\gamma]}(\mathbb{R})$, $\phi \in [0,1)$, and $v \in \mathbb{R}^N$ with $|v| = 1$ and

$$\big|\big(L - \exp\big(2\pi ij/[5/\gamma]\big)\big)v\big| \leqslant 2\gamma.$$

Let $w = (L - \exp(2\pi ij/[5/\gamma]))v$ and let $M \in M_N(\mathbb{R})$ be the matrix whose $k$th row is $w_k v$, where $w_k$ is the $k$th coordinate of $w$. Then the Euclidean norm of $M$ is $|w| \leqslant 2\gamma$ and $Mv = w$, so that $(L - M - \exp(2\pi ij/[5/\gamma]))v = 0$ and hence $L - M \in NH_N^{0,j/[5/\gamma]}(\mathbb{R})$.

We complete the estimate (165) by estimating for each $j$ the number of $\gamma$-balls needed to cover $NH_N^{0,j/[5/\gamma]}(\mathbb{R})$ within an appropriate bounded domain. It then follows from the previous paragraph that if we inflate each of these balls to the concentric ball of radius $3\gamma$, the collection of larger balls will cover $NH_N^{2\gamma,j/[5/\gamma]}(\mathbb{R})$, and from the paragraph before that the union over $j$ of these covers will then cover $NH_N^{\gamma}(\mathbb{R})$. To this end, we show that each $NH_N^{0,j/[5/\gamma]}(\mathbb{R})$ is a real algebraic set and compute its codimension.[12] Then we will apply an estimate of Yomdin [42] on the number of $\gamma$-balls necessary to cover a given algebraic set by polynomials of known degree.

Notice that

$$NH_N^{0,\phi}(\mathbb{R}) = \big\{L \in M_N(\mathbb{R}) \mid \det\big(L - \exp(2\pi i\phi)Id\big) = 0\big\}. \tag{202}$$

[12] Unfortunately $NH_N^{0}(\mathbb{R})$, in contrast to $NH_N^{0,j/[5/\gamma]}(\mathbb{R})$, is not algebraic.

We split into the two cases $\exp(2\pi i\phi) \in \mathbb{R}$ (that is, $\phi = 0$ or $1/2$) and $\exp(2\pi i\phi) \notin \mathbb{R}$. In the first case, the equation $\det(L \pm Id) = 0$ is a polynomial of degree $N$ in the entries of $L$, so $NH_N^{0,0}(\mathbb{R})$ and $NH_N^{0,1/2}(\mathbb{R})$ are real algebraic sets defined by a single polynomial of degree $N$.

In the second case, decompose the equation $\det(L - \exp(2\pi i\phi)Id) = 0$ into two parts: $\mathrm{Re}[\det(L - \exp(2\pi i\phi)Id)] = 0$ and $\mathrm{Im}[\det(L - \exp(2\pi i\phi)Id)] = 0$. Each part is given by a real polynomial of degree $N$. Furthermore, these two polynomials are algebraically independent, since otherwise $\mathrm{Re}[\det(L - \exp(2\pi i\phi)Id)]$ and $\mathrm{Im}[\det(L - \exp(2\pi i\phi)Id)]$ would satisfy some polynomial relationship which would imply that $\det(L - \exp(2\pi i\phi)Id)$ takes on values only in some real algebraic subset of the complex plane. However, for $N \geqslant 2$ (which is necessary for complex eigenvalues), by considering real diagonal matrices $L$ we see that the values of $\det(L - \exp(2\pi i\phi)Id)$ contain an open set in $\mathbb{C}$. Therefore, $NH_N^{0,\phi}(\mathbb{R})$ is a real algebraic set given by two algebraically independent polynomials of degree $N$.

**Covering Lemma for Algebraic Sets.** [42, Lemma 4.6] *Let $V \subset \mathbb{R}^m$ be an algebraic set given by $k$ algebraically independent polynomials $p_1, \dots, p_k$ of some degrees $d_1, \dots, d_k$ respectively, i.e. $V = \{x \in \mathbb{R}^m \mid p_1(x) = 0, \dots, p_k(x) = 0\}$. Let $C_A^m(s)$ be the cube in $\mathbb{R}^m$ with side $2s$ centered at some point $A$. Then for $\gamma \leqslant s$, the number of $\gamma$-balls necessary to cover $V \cap C_A^m(s)$ does not exceed $C(D,m)(2s/\gamma)^{m-k}$, where the constant $C(D,m)$ depends only on the dimension $m$ and product of degrees $D = \prod_i d_i$.*

**Remark 18.** Some additional arguments based on Bezout's Theorem give an upper estimate of $C(D,m)$ by $2^m D$ for $\gamma$ sufficiently small.

To complete the proof of Proposition 5, we apply the Covering Lemma for Algebraic Sets to each $NH^{0,j/[5/\gamma]}(\mathbb{R})$, where $j = 0, \dots, [5/\gamma] - 1$, with $m = N^2$, $s = Kr$, and $A$ as in the statement of the proposition. (Notice that if $U \in C^{N^2}(r)$ then $A + UB \in C_A^{N^2}(Kr)$, so we need only cover the part of $NH^{0,j/[5/\gamma]}(\mathbb{R})$ lying in the latter set.) In the case that $j/[5/\gamma] = 0$ or $1/2$, we have $k = 1$, $d_1 = N$, and $D = N$, so the number of covering $\gamma$-balls is bounded by $C(N, N^2)(2Kr/\gamma)^{N^2-1}$. In the case of other $j$, we have $k = 2$, $d_1 = d_2 = N$, and $D = N^2$, so the number of covering $\gamma$-balls is bounded by $C(N^2, N^2)(2Kr/\gamma)^{N^2-2}$. The number of $j$'s of the second type is less than $5/\gamma$. Combining all these estimates along with (200) we get that $NH^\gamma(\mathbb{R}) \cap C_A^{N^2}(Kr)$ can be covered by $C(N^2, N^2)(2 + 5/(2Kr))(2Kr/\gamma)^{N^2-1}$ balls of radius $3\gamma$.

Finally, notice that the preimage of a ball of radius $3\gamma$ under the map $U \mapsto A + UB$ is contained in a ball of radius $3K\gamma$, whose $\mu_{r,N^2}$-measure is less than $(3K\gamma/r)^{N^2}$. Therefore the total measure of $3K\gamma$-balls needed to cover the set $\{U \in C^{N^2}(r) \mid \mathrm{Hyp}(A + UB) \leqslant \gamma\}$ is at most $C(N)K^{2N^2}\gamma/r^2$, where the constant $C(N)$ depends only on $N$.

Proposition 5 is proved. □

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