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Certain Properties of Skew Products over a Horseshoe and a Solenoid

A. S. Gorodetski[1] **and Yu. S. Ilyashenko**[2]

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Abstract—The skew products are investigated over the Bernoulli shift and the Smale–Williams solenoid with a fiber S^1. It is assumed that the mapping in the fiber Hölder continuously depends on a point in the base (it is these skew products that arise in the study of partially hyperbolic sets). It is proved that, in the space of skew products with this property, there exists an open domain such that the mappings from this domain have dense sets of periodic orbits that are attracting and repelling along the fiber, as well as the dense orbits with the zero (along the fiber) Lyapunov exponent.

1. INTRODUCTION

1.1. Actions of free semigroups and cascades. There is a relation between the dynamics generated by the action of free semigroups by diffeomorphisms on compact manifolds, on the one hand, and the classical dynamical systems—iterations of diffeomorphisms on large-dimension manifolds, on the other.

At first glance, the orbits of actions of a free semigroup may be much more complicated as compared with the orbits of action of a cyclic group. However, at the cost of increasing the dimension of the phase space, *all phenomena observed in the dynamics generated by the action of a free semigroup can be translated to diffeomorphisms.*

This heuristic principle is realized by the following construction. The action of a free semigroup with N generators f_i—the diffeomorphisms of a compact manifold M—are closely related to the action of one homeomorphism on the Cartesian product of the space Σ^N of bi-infinite sequences containing N symbols by the manifold M. The space Σ^N is the base of this product, while the manifold M is its fiber. The aforementioned homeomorphism—a skew product over the Bernoulli shift $\sigma\colon \Sigma^N \to \Sigma^N$—is defined by the formula

$$F\colon\ \Sigma^N \times M \to \Sigma^N \times M, \quad (\omega, \varphi) \mapsto (\sigma\omega, f_{\omega_0}(\varphi)). \tag{1.1}$$

The action orbits of a semigroup with generators f_i coincide with the projections of positive semitrajectories of the mapping F onto the fiber along the base. Thus, all phenomena observed in the dynamics of the action of a free semigroup on M are also observed in the cascade generated by the homeomorphism (1.1).

[1]Independent Moscow University, Bol'shoi Vlas'evskii per. 11, Moscow, 121002 Russia. E-mail: asgor@mccme.ru
[2]Independent Moscow University, Bol'shoi Vlas'evskii per. 11, Moscow, 121002 Russia, and Moscow State University, Vorob'evy gory, Moscow, 119899 Russia. E-mail: yulijs@mccme.ru

The skew product (1.1) is called *step* for the following reason. Consider the mapping that differs from (1.1) by the fact that a mapping in the fiber depends on the *entire sequence in the base* rather than on its zero element:

$$G\colon\ \Sigma^N \times M \to \Sigma^N \times M, \quad (\omega, \varphi) \mapsto (\sigma\omega, f_\omega(\varphi)). \tag{1.2}$$

Such skew products are called *soft*. The step products are distinguished among soft products by the requirement

$$f_\omega = f_{\omega_0}. \tag{1.3}$$

For these products, a mapping in the fiber is constant on the cylinder consisting of all sequences with the common zero element. That is why they are called step products.

One can construct a smooth implementation of the homeomorphism (1.1). Consider N nonintersecting "horizontal" rectangles D_i and the mapping of Smale's horseshoe S of the union D of these rectangles onto the union D' of N nonintersecting "vertical" rectangles. The diffeomorphism S has a hyperbolic invariant set Λ such that the restriction of this diffeomorphism onto this set is topologically equivalent to the Bernoulli shift $\sigma\colon \Sigma^N \to \Sigma^N$. Consider the diffeomorphism

$$\mathfrak{F}\colon\ D \times M \to D' \times M, \quad \mathfrak{F}(x, \varphi) = (S(x), f_x(\varphi)), \tag{1.4}$$

$$f_x = f_i \qquad \text{for} \quad x \in D_i. \tag{1.5}$$

The mapping $\mathfrak{F}$ has the invariant set $\Lambda \times M$. A restriction of $\mathfrak{F}$ onto this set is conjugate to the mapping (1.1). Thus, the properties found during the investigation of a semigroup generated by the diffeomorphisms $f_i\colon M \to M$ are realized by the smooth mapping $\mathfrak{F}$. However, this mapping is extremely degenerate. The last (and the most difficult) step of the program is to prove that these properties are preserved under a small perturbation of the mapping $\mathfrak{F}$.

The results obtained in this course are presented in [1]. In that work, theorems are either supplied with the sketches of proofs or presented without proofs at all. The present paper is the first one in the series of works that discuss this problem in detail. Here, we investigate the skew products (1.4) over the Smale horseshoe and analogous but more complicated skew products over the solenoid. The skew products over the horseshoe have invariant sets; almost all points that do not belong to these sets spend only finite time in their neighborhoods. In contrast, the invariant sets of skew products over a solenoid that are considered below are maximal attractors, and their investigation is of special interest.

1.2. Smooth dynamical systems and soft skew products. Small perturbations of mappings (1.4) satisfying certain expansion and contraction requirements have invariant partially hyperbolic sets that are homeomorphic to the product $\Sigma^N \times M$ (see Theorem 7.1 in [2] as well as Theorem 2 in [1]). The fibers of this invariant set Hölder continuously depend on the base points (Theorem 3 in [1]). This result, proved by the first author, makes up the content of a separate paper.

The restriction of a small perturbation of mapping (1.4) onto the invariant set described above has the form (1.2). The aforementioned Hölder dependence of fibers on a base point motivates the following requirement for the soft skew products that will be considered below:

$$d_{C^r}(f_\omega, f_{\omega'}) \le C(d_{\Sigma^N}(\omega, \omega'))^\alpha, \tag{1.6}$$

where C and α are certain positive constants. A more detailed motivation of this requirement of the Hölder dependence of a mapping in a fiber on a base point is discussed in Subsection 4.1.

1.3. Main results. In Section 2, we describe the properties of step skew products and, in Section 3, the properties of soft skew products. It turns out that the Hölder property (1.6) allows one to translate the properties of step skew products to the case of soft products.

Theorem 1 from Section 2 states that there exists an open set in the space of step systems (1.1) such that the mappings from this set have

(i) dense sets of periodic points of different indices and

(ii) dense orbits with the zero Lyapunov exponent.

Theorem 2 from Section 3 states that there exists an open set in the space of soft systems (1.2) with the Hölder property (1.6) that has the same properties. Together with the theory of invariant partially hyperbolic sets mentioned in Subsection 1.2, these results allow one to prove that properties (i) and (ii) for smooth dynamical systems are robust.

In Section 4, we consider the soft dynamical systems that can be implemented by the smooth systems

$$\mathfrak{F}\colon\ T\times M\to T\times M,\quad \mathfrak{F}(x,\varphi)=(S(x),f_x(\varphi)),$$

under the assumption that T is a solid torus and $S\colon T\to T$ is a Smale–Williams solenoid rather than the Smale horseshoe. Property (i) is proved for these systems. Combined with the theory mentioned in Subsection 1.2, this allows us to prove the existence of attractors of smooth mappings with dense sets of periodic points of different indices and argue that this property is robust. The conjecture about the existence of such mappings was formulated in [3] and proved in [4]. Our methods, in particular, allow one to add to this property the presence of a dense orbit with the zero intermediate Lyapunov exponent.

The results of Sections 2 and 3 and Subsection 4.1 are due to the first author, while the other results of Section 4 are due to the second.

2. ROBUST PROPERTIES OF STEP SKEW PRODUCTS

2.1. Properties of step systems: Statement. In this section, we consider the iterations of step mappings (1.1). It is obvious that, for $n>0$,

$$F^n(\omega,\varphi)=(\sigma^n\omega,\bar f_n[\omega](\varphi)),$$

where

$$\bar f_n[\omega]=f_{\omega_{n-1}}\circ\ldots\circ f_{\omega_0}, \tag{2.1}$$

and

$$F^{-n}(\omega,\varphi)=(\sigma^{-n}\omega,\bar f_{-n}[\omega](\varphi)),$$

where

$$\bar f_{-n}[\omega]=f^{-1}_{\omega_{-n}}\circ\ldots\circ f^{-1}_{\omega_{-1}}. \tag{2.2}$$

Let $\bar f_0[\omega]=\mathrm{id}$. We will also use the notation $C_{\bar\alpha}=\{\omega\in\Sigma^2\mid\omega_j=\alpha_j,\ -n\le j\le n-1\}$, where $\bar\alpha=\alpha_{-n}\ldots\alpha_0\ldots\alpha_{n-1}$ is a segment of 0 and 1.

The proofs of the exotic properties of orbits of the mapping F are based on the fact that, choosing the initial condition, we first choose a sequence ω; it is our prerogative to choose this sequence. Using the freedom of action provided by the action of a free semigroup on a fiber, we can add zeros and ones to the constructed segment of the sequence ω so that the superpositions (2.1) and (2.2) have the required properties.

Theorem 1. *There exist an interval $I \subset \mathbb{R}$, $0 \in I$, and open sets $U_0, U_1 \subset \mathrm{Diff}^1(S^1)$ such that, for any f_0 and f_1, $f_j \in U_j$, the mapping F possesses the following properties.*

(i) *Periodic orbits of the mapping F with the absolute value of multiplier along the circle greater than one are dense in $\Sigma^2 \times S^1$. The same property holds for the periodic orbits with the absolute value of multiplier less than one.*

(ii) *For any $\lambda \in I$, there exists an orbit, dense in $\Sigma^2 \times S^1$, whose Lyapunov exponent along the circle is equal to λ.*

(iii) *If the orbit of the sequence $\omega \in \Sigma^2$ under the Bernoulli shift σ is dense in Σ^2, then, for any $\varphi \in S^1$, the orbit of the point (ω, φ) under the action of F is dense in $\Sigma^2 \times S^1$.*

As the sets U_0 and U_1, we take small C^1-neighborhoods of the following diffeomorphisms g_0 and g_1.

Let g_0 be a rotation of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ by a small angle $b < \frac{1}{20}$. Let g_1 be an orientation-preserving diffeomorphism with two fixed points, an attractor p_0 and repeller q_0; all other points move from q_0 to p_0 under the action of positive iterations of g_1. Suppose also that $q_0 = 0$, $p_0 = \frac{1}{2}$, and g_1 is linear in certain neighborhoods of q_0 and p_0; namely, on $O(q_0) = (-\frac{1}{8}, \frac{1}{8}) \subset S^1$, the mapping g_1 is an expansion: $\varphi \mapsto a\varphi$, $a > 1$; on $O(p_0) = (\frac{3}{8}, \frac{5}{8}) \subset S^1$, the mapping g_1^{-1} has a similar form: $\varphi \mapsto a(\varphi - \frac{1}{2}) + \frac{1}{2}$; and $\max_{S^1} |Dg_1| \le a$, $\max_{S^1} |Dg_1^{-1}| \le a$.

In Theorem 1, we take, as the neighborhood U_i, $i = 0, 1$, a set of diffeomorphisms f_i that satisfy the condition

$$d_{C^1}(f_i, g_i) < \delta, \qquad i = 0, 1, \tag{2.3}$$

where δ is small as compared with $\ln a$ and b. Thus, the mapping f_0 is close to the rotation, and the mapping f_1 has two fixed points: an attractor p and repeller q. Denote by W^+ the arc $(-\frac{1}{8} + \delta, \frac{1}{8} - \delta)$ and by W^- the arc $(\frac{3}{8} + \delta, \frac{5}{8} - \delta)$. The mapping f_1 expands the arc W^+ (namely, $|f_1'|_{W^+}| > a - \delta$) and contracts the arc W^- (namely, $|f_1'|_{W^-}| < a^{-1} + \delta$).

Let us choose and fix arbitrary $f_0 \in U_0$ and $f_1 \in U_1$ and prove that the corresponding mapping F (1.1) possesses properties (i), (ii), and (iii).

2.2. Distortion of arcs.

Lemma 2.1 (on the distortion of arcs). *Suppose that a segment $\bar\alpha = \alpha_{-n} \dots \alpha_{-1}\alpha_0 \dots \alpha_{n-1}$ (of 0 and 1) and an arc $J \subset S^1$ are given. Then, there exist segments $\bar\beta = \beta_{-m} \dots \beta_0 \dots \beta_{m-1}$, $\bar\beta' = \beta'_{-m'} \dots \beta'_0 \dots \beta'_{m'-1}$, and $\bar\beta'' = \beta''_{-m''} \dots \beta''_0 \dots \beta''_{m''-1}$, consisting of 0 and 1, such that $\beta_i = \beta'_i = \beta''_i = \alpha_i$ for $-n \le i \le n-1$ and the following conditions hold:*

(1) *if $\omega \in C_{\bar\beta}$, then $\bar f_m[\omega](J) \subset W^- \subset \bar f_{-m}[\omega](J)$, where*

$$|(\bar f_m[\omega])'|_J| < 1, \qquad |(\bar f_{-m}[\omega])'|_{(\bar f_{-m}[\omega])^{-1}(W^-)}| > 1;$$

(2) *if $\omega' \in C_{\bar\beta'}$, then $\bar f_{-m'}[\omega'](J) \subset W^+ \subset \bar f_{m'}[\omega'](J)$, where*

$$|(\bar f_{-m'}[\omega'])'|_J| < 1, \qquad |(\bar f_{m'}[\omega'])'|_{(\bar f_{m'}[\omega'])^{-1}(W^+)}| > 1;$$

(3) *if $\omega'' \in C_{\bar\beta''}$, then $S^1 - W^- \subset \bar f_{m''}[\omega''](J)$ and $S^1 - W^+ \subset \bar f_{-m''}[\omega''](J)$.*

Proof. Let $s = [\frac{1}{b-\delta}] + 1$. Under the action of s iterations of the mapping f_0, any point $\varphi \in S^1$ makes more than one full rotation moving in small steps $f_0(\varphi) - \varphi \in (b - \delta, b + \delta)$.

Proposition 2.1. *Suppose that an arc $J^* \subset S^1$ is given. There exists a finite composition h of the mappings f_0 and f_1 such that the length of the arc $h(J^*)$ is at least $2b$.*

Proof. If the length of the arc J^* is less than $2b$, then, by at most s applications of f_0, the arc J^* can be mapped into the interior of the arc W^+; then, the application of f_1 expands the image of the arc J^* with the multiplier at least $(a-\delta)$. After k applications of this procedure, we obtain the arc of length at least $(1-\delta)^{sk}(a-\delta)^k|J^*|$. For sufficiently large k, we obtain the arc of length greater than $2b$. Proposition 2.1 is proved.

Remark. Proposition 2.1 remains valid if we take the mappings f_0^{-1} and f_1^{-1} instead of f_0 and f_1.

Let us continue the proof of Lemma 2.1. Let $f^+ = f_{\alpha_{n-1}} \circ \ldots \circ f_{\alpha_0}$ and $f^- = f_{\alpha_{-n}}^{-1} \circ \ldots \circ f_{\alpha_{-1}}^{-1}$. Denote $J^+ = f^+(J)$ and $J^- = f^-(J)$.

Now, let us construct a segment $\bar{\beta}$ satisfying condition (1). This is equivalent to the construction of a finite composition $h^+ = f_{\beta_{m-1}} \circ \ldots \circ f_{\beta_n}$ of the mappings f_0 and f_1 and a finite composition $h^- = f_{\beta_{-m}}^{-1} \circ \ldots \circ f_{\beta_{-n-1}}^{-1}$ of the mappings f_0^{-1} and f_1^{-1}, so that the mappings $h^+ \circ f^+$ and $h^- \circ f^-$ satisfy property (1); namely, the inclusions $h^+ \circ f^+(J) \subset W^- \subset h^- \circ f^-(J)$ are valid, where $|(h^+ \circ f^+)'|_J| < 1$ and $|(h^- \circ f^-)'|_{(h^- \circ f^-)^{-1}(W^-)}| > 1$.

To construct h^+, we apply Proposition 2.1 to the arc $J^* = S^1 - \bar{J}^+$. Then, by at most s mappings f_0, we cover the repeller q by the image of the arc $S^1 - \bar{J}^+$ (whose length is now at least $2b$). Then, we apply as many number of mappings f_1 as it is necessary to expand the image of the arc $S^1 - \bar{J}^+$ to cover $S^1 - W^-$. In this case, the image of the arc J^+ gets into the interior of W^-, and, since the mapping f_1 contracts the arc W^-, the application of an additional number of mappings f_1 guarantees that the condition for the derivative holds (namely, $|(h^+ \circ f^+)'|_J| < 1$).

Now, let us construct the composition h^- of the mappings f_0^{-1} and f_1^{-1}. To this end, we apply Proposition 2.1 to the arc $J^* = J^-$ and the mappings f_0^{-1} and f_1^{-1}. In this case, the image of the arc J^- has the length at least $2b$. Then, by at most s mappings f_0^{-1}, we cover the point p (which is a repeller for the mapping f_1^{-1}) by the image of the arc J^-. Then, we apply as large number of mappings f_1^{-1} as is necessary for the image of the arc J^- to cover the arc W^-. Since the mapping f_1^{-1} expands the arc W^-, the application of some more number of mappings f_1^{-1} guarantees that the condition for the derivative holds (namely, $|(h^- \circ f^-)'|_{(h^- \circ f^-)^{-1}(W^-)}| > 1$).

It remains to note that one can apply an arbitrary number of mappings f_1 to the constructed composition of mappings f_0 and f_1, preserving the required properties; accordingly, we can apply the mapping f_1^{-1} to the composition of mappings f_0^{-1} and f_1^{-1}. Hence, we can assume that the compositions h^+ and h^- have equal lengths (in the case of step systems, this is of no importance; however, it is important when generalizing the procedure to soft systems). The segment $\bar{\beta}$ is constructed.

The segment $\bar{\beta}'$ can be constructed analogously.

The segment $\bar{\beta}''$ can also be constructed analogously; here, the situation is simplified since there is no additional requirement for the derivatives. Lemma 2.1 is proved.

2.3. Periodic orbits. In this subsection, we prove assertion (i) of Theorem 1.

Let us prove that the periodic orbits of the mapping F (1.1) with the absolute value of the multiplier along the circle less than one are dense in $\Sigma^2 \times S^1$. First, note that the sets of the form $C_{\bar{\alpha}} \times J \subset \Sigma^2 \times S^1$ (where $\bar{\alpha} = \alpha_{-n} \ldots \alpha_0 \ldots \alpha_{n-1}$ is a segment of 0 and 1 and J is an arc of the circle S^1) form a base of the topology of the space $\Sigma^2 \times S^1$. Therefore, it suffices to show that the required periodic trajectory passes through any set of the form $C_{\bar{\alpha}} \times J$. To this end, we employ assertion (1) of Lemma 2.1: there exists a segment $\bar{\beta} = \beta_{-m} \ldots \beta_0 \ldots \beta_{m-1}$ such that $\bar{f}_m[\omega](J) \subset W^- \subset \bar{f}_{-m}[\omega](J)$ whenever $\omega \in C_{\bar{\beta}}$; here, $|(\bar{f}_m[\omega])'|_J| < 1$ and $|(\bar{f}_{-m}[\omega])'|_{(\bar{f}_{-m}[\omega])^{-1}(W^-)}| > 1$. Note that the second inequality for the derivative implies that $|((\bar{f}_{-m}[\omega])^{-1})'|_{(W^-)}| < 1$. Take a sequence

$\omega \in C_{\bar\beta}$ that is periodic with the period $\bar\beta$ (i.e., $\sigma^{2m}\omega = \omega$, $\omega \in C_{\bar\beta}$). The mapping $\bar f_{2m}[\omega]\colon S^1 \to S^1$ has a fixed point $\varphi_0 \in J$ with the multiplier less than one. Indeed, $\bar f_{2m}[\omega] = (\bar f_{-m}[\omega])^{-1} \circ \bar f_m[\omega]$, and the arc J under the action of the mapping $\bar f_m[\omega]$ gets into the interior of the arc W^- and is contracted. However, the entire arc W^- under the action of the mapping $(\bar f_{-m}[\omega])^{-1}$ is mapped with contraction into the interior of the arc J. Thus, $\bar f_{2m}[\omega]$ maps the arc J into itself with contraction; hence, it has a fixed point $\varphi_0 \in J$ with the multiplier less than one. However, then the point (ω, φ_0) is the sought-for periodic point for the mapping F.

The existence of a dense set of periodic orbits with the multiplier along a circle greater than one can be proved similarly with the use of assertion (2) of Lemma 2.1.

2.4. Density segments.

Lemma 2.2. *Let a segment $\bar\alpha = \alpha_{-n} \dots \alpha_{-1}\alpha_0 \dots \alpha_{n-1}$ (of 0 and 1) and an arc $J \subset S^1$ be given. Then, there exists a finite segment $\bar\gamma = \gamma_0\gamma_1 \dots \gamma_l$ consisting of 0 and 1 such that, if a sequence $\omega \in \Sigma^2$ contains the segment $\bar\gamma$, then, for any point $\varphi \in S^1$, the orbit of the point $(\omega, \varphi) \in \Sigma^2 \times S^1$ under the action of the mapping F intersects the set $C_{\bar\alpha} \times J \subset \Sigma^2 \times S^1$.*

Proof. Let us apply assertion (3) of Lemma 2.1: there exists a segment $\bar\beta = \beta_{-m} \dots \beta_0 \dots \beta_{m-1}$, $\beta_i = \alpha_i$ for $-n \le i \le n-1$, such that, if $\omega \in C_{\bar\beta}$, then $S^1 - W^- \subset \bar f_m[\omega](J)$ and $S^1 - W^+ \subset \bar f_{-m}[\omega](J)$. Note that $(S^1 - W^-) \cup (S^1 - W^+) = S^1$. It remains to note that one can take as the segment $\bar\gamma$ a doubly written segment $\bar\beta$, i.e., $\gamma_{-2m} \dots \gamma_{-1} = \gamma_0 \dots \gamma_{2m-1} = \beta_{-m} \dots \beta_{m-1}$. Indeed, let $\omega^* \in \Sigma^2$ contain the segment $\bar\gamma$. Without loss of generality, we can assume that $\bar\gamma$ is the "central" segment of the sequence ω^*: $\bar\gamma = \omega^*_{-2m} \dots \omega^*_0 \dots \omega^*_{2m-1}$. Note that, in this case, $\sigma^m\omega^* \in C_{\bar\alpha}$ and $\sigma^{-m}\omega^* \in C_{\bar\alpha}$. Note also that, for any sequence $\omega \in C_{\bar\beta}$, $\bar f_m[\omega^*] = (\bar f_{-m}[\omega])^{-1}$, while $\bar f_{-m}[\omega^*] = (\bar f_m[\omega])^{-1}$. Now, take an arbitrary point $\varphi \in S^1$. Since $(S^1 - W^-) \cup (S^1 - W^+) = S^1$, the point φ lies at least on one of the arcs $S^1 - W^-$ or $S^1 - W^+$. If $\varphi \in S^1 - W^-$, then $\bar f_{-m}[\omega^*](\varphi) \in \bar f_{-m}[\omega^*](S^1 - W^-) = (\bar f_m[\omega])^{-1}(S^1 - W^-) \subset J$. If $\varphi \in S^1 - W^+$, then $\bar f_m[\omega^*](\varphi) \in \bar f_m[\omega^*](S^1 - W^+) = (\bar f_{-m}[\omega])^{-1}(S^1 - W^+) \subset J$. Thus, in any case, the orbit of the point (ω^*, φ) under the action of the mapping F intersects $C_{\bar\alpha} \times J$. Lemma 2.2 is proved.

2.5. The Lyapunov exponents. Given a point $\varphi \in S^1$ and a sequence $\omega \in \Sigma^2$, for $n \in \mathbb{N}$ we set

$$\lambda_n(\varphi) = \ln|\bar f_n'(\varphi)| = \sum_{i=0}^{n-1} \ln|f'_{\omega_i}(\bar f_i(\varphi))|, \qquad \lambda_{-n}(\varphi) = \ln|\bar f'_{-n}(\varphi)| = -\sum_{i=0}^{n-1} \ln|f'_{\omega_{-i}}(\bar f_{-i}(\varphi))|.$$

Denote also

$$s = \left[\frac{1}{b-\delta}\right] + 1, \qquad \mu = (\ln(a-\delta) + s\ln(1-\delta)), \qquad I = \frac{1}{s}(-\mu, \mu). \tag{2.4}$$

Lemma 2.3. *There exists a constant $C > 0$ such that the following holds. For any point $\varphi \in S^1$, any $f_0 \in U_0$ and $f_1 \in U_1$, and any $\lambda \in I$, there exists a sequence $\omega(\varphi) \in \Sigma^2$ such that $|\lambda_n(\varphi) - \lambda n| < C$ and $|\lambda_{-n}(\varphi) + \lambda n| < C$.*

Proof. Fix an arbitrary point $\varphi \in S^1$ and construct the right-hand side of the required sequence $\omega(\varphi)$: $\omega_0\omega_1 \dots \omega_k \dots$. The left-hand side is constructed analogously since the mappings (f_0^{-1}, f_1^{-1}) possess the same properties as (f_0, f_1).

For convenience, let us set

$$\nu_k = \lambda_k(\varphi) - \lambda k.$$

First, note that the following inequality holds for any k:

$$|\nu_{k+1} - \nu_k| < \ln L + |I|. \tag{2.5}$$

The constant L is defined by (3.3). Note that, if a finite part of the sequence $\omega(\varphi)$, i.e., $\omega_0, \dots, \omega_{n-1}$, is constructed, then $\lambda_k(\varphi)$ and ν_k as well as $\varphi_k := \bar{f}_k[\omega](\varphi)$ are defined for $1 \le k \le n$.

We describe the algorithm for constructing a sequence $\omega(\varphi)$ that employs at every step the values of the functions ν_k.

Suppose that the segment $\omega_0 \dots \omega_{k-1}$ is constructed.

In the case $\nu_k \ge 0$, set $\omega_k = 1$ if $\varphi_k \in W^-$ and $\omega_k = 0$ if $\varphi_k \in S^1 - W^-$.

In the case $\nu_k < 0$, set $\omega_k = 1$ if $\varphi_k \in W^+$ and $\omega_k = 0$ if $\varphi_k \in S^1 - W^+$.

The element ω_k is determined; increase k by one and repeat the procedure.

Let us show that, as a result of this procedure, we obtain a sequence $\omega_0 \dots \omega_k \dots$ such that $|\nu_k|$ is no greater than $C = 2s(\ln L + |I|)$.

Proposition 2.2. *If* $|\nu_k| \ge s(\ln L + |I|)$, *then* $|\nu_{k+s}| < |\nu_k|$.

Proof. Let $\nu_k \ge s(\ln L + |I|)$ (the case of $\nu_k \le -s(\ln L + |I|)$ is analyzed similarly). Then, by (2.5), $\nu_{k+l} > 0$ for any l: $0 < l \le s$.

Among the elements $\omega_k, \omega_{k+1}, \dots, \omega_{k+s-1}$, there certainly exists at least one 1. Indeed, if all elements are equal to 0, then $\varphi_{k+l} = f_0(\varphi_{k+l-1})$ for any l: $0 < l \le s$. The mapping f_0 is close to the rotation by b; therefore, there exists $t < s$ such that $\varphi_{k+t} \in W^-$; hence, by our algorithm, we have to choose $\omega_{k+t} = 1$. Contradiction.

Note that, for any l: $0 < l \le s$, if $\omega_{k+l} = 1$, then λ_{k+l+1} decreases as compared with λ_{k+l} at least by $\ln(a - \delta)$ (since the mapping f_1 contracts the arc W^-), whereas, if $\omega_{k+l} = 0$, then λ_{k+l+1} cannot increase as compared with λ_{k+l} by more than $|\ln(1-\delta)|$. Since at least one element ω_{k+l}, $0 < l \le s$, is equal to 1, we have $\lambda_{k+s} \le \lambda_k - \mu$. However, the segment I (2.4) is chosen so that $\mu > |\lambda s|$; therefore, $\nu_{k+s} = \lambda_{k+s} - (k+s)\lambda < \lambda_k - k\lambda = \nu_k$. Proposition 2.2 is proved.

Now, Proposition 2.2, combined with (2.5), implies that $|\nu_k| \le 2s(\ln L + |I|)$. Indeed, if there exists k such that $|\nu_k| > 2s(\ln L + |I|)$, then (2.5) implies that $|\nu_{k-s}| > s(\ln L + |I|)$; but then $|\nu_{k-s}| > |\nu_k| > 2s(\ln L + |I|)$ by Proposition 2.2. We obtain by induction that, if $k - ms > 0$ for a certain $m \in \mathbb{N}$, then $|\nu_{k-ms}| > |\nu_k| > 2s(\ln L + |I|)$. However, $|\nu_l| \le s(\ln L + |I|)$ certainly for $|l| \le s$. Contradiction. Lemma 2.3 is proved.

Now, we pass directly to the proof of property (ii) in Theorem 1. Let us fix $\lambda \in I$ and construct a point (ω^*, φ) that has an orbit dense in $\Sigma^2 \times S^1$ and the Lyapunov exponent along the circle equal to λ.

Let $\{C_{\bar{\alpha}_i} \times J_i\}$ be a countable base of the topology of the space $\Sigma^2 \times S^1$. By Lemma 2.2, for the set $C_{\bar{\alpha}_i} \times J_i$ there exists a segment $\bar{\gamma}_i$ such that, if the sequence $\omega \in \Sigma^2$ contains $\bar{\gamma}_i$, then, for any $\varphi \in S^1$, the orbit of the point (ω, φ) under the action of F (1.1) passes through $C_{\bar{\alpha}_i} \times J_i$. Denote the length of the segment $\bar{\gamma}_i$ by k_i.

Choose a certain sequence $\{n_i\}$, $i \ge 0$, such that

$$\frac{(i+2)C + (k_1 + k_2 + \ldots + k_{i+1})(\ln L + |I|)}{n_i} < \frac{1}{i}, \tag{2.6}$$

where C is a constant from Lemma 2.3. Define a sequence $\{N_i\}$: $N_0 = n_0$, $N_l = N_{l-1} + k_l + n_l$.

Fix a point $\varphi \in S^1$ and construct the right-hand side of the sequence ω^* by induction. As the basis of induction, consider a sequence $\omega(\varphi)$ that satisfies the requirements of Lemma 2.3 and take the first n_0 elements of this sequence as elements of ω^*. Thus, the segment $\omega^*_0 \dots \omega^*_{N_0-1}$ is constructed.

Induction step. Suppose that the sequence $\omega^*_0 \dots \omega^*_{N_i-1}$ is constructed. Let us add the segment $\bar\gamma_{i+1}$ to this sequence. Consider the point $\varphi_{N_i+k_{i+1}} = \bar f_{N_i+k_{i+1}}[\omega^*](\varphi)$ and the sequence $\omega(\varphi_{N_i+k_{i+1}})$ that satisfies the requirements of Lemma 2.3. Take its first n_{i+1} elements as the subsequent n_{i+1} elements of the sequence ω^*. The sequence $\omega^*_0 \dots \omega^*_{N_{i+1}-1}$ is constructed.

Let us show that the sequence thus constructed is the sought-for one. As above, we set $\nu_l = (\lambda_l(\varphi) - l\lambda)$. By (2.5), adding the segment $\bar\gamma_{i+1}$ to the sequence $\omega^*_0 \dots \omega^*_{N_i-1}$, we increase $|\nu_{N_i}|$ by at most $k_{i+1}(\ln L + |I|)$: $|\nu_l| < |\nu_{N_i}| + k_{i+1}(\ln L + |I|)$ for $N_i \le l \le N_i + k_{i+1}$. Then, adding an arbitrary segment of the sequence $\omega(\varphi_{N-i+k_{i+1}})$, we increase $|\nu_{N_i+k_{i+1}}|$ by at most C by Lemma 2.3. Hence, for $N_i \le l \le N_i + k_{i+1} + n_{i+1}$, we have $|\nu_l| \le |\nu_{N_i}| + k_{i+1}(\ln L + |I|) + C \le (i+2)C + (k_1 + k_2 + \dots + k_{i+1})(\ln L + |I|)$. The last inequality is obtained by induction on i. Note that, under the assumption made, $l > n_i$. Hence, by (2.6), $|\nu_l|/l < \frac{1}{i}$; i.e., the Lyapunov exponent for the positive semitrajectory of the orbit of ω^* is equal to λ.

The left-hand side of the sequence ω^* is constructed analogously.

2.6. Density. As we noted above, open sets of the form $C_\alpha \times J$, where $C_\alpha = \{\omega \mid \omega_j = \alpha_j,\ j \in \{-n, \dots, n-1\}\}$ and J is an arc of the circle S^1, form a base of the topology of the space $\Sigma^2 \times S^1$. If the orbit of the sequence $\omega \in \Sigma^2$ under the action of the Bernoulli shift σ is dense in Σ^2, then this orbit contains any predetermined segment consisting of 0 and 1. Therefore, property (iii) of Theorem 1 directly follows from Lemma 2.2.

3. ROBUST PROPERTIES OF SOFT RANDOM DYNAMICAL SYSTEMS

In this section, we consider the soft skew products

$$G\colon\ \Sigma^2 \times S^1 \to \Sigma^2 \times S^1, \quad (\omega, \varphi) \mapsto (\sigma\omega, f_\omega(\varphi)). \tag{3.1}$$

For these products, we prove that the properties similar to the properties of step systems (see Subsection 2.1) are robust. The scheme of the proof for soft systems is the same as for step systems: to obtain a product of mappings $f_{\sigma^i\omega}$ with required properties, the sequence ω is constructed inductively—zeros and ones are added to the left and right ends of the already written segment of this sequence. The difficulty consists in the fact that *no finite segment of the sequence ω allows one to determine exactly the mapping $f_{\sigma^i\omega}$*; now, this mapping depends on the whole sequence rather than on one element or even on a long segment of this sequence.

This difficulty is surmounted as follows. If the segment $\bar\beta = \beta_{-m} \dots \beta_{m-1}$ of the sequence ω is fixed and sufficiently long, then any two sequences ω and ω' with such a central segment are exponentially close to each other. Inequality (1.6) implies that then the mappings $f_{\sigma^i\omega}$ and $f_{\sigma^i\omega'}$ are also close for $i \ll m$. The difference between these mappings makes an exponentially small contribution to the difference between the φ-components of the points $G^m(\omega, \varphi)$ and $G^m(\omega', \varphi)$; therefore, these components themselves differ not too strongly. These considerations are formalized in Lemma 3.1 from Subsection 3.2.

3.1. Statement of the results.

Theorem 2. *There exist diffeomorphisms $g_0, g_1\colon S^1 \to S^1$ and an interval $I \subset \mathbb{R}$, $0 \in I$, such that, for any $C > 1$ and $\alpha > 0$, there are neighborhoods $U_0(g_0)$ and $U_1(g_1)$ such that the following* *holds. Suppose that the mapping G of the form* (3.1) *satisfies the conditions*

$$f_\omega \in U_{\omega_0} \qquad \text{for any} \ \ \omega \in \Sigma^2, \tag{3.2}$$

$$L = \max_{\omega\in\Sigma^2}\max_{\varphi\in S^1}(\|Df_\omega(\varphi)\|, \|Df_\omega^{-1}(\varphi)\|), \tag{3.3}$$

$$d_{C^1}(f_\omega, f_{\omega'}) \le C(d_{\Sigma^2}(\omega,\omega'))^\alpha \tag{3.4}$$

for any $\omega, \omega' \in \Sigma^2$, and

$$L\cdot 2^{-\alpha} < 1. \tag{3.5}$$

Then, the mapping G has the following properties.

(i) *The periodic orbits of G with the absolute value of multiplier along the circle greater than one are dense in $\Sigma^2 \times S^1$. The same is valid for the periodic orbits with the absolute value of multiplier less than one.*

(ii) *For any $\lambda \in I$, there exists an orbit, dense in $\Sigma^2 \times S^1$, with the Lyapunov exponent along the circle equal to λ.*

(iii) *If the orbit of a sequence $\omega \in \Sigma^2$ under the Bernoulli shift σ is dense in Σ^2, then, for any $\varphi \in S^1$, the orbit of the point (ω, φ) under the action of G is dense in $\Sigma^2 \times S^1$.*

As the sought-for diffeomorphisms, we take the diffeomorphisms g_0 and g_1 defined in Subsection 2.1. As the neighborhoods $U_0(g_0)$ and $U_1(g_1)$, we take their sufficiently small C^2-neighborhoods:

$$d_{C^2}(f_\omega, g_{\omega_0}) < \delta. \tag{3.6}$$

Introduce the notations

$$\bar f_m[\omega] = f_{\sigma^{m-1}\omega}\circ\ldots\circ f_{\sigma\omega}\circ f_\omega,$$
$$\bar f_{-m}[\omega] = f^{-1}_{\sigma^{-m}\omega}\circ\ldots\circ f^{-1}_{\sigma^{-1}\omega}.$$

These are the φ-components of the points $G^{\pm m}(\omega,\varphi)$. Let us set

$$\bar f_0[\omega] = \mathrm{id}\,.$$

3.2. Lemmas on error.

Lemma 3.1 (on the error in the coordinate along the fiber)**.** *Suppose that mapping* (3.1) *satisfies conditions* (3.3)–(3.6). *Then, there exists $K > 0$, $K = K(L, C, \alpha)$, independent of δ, such that for any $m \in \mathbb{N}$ the inequality*

$$d_{\Sigma^2}(\omega,\omega') \le 2^{-m} \tag{3.7}$$

implies

$$d_{C^0}(\bar f_{\pm m}[\omega], \bar f_{\pm m}[\omega']) \le \gamma := K\delta^\beta, \tag{3.8}$$

where $\beta = 1 - \frac{\ln L}{\ln 2^\alpha}$.

Proof. Let us prove the lemma for $\bar f_m$; the proof for $\bar f_{-m}$ is absolutely analogous. To facilitate the references, let us divide the proof into two steps.

Step 1. We begin with the inequality valid for arbitrary diffeomorphisms of a circle onto itself that satisfy the following requirements:

$$\|Dh\| \le L \qquad \text{for} \ \ h = f, g, f^{-1}, g^{-1};$$
$$|f(\varphi) - g(\psi)| \le L|\varphi-\psi| + d_C(f,g); \tag{3.9}$$

a similar inequality holds for the inverse mappings. Set

$$\delta_l = d_{C^0}(\bar{f}_l[\omega], \bar{f}_l[\omega']). \tag{3.10}$$

It is obvious that $\delta_0 = 0$. Inequality (3.7) implies

$$d_{\Sigma^2}(\sigma^l\omega, \sigma^l\omega') \le 2^{-m+l}.$$

Now, from condition (3.4), we obtain

$$d_{C^0}(f_{\sigma^l\omega}, f_{\sigma^l\omega'}) \le C \cdot 2^{(-m+l)\alpha}. \tag{3.11}$$

Inequality (3.9) implies

$$\delta_{l+1} \le L\delta_l + C \cdot 2^{(-m+l)\alpha}.$$

Let us find the terms of the recurrent sequence that satisfy the conditions

$$x_{l+1} = Lx_l + C \cdot 2^{(-m+l)\alpha}, \qquad x_0 = 0.$$

It is obvious that $x_l \ge \delta_l$. We have

$$x_l = C \cdot 2^{-m\alpha}\frac{2^{l\alpha} - L^l}{2^\alpha - L}. \tag{3.12}$$

Take k such that

$$C \cdot 2^{(-m+k)\alpha} \le 2\delta \le C \cdot 2^{(-m+k+1)\alpha}. \tag{3.13}$$

Then,

$$\delta_k \le C\frac{2^{(-m+k)\alpha}}{2^\alpha - L} \le \frac{2\delta}{2^\alpha - L}.$$

Step 2. Note that (3.6) implies

$$\max_{\{\omega,\omega' \mid \omega_0 = \omega'_0\}} d_{C^0}(f_\omega, f_{\omega'}) \le 2\delta. \tag{3.14}$$

It is convenient to estimate the terms of the sequence after δ_k by the inequality

$$\delta_{l+1} \le L\delta_l + 2\delta. \tag{3.15}$$

This inequality follows from (3.14) and (3.9). Let us find a term of the recurrent sequence $\{y_l\}$ that satisfies the conditions

$$y_{k+i+1} = Ly_{k+i} + 2\delta, \qquad y_k = \frac{2\delta}{2^\alpha - L}.$$

We have

$$y_{k+i} = \frac{2\delta}{1-L} + C(L,\alpha)\delta L^i, \qquad C(L,\alpha) = \frac{2(2^\alpha - 1)}{(2^\alpha - L)(L-1)}. \tag{3.16}$$

By the right inequality in (3.13),

$$2^{(m-k)\alpha} \le 2^\alpha\frac{C}{2\delta}, \qquad L^{m-k} \le \left(2^\alpha\frac{C}{2\delta}\right)^{\ln L/\alpha\ln 2}.$$

Then,

$$y_m \le C(L,\alpha)\delta L^{m-k} \le C(L,\alpha)\delta\left(2^\alpha\frac{C}{2\delta}\right)^{\ln L/\alpha\ln 2} = K(L,\alpha,C)\delta^\beta.$$

The lemma is proved.

Lemma 3.2 (on the error for the Lyapunov exponents). *Suppose that the mapping G (3.1) satisfies conditions (3.3)–(3.6) and, in addition, the mappings f_ω and f_ω^{-1} are twice continuously differentiable with respect to φ, the second derivative being uniformly bounded in ω: $|D^2_\varphi f_\omega^{\pm 1}| < T$.*

Then, there exists $R > 0$, independent of δ, such that for any $m \in \mathbb{N}$ inequality (3.7) implies

$$\left|\ln \bar{f}'_{\pm m}[\omega](\varphi) - \ln \bar{f}'_{\pm m}[\omega'](\varphi)\right| \le R\delta^\beta,$$

where $\beta = 1 - \frac{\ln L}{\ln 2^\alpha}$ and δ is the constant from (3.6).

Proof. We have to estimate

$$Z_m := \left|\ln \bar{f}'_m[\omega](\varphi) - \ln \bar{f}'_m[\omega'](\varphi)\right|$$

from above. For short, we set

$$z_l := \left|f'_{\sigma^l\omega}(\bar{f}_l[\omega](\varphi)) - f'_{\sigma^l\omega'}(\bar{f}_l[\omega'](\varphi))\right|. \tag{3.17}$$

Let us avail ourselves of the inequality valid for all $0 < w_1 < w_2$: $|\ln w_1 - \ln w_2| \le \frac{w_2 - w_1}{w_1}$. We obtain

$$Z_m \le \sum_{l=0}^{m-1} \left|\ln f'_{\sigma^l\omega}(\bar{f}'_l[\omega](\varphi)) - \ln f'_{\sigma^l\omega'}(\bar{f}'_l[\omega'](\varphi))\right| \le L \sum_{l=0}^{m-1} z_l.$$

By estimate (3.9) applied to $f = f'_{\sigma^l\omega}$ and $g = f'_{\sigma^l\omega'}$, using notations (3.17) and (3.10), we obtain $z_l \le T\delta_l + d_{C^1}(f_{\sigma^l\omega}, f_{\sigma^l\omega'})$. Here, T is the same as in the hypothesis of the lemma. Choose k such that (3.13) holds. Divide the sum in the estimate for Z_m into two: $\sum_{l=0}^{k} z_l$ and $\sum_{l=k+1}^{m-1} z_l$. Estimate the first sum using (3.11):

$$\begin{aligned}\sum_{l=0}^{k} z_l &\le \sum_{l=0}^{k} (T\delta_l + C\cdot 2^{(-m+l)\alpha}) \le C\cdot 2^{-m\alpha} \sum_{l=0}^{k}\left(T\frac{2^{l\alpha} - L^l}{2^\alpha - L} + 2^{l\alpha}\right) \\ &= R_0(C,T,\alpha,L)\cdot 2^{(-m+k)\alpha} \le R_1(C,T,\alpha,L)\delta.\end{aligned}$$

To estimate the second sum, we use the inequalities $z_l \le T\delta_l + 2\delta$ and $\delta_l \le C(L,\alpha)\delta L^l$ for $k < l < m$ (see (3.16)) and $m - k \le C_1 + \frac{\ln\delta^{-1}}{\ln 2^\alpha}$ (see (3.13)). We obtain

$$\sum_{l=k+1}^{m-1} z_l \le \sum_{l=k+1}^{m-1} (T\delta_l + 2\delta) \le \sum_{l=k+1}^{m-1} (C(L,\alpha)\delta L^{l-k} + 2\delta) \le R_2(C,L,\alpha)\delta^\beta + R_3\delta\ln(\delta^{-1}).$$

Finally, we have $Z_m \le R\delta^\beta$.

A similar estimate for the inverse mappings is proved analogously.

3.3. Lemma on separation of points on a fiber. Let $\bar\beta = \beta_{-m}\dots\beta_{m-1}$. Let us apply the lemma on error to estimate how far apart the images of the point $\varphi \in S^1$ are under the action of the mappings $\bar{f}_{\pm l}[\omega]$, $\omega \in C_{\bar\beta}$. To this end, we define the sets $V_\pm[\bar\beta](\varphi) = \{f_{\pm m}[\omega](\varphi) | \omega \in C_{\bar\beta}\}$.

Lemma 3.1 states that $\operatorname{diam} V_\pm[\bar\beta](\varphi) \le K\delta^\beta$ irrespective of the length of $\bar\beta$.

Given a segment $\bar\beta = \beta_{-l}\dots\beta_0\dots\beta_{l-1}$, we set $\Gamma_l = C_{\beta_{-l}\dots\beta_0\dots\beta_{l-1}}$.

For a fixed segment $\bar\beta$, define the sets $V_l(\varphi)$ and $V_{-l}(\varphi)$: $V_{\pm l}(\varphi) = \{\bar{f}_{\pm l}[\omega](\varphi) \mid \omega \in \Gamma_l\}$. Assume that δ is so small (which is attained by choosing, as U_0 and U_1, sufficiently small neighborhoods of the mappings g_0 and g_1) that $\gamma = K\delta^\beta < \frac{b}{40}$. Then, by Lemma 3.1 on error,

$$\operatorname{diam} V_{\pm l}(\varphi) < \gamma < \frac{b}{40}. \tag{3.18}$$

Note also that there exist arcs $W^+, W^- \subset S^1$, $|W^\pm| \geq \frac{1}{4} - \delta$, such that, for any sequence ω with $\omega_0 = 1$, the mapping f_ω expands the arc W^+ (namely, $|f'_\omega|_{W^+}| > a - \delta$) and contracts the arc W^- (namely, $|f'_\omega|_{W^-}| < a^{-1} + \delta$).

Finally, define the sets

$$P = \{p \text{ is an attractor of the mapping } f_\omega \mid \omega \in \Sigma^2,\ \omega_0 = 1\},$$
$$Q = \{q \text{ is a repeller of the mapping } f_\omega \mid \omega \in \Sigma^2,\ \omega_0 = 1\}.$$

The diameters $\operatorname{diam} P$ and $\operatorname{diam} Q$ of these sets are of the order of δ; therefore, choosing again δ sufficiently small, we can assume that $\operatorname{diam} P < \gamma$ and $\operatorname{diam} Q < \gamma$.

In the remaining part of Section 3, we consider the mapping G (3.1) satisfying conditions (3.3)–(3.6) under the assumption that the constant δ is small as compared with $\ln a$ and b.

Lemma 3.3. *Let $\bar{\alpha} = \alpha_{-n} \dots \alpha_{n-1}$ be an arbitrary segment and $\varphi_1, \varphi_2 \in S^1$ be arbitrary points. Then, there exists a segment $\bar{\beta}_m = \beta_{-m} \dots \beta_{m-1}$, $C_{\bar{\beta}} \subset C_{\bar{\alpha}}$, such that*

$$d_{S^1}(V_\pm[\bar{\beta}](\varphi_1), V_\pm[\bar{\beta}](\varphi_2)) \geq 2b. \tag{3.19}$$

Here, by the distance between two sets on the circle, we mean the length of the minimal arc that connects the points of these sets. The constant δ is assumed to be as small as is necessary in the proof.

Proof of Lemma 3.3. Construct the sequence $\bar{\beta}_l = \beta_{-l} \dots \beta_{l-1}$ by induction, setting $l \geq n$, $\bar{\beta}_n = \bar{\alpha}$ and adding one element to each side of the segment $\bar{\beta}_l$ at every step. The induction step is defined by the algorithm described below; when the algorithm stops, the construction is completed. The algorithm consists of two steps.

Step 1. Set

$$M_{\pm l} = \min_{\omega \in C_{\bar{\beta}_l}} \left(\bar{f}_{\pm l}[\omega](\varphi_1), \bar{f}_{\pm l}[\omega](\varphi_2)\right).$$

Check if the following conditions are fulfilled:

$$M_l > 3b, \quad \text{(A1)} \qquad\qquad M_{-l} > 3b. \quad \text{(A2)}$$

If both conditions hold, then the algorithm stops; below, we prove that the segment $\bar{\beta}_m$ $(m = l)$ thus obtained is the required one. If at least one of conditions (A1) or (A2) is violated, go to Step 2.

Step 2. Set

$$W_{\pm l} = V_\pm[\bar{\beta}_l](\varphi_1) \cup V_\pm[\bar{\beta}_l](\varphi_2).$$

Check if the following conditions are fulfilled:

$$W_l \subset W^+, \quad \text{(B1)} \qquad\qquad W_{-l} \subset W^-. \quad \text{(B2)}$$

If condition (B1) (respectively, (B2)) holds, set $\beta_l = 1$; otherwise, $\beta_l = 0$ (respectively, $\beta_{-(l+1)} = 1$; otherwise, $\beta_{-(l+1)} = 0$). The segment $\bar{\beta}_{l+1}$ is constructed. Increase l by 1 and return to Step 1.

Proposition 3.1. *The segment $\bar{\beta}_m$ constructed by the moment when the algorithm stops satisfies the requirements of Lemma* 3.3.

Proof. By the hypothesis, the segment $\bar{\beta}_m$ satisfies conditions (A1) and (A2). Let us prove that these conditions imply inequality (3.19) for V_+ and V_-, respectively. Consider only the case of V_+; the case of V_- is analogous.

The hypotheses of Lemma 3.3 imply the hypotheses of Lemma 3.1 (on error). Therefore,

$$\operatorname{diam} V_+[\bar\beta_m]\varphi_j \le K\delta^\beta, \qquad j = 1, 2.$$

It follows from condition (A1) that each set $V_+[\bar\beta_m]\varphi_j$ contains a point such that the distance between these points is greater than $3b$. Hence, the distance between the sets themselves is at least $3b - 2K\delta^\beta \ge 2b$. This proves (3.19).

Proposition 3.2. *The algorithm described above stops in a finite number of steps.*

Proof. Suppose the contrary. Then, the increasing sequence of segments $\bar\beta_l$ defines a sequence ω; the analysis of the latter leads to a contradiction. Indeed, let $\bar\beta_l$ be the segments constructed above, and

$$\omega = \bigcap_{l \ge n} C_{\bar\beta_l}.$$

Set

$$\varphi_{jl}^\pm = \bar f_{\pm l}[\omega](\varphi_j), \qquad j = 1, 2,$$
$$\delta_l^\pm = d_{S^1}(\varphi_{1l}^\pm, \varphi_{2l}^\pm).$$

It follows from the definition of the sequence ω that $\varphi_{jl}^\pm \in V_\pm[\bar\beta_l](\varphi_j)$. Therefore,

$$\delta_l^\pm - 2K\delta^\beta \le M_{\pm l} \le \delta_l^\pm.$$

In addition, set

$$W_l^\pm = V_\pm[\bar\beta_l](\varphi_1) \cup V_\pm[\bar\beta_l](\varphi_2), \qquad d_l^\pm = \operatorname{diam} W_l^\pm.$$

Proposition 3.3. *There exists m such that $\delta_m^\pm > \frac{1}{16}$.*

Proposition 3.2 follows from Proposition 3.3 since $M_{\pm m} > 3b$ for such an m; the algorithm stops—a contradiction. We analyze only the sequences δ_m^+; the sequences with the minus superscript are analyzed similarly.

Proof of Proposition 3.3. Let s be the same as in (2.4). For $\omega_l = 0$, the mapping $f_{\sigma^l\omega}$ translates each point φ into the arc $[\varphi + b - \delta, \varphi + b + \delta]$ and each arc of length λ to the arc of length $\lambda' \in ((1-\delta)\lambda, (1+\delta)\lambda)$. For $\omega_l = 1$, the mapping $f_{\sigma^l\omega}$ translates each arc of length λ that belongs to W^+ to the arc of length $\lambda' \in ((a-\delta)\lambda, (a+\delta)\lambda)$. Note that, when $\omega_l = 1$, both points φ_{1l}^+ and φ_{2l}^+ lie on W^+ by construction.

Hence, if $\delta_l < \frac{1}{8}$ (i.e., less than half the length of the arc W^+), then the sequence ω cannot contain s zeros in succession after ω_l. Indeed, the application of s mappings that are δ-close to the rotation by b to the lesser of the arcs $(\varphi_{1l}^+, \varphi_{2l}^+)$ translates this arc onto the arc W^+ in less than s steps; after that, the corresponding element of the sequence ω must be one rather than zero. Hence,

$$\delta_{l+s}^+ > (a-\delta)(1-\delta)^s \delta_l^+ \qquad \text{for } \delta_l^+ < \frac{1}{8}. \tag{3.20}$$

Hence, the sequence (δ_n^+) must contain an element $\delta_n^+ \ge \frac{1}{8}$. Let us prove that

$$\delta_k^+ > \frac{1}{16} \qquad \text{for all } k > n. \tag{3.21}$$

Suppose the contrary. Let there exist $\delta_k^+ \le \frac{1}{16}$, $k > n$. Let δ_t^+ be the last element of the sequence between δ_n^+ and δ_k^+ that is at least $\frac{1}{8}$. Then, $k - t < s + 1$ by (3.20):

$$\delta_{t+s+1}^+ > (a-\delta)(1-\delta)^s \delta_{t+1} \ge (a-\delta)(1-\delta)^{s+1}\delta_t > \frac{1}{8}.$$

The assumption $k - t \geq s + 1$ contradicts the choice of δ_t^+. For the same reason, there are no ones among the elements ω_l, $t+1 \leq l \leq k$. But then $\delta_k \geq (1-\delta)^{s+2}\frac{1}{8} > \frac{1}{16}$—contradiction. Inequality (3.21) is proved. Similarly, there exist n' such that $\delta_k^- > \frac{1}{16}$ for all $k > n'$. This proves Proposition 3.3 and, hence, also Lemma 3.3.

3.4. Distortion of arcs. For a soft system, the mapping $\bar{f}_m[\omega]$ is not determined by a finite set of elements $\omega_0 \dots \omega_{m-1}$ of the sequence $\omega \in \Sigma^2$ as it was in the case of step systems but depends on the entire sequence ω. Therefore, in spite of the similarity of statements, the assertion of Lemma 3.4 is stronger than that of Lemma 2.1.

Lemma 3.4 (an analogue of Lemma 2.1). *Suppose that a segment $\bar{\alpha} = \alpha_{-n} \dots \alpha_{-1}\alpha_0 \dots \alpha_{n-1}$ (of 0 and 1) and an arc $J \subset S^1$ are given. Then, there exist segments $\bar{\beta} = \beta_{-m} \dots \beta_0 \dots \beta_{m-1}$, $\bar{\beta}' = \beta'_{-m'} \dots \beta'_0 \dots \beta'_{m'-1}$, and $\bar{\beta}'' = \beta''_{-m''} \dots \beta''_0 \dots \beta''_{m''-1}$ that consist of 0 and 1 such that $\beta_i = \beta'_i = \beta''_i = \alpha_i$ for $-n \leq i \leq n-1$ and the following is valid:*

(1) *if $\omega \in C_{\bar{\beta}}$, then $\bar{f}_m[\omega](J) \subset W^- \subset \bar{f}_{-m}[\omega](J)$, and*

$$\left|(\bar{f}_m[\omega])'|_J\right| < 1, \qquad \left|(\bar{f}_{-m}[\omega])'|_{(\bar{f}_{-m}[\omega])^{-1}(W^-)}\right| > 1;$$

(2) *if $\omega' \in C_{\bar{\beta}'}$, then $\bar{f}_{-m'}[\omega'](J) \subset W^+ \subset \bar{f}_{m'}[\omega'](J)$, and*

$$\left|(\bar{f}_{-m'}[\omega'])'|_J\right| < 1, \qquad \left|(\bar{f}_{m'}[\omega'])'|_{(\bar{f}_{m'}[\omega'])^{-1}(W^+)}\right| > 1;$$

(3) *if $\omega'' \in C_{\bar{\beta}''}$, then $S^1 - W^- \subset \bar{f}_{m''}[\omega''](J)$ and $S^1 - W^+ \subset \bar{f}_{-m''}[\omega''](J)$.*

Proof. Let us construct a segment $\bar{\beta}$ satisfying property (1). Let s be the same as in (2.4). Denote the ends of the arc J by φ_1 and φ_2. By Lemma 3.3 on separation, there exists a segment $\bar{\beta}_1 = \beta_{-k_1} \dots \beta_0 \dots \beta_{k_1-1}$, $\Gamma_{k_1} \subset \Gamma_n$, such that the distance between the sets $V_{k_1}(\varphi_1)$ and $V_{k_1}(\varphi_2)$, as well as the distance between the sets $V_{-k_1}(\varphi_1)$ and $V_{-k_1}(\varphi_2)$ is at least $2b$.

For $l \geq k_1$, denote by X_{-l} the interval between $V_{-l}(\varphi_1)$ and $V_{-l}(\varphi_2)$ that is contained in the images of the arc J under the mapping $\bar{f}_{-l}[\omega]$, $\omega \in \Gamma_l$ (i.e., $X_{-l} = \bigcap_{\omega \in \Gamma_l} \bar{f}_{-l}[\omega](J)$), and denote by Y_l the interval between $V_l(\varphi_1)$ and $V_l(\varphi_2)$ contained in the images of the arc $S^1 - \bar{J}$ under the action of $\bar{f}_l[\omega]$.

Let us construct a segment $\bar{\beta}_2 = \beta_{-k_2} \dots \beta_0 \dots \beta_{k_2-1}$, $\Gamma_{k_2} \subset \Gamma_{k_1}$, such that $Q \subset Y_{k_2}$ and $P \subset X_{-k_2}$. For this purpose, we consider the following procedure of inductive construction of the sets Γ_l.

A set Γ_{k_1} is given.

Suppose that a set Γ_l is constructed. Let us check if the following conditions are fulfilled:

$$Q \subset Y_l, \quad \text{(C1)} \qquad\qquad P \subset X_{-l}. \quad \text{(C2)}$$

If both conditions are fulfilled, then the construction is completed; otherwise, define β_l and β_{-l-1} as follows. If condition (C1) is fulfilled, then set $\beta_l = 1$; otherwise, $\beta_l = 0$. If condition (C2) is fulfilled, then set $\beta_{-l-1} = 1$; otherwise, $\beta_{-l-1} = 0$. Increase l by one and repeat the construction procedure.

Let us show that this construction is completed in a finite number of steps.

Proposition 3.4. *If condition* (C1) *or* (C2) *is fulfilled at a certain step, then this condition will hold up to the end of construction.*

Proof. If $Q \subset Y_l$ and the construction is not completed, then $\beta_l = 1$ and, for any $\omega \in \Gamma_{l+1}$, the mapping $f_{\sigma^l\omega}$ is the Morse–Smale mapping with the repeller lying on the arc Y_l and, consequently, on the arc $\bar{f}_l[\omega](S^1 - \bar{J})$. But then $\bar{f}_l[\omega](S^1 - \bar{J}) \subset f_{\sigma^l\omega} \circ \bar{f}_l[\omega](S^1 - \bar{J}) = \bar{f}_{l+1}[\omega](S^1 - \bar{J})$. Since this is valid for any $\omega \in \Gamma_{l+1}$ and we have $Y_l = \bigcap_{\omega\in\Gamma_l} \bar{f}_l[\omega](S^1 - \bar{J})$, then $Y_l \subset Y_{l+1}$. But then $Q \subset Y_l \subset Y_{l+1}$, and condition (C1) still holds. The situation is analogous for condition (C2).

Proposition 3.5. *Each condition* (C1) *and* (C2) *is fulfilled at a certain step of our construction.*

Proof. Assume the contrary, i.e., that condition (C1) is never fulfilled. Then the construction procedure will never end and thus defines a certain sequence $\omega \in C_{\beta_1}$ such that $\omega_l = 0$ for $l \geq k_1$. Then, the mappings $\bar{f}_l[\omega]$ are close to the rotation by b for $l > k_1$. Here, since k_1 was chosen so that the distance between $V_{k_1}(\varphi_1)$ and $V_{k_1}(\varphi_2)$ is at least $2b$, we have $|\bar{f}_{k_1}[\omega](S^1 - \bar{J})| \geq 2b$. But then one of the arcs $\bar{f}_{k_1}[\omega](S^1 - \bar{J}), \dots, \bar{f}_{k_1+s}[\omega](S^1 - \bar{J})$ completely covers the set Q so that the distance from Q to the ends of this arc will be greater than γ. Suppose that this arc is $\bar{f}_{k_1+t}[\omega](S^1 - \bar{J})$. The sets $V_{k_1+t}(\varphi_1)$ and $V_{k_1+t}(\varphi_2)$ lie in the γ-neighborhoods of the ends of this arc by Lemma 3.1 on error. Therefore, $Q \subset Y_{k_1+t}$; i.e., condition (C1) is fulfilled. We arrived at a contradiction, which proves Proposition 3.5 for condition (C1). For condition (C2), the proposition is proved analogously.

It follows from Propositions 3.4 and 3.5 that the inductive construction ends at a certain step k_2; in this case, a segment $\bar{\beta}_2 = \beta_{-k_2} \dots \beta_0 \dots \beta_{k_2-1}$ is constructed such that conditions (C1) and (C2) are fulfilled for $l = k_2$.

Next, there exists a segment $\bar{\beta}_3$, $C_{\bar{\beta}_3} \subset C_{\bar{\beta}_2}$, such that the inclusions $\bar{f}_{k_3}[\omega](J) \subset W^- \subset \bar{f}_{-k_3}[\omega](J)$ hold for any $\omega \in C_{\bar{\beta}_3}$. Indeed, consider the sequence $\omega^* \in C_{\bar{\beta}_2}$ that is obtained from the segment $\bar{\beta}_2$ by adding an infinite number of ones to both ends of the latter. Since $\omega^* \in C_{\bar{\beta}_2}$, then $Q \subset \bar{f}_{k_2}[\omega^*](S^1 - \bar{J})$ and $P \subset \bar{f}_{-k_2}[\omega^*](J)$. But then the inclusions $\bar{f}_{k_3}[\omega^*](J) \subset W^- \subset \bar{f}_{-k_3}[\omega^*](J)$ will hold for a certain $k_3 \geq k_2$; here, one can assume that the ends of the arc W^- are at a distance greater than γ from the ends of the arcs $\bar{f}_{k_3}[\omega^*](J)$ and $\bar{f}_{-k_3}[\omega^*](J)$. But then, by Lemma 3.1 on error, the inclusions $\bar{f}_{k_3}[\omega](J) \subset W^- \subset \bar{f}_{-k_3}[\omega](J)$ will hold for any $\omega \in C_{\bar{\beta}_3}$ where $\bar{\beta}_3$ is the segment $\bar{\beta}_2$ with $(k_3 - k_2)$ ones added to its left and right ends. Finally, adding, if necessary, some more ones to the left and right ends of $\bar{\beta}_3$, we construct a segment $\bar{\beta}$ that also satisfies the conditions imposed on the derivatives ($|\bar{f}'_m[\omega]|_J| < 1$ and $|\bar{f}'_{-m}[\omega]|_{\bar{f}^{-1}_{-m}(W^-)}| > 1$). Indeed, this directly follows from the fact that any mapping f_ω^{-1} with $\omega_0 = 1$ contracts on the arc W^+, while any mapping f_ω with $\omega_0 = 1$ expands on the arc W^+, the contraction and expansion coefficients being (uniformly in ω) separated from 1.

The segments $\bar{\beta}'$ and $\bar{\beta}''$ are constructed analogously.

3.5. Periodic orbits. The proof of property (i) in Theorem 2 is a word-for-word repetition of the proof of property (i) in Theorem 1 (Subsection 2.3) (instead of Lemma 2.1, one should apply Lemma 3.4).

3.6. Density segments.

Lemma 3.5 (an analogue of Lemma 2.2). *Let a segment $\bar{\alpha} = \alpha_{-n} \dots \alpha_{-1}\alpha_0 \dots \alpha_{n-1}$ (of 0 and 1) and an arc $J \subset S^1$ be given. Then, there exists a finite segment $\bar{\gamma} = \gamma_0\gamma_1 \dots \gamma_l$ consisting of 0 and 1 such that, if a sequence $\omega \in \Sigma^2$ contains the segment $\bar{\gamma}$, then, for any point $\varphi \in S^1$, the orbit of the point $(\omega, \varphi) \in \Sigma^2 \times S^1$ under the action of the mapping G intersects the set $C_{\bar{\alpha}} \times J \subset \Sigma^2 \times S^1$.*

Proof of Lemma 3.5 word for word repeats the proof of Lemma 2.2 (one should apply Lemma 3.4 instead of Lemma 2.1).

3.7. Lyapunov exponents. As in Subsection 2.5, given a point $\varphi \in S^1$ and a sequence $\omega \in \Sigma^2$, for $n \in \mathbb{N}$ we set

$$\lambda_n(\varphi) = \ln|\bar{f}_n'(\varphi)| = \sum_{i=0}^{n-1} \ln|f_{\omega_i}'(\bar{f}_i(\varphi))|, \qquad \lambda_{-n}(\varphi) = \ln|\bar{f}_{-n}'(\varphi)| = -\sum_{i=0}^{n-1} \ln|f_{\omega_{-i}}'(\bar{f}_{-i}(\varphi))|.$$

If we want to emphasize the dependence of $\lambda_{\pm n}(\varphi)$ on ω, we will write $\lambda_{\pm n}[\omega](\varphi)$.

Lemma 3.6 (an analogue of Lemma 2.3). *There exists a constant $C > 0$ such that the following holds. For any segment $\bar{\alpha} = \alpha_{-n}\dots\alpha_0\dots\alpha_{n-1}$, any point $\varphi \in S^1$, and any $\lambda \in I$, there exists a sequence $\omega(\varphi) \in C_{\bar{\alpha}}$ such that $|\lambda_k(\varphi) - \lambda_n(\varphi) - (k-n)\lambda| < C$ and $|\lambda_{-k}(\varphi) - \lambda_{-n}(\varphi) + (k-n)\lambda| < C$ for any $k \geq n$.*

Proof. Denote, for convenience,

$$\nu_{k,n}[\omega] = \lambda_k[\omega](\varphi) - \lambda_n[\omega](\varphi) - (k-n)\lambda, \qquad \nu_{-k,-n}[\omega] = \lambda_{-k}[\omega](\varphi) - \lambda_{-n}[\omega](\varphi) + (k-n)\lambda.$$

We will construct the sequence $\bar{\beta}_k = \beta_{-k}\dots\beta_{k-1}$ inductively, assuming $k \geq n$, $\bar{\beta}_n = \bar{\alpha}$ and adding, at every step, one symbol to each end of the segment $\bar{\beta}_k$. As usually, denote $\Gamma_k = C_{\bar{\beta}_k}$. A set $\Gamma_n = C_{\bar{\alpha}}$ is given. Suppose that Γ_k is constructed.

If the inequality $\nu_{k,n}[\omega^*] > 0$ holds for a certain $\omega^* \in \Gamma_k$, then we set $\omega_k = 1$ if $V_k(\varphi) \subset W^-$ and $\omega_k = 0$ otherwise.

If the inequality $\nu_{k,n} \leq 0$ holds for all sequences from Γ_k, then we set $\omega_k = 1$ if $V_k(\varphi) \subset W^+$ and $\omega_k = 0$ otherwise.

If the inequality $\nu_{-k,-n}[\omega^*] > 0$ holds for a certain $\omega^* \in \Gamma_k$, then we set $\omega_{-k-1} = 1$ if $V_{-k}(\varphi) \subset W^+$ and $\omega_{-k-1} = 0$ otherwise.

If the inequality $\nu_{-k,-n} \leq 0$ holds for all sequences from Γ_k, then we set $\omega_{-k-1} = 1$ if $V_{-k}(\varphi) \subset W^-$ and $\omega_{-k-1} = 0$ otherwise.

We constructed the set Γ_{k+1}. Let us increase k by one and repeat the construction procedure.

Let us show that, for the sequence $\omega(\varphi)$ constructed by this procedure, the quantity $|\nu_{k,n}|$ does not exceed $C = 2s(\ln L + |I|)$, where the segment I is the same as in (2.4). (The inequality $|\nu_{-k,-n}| < C$ is verified analogously.) Note that

$$|\nu_{k+1,n} - \nu_{k,n}| < (\ln L + |I|). \tag{3.22}$$

Proposition 3.6. *If $|\nu_{k,n}| \geq s(\ln L + |I|)$, then $|\nu_{k+s,n}| < |\nu_{k,n}|$.*

Proof. Let $\nu_{k,n} \geq s(\ln L + |I|)$ (the case $\nu_{k,n} \leq -s(\ln L + |I|)$ is considered analogously). Then, (3.22) implies that $\nu_{k+l} > 0$ for any l: $0 < l \leq s$.

Among the elements $\omega_k, \omega_{k+1}, \dots, \omega_{k+s-1}$, there certainly exists at least one 1. Indeed, if all elements are equal to 0, then the mappings $f_{\sigma^{k-1}\omega}, f_{\sigma^k\omega}, \dots, f_{\sigma^{k+s-2}\omega}$ are close to the rotation by b. Then, there exists $t < s$ such that the point $\bar{f}_{k+t}[\omega](\varphi)$ lies on the arc W^- and is at least at a distance γ from its ends. But then $V_{k+t}(\varphi) \subset W^-$. Note also that, by Lemma 3.2 on error for the Lyapunov exponents, the quantities $\lambda_{k+t}[\omega](\varphi)$ and $\lambda_{k+t}[\omega'](\varphi)$ differ little for $\omega, \omega' \in \Gamma_k$; therefore, $\nu_{k+t}[\omega]$ and $\nu_{k+t}[\omega']$ have the same sign for any $\omega' \in \Gamma_k$. Hence, by our algorithm, we have to choose $\omega_{k+t} = 1$. Contradiction. The remaining part of the proof is parallel to the corresponding part of the proof of Proposition 2.2.

Now, Proposition 3.6, combined with (3.22), implies Lemma 3.6 just as Proposition 2.2, combined with (2.5), implied Lemma 2.3.

Now, property (ii) of Theorem 2 can be proved by obvious modifications from the proof of property (ii) in Theorem 1 (Subsection 2.5). Namely, instead of Lemma 2.2, we have to apply Lemma 3.5 and, instead of Lemma 2.3, apply Lemma 3.6. In this case, the required sequence should be constructed inductively toward both ends simultaneously; Lemma 3.2 on error for the Lyapunov exponents allows one to use, without substantial changes, the proof of the fact that the sequence constructed possesses the required properties.

3.8. Density. Property (iii) in Theorem 2 directly follows from Lemma 3.5 just as property (iii) in Theorem 1 followed from Lemma 2.2 in the case of step systems.

4. PERIODIC POINTS OF SOFT SKEW PRODUCTS OVER A SOLENOID

In this section, we construct dense sets of periodic points of different indices for the skew products over a solenoid. We begin with a smooth case.

4.1. Smooth skew products over a solenoid and their perturbations. Let T be a solid torus: $T = S^1 \times D$, $S^1 = \mathbb{R}/\mathbb{Z}$, and $D = \{z \in \mathbb{C} \mid |z| \le 2\}$. Denote by S the mapping of a solenoid

$$S\colon\ T \to T, \quad (s, z) \mapsto \left(2s, \frac{1}{3}z + \exp 2\pi i s\right).$$

The Smale–Williams solenoid is the maximal attractor of this mapping; denote it by Λ.

Consider a smooth skew product over the solid torus with the mapping of the solenoid in the base

$$H\colon\ T \times S^1 \to T \times S^1, \quad (x, \varphi) \mapsto (S(x), f_x(\varphi)), \qquad x \in T, \quad \varphi \in S^1. \tag{4.1}$$

The maximal attractor of the mapping H,

$$\Delta = \bigcap_{k \ge 0} H^k(T \times S^1),$$

is given by $\Delta = \Lambda \times S^1$. The restriction $G = H|_\Delta$ is a skew product over the solenoid:

$$G\colon\ \Lambda \times S^1 \to \Lambda \times S^1, \quad (x, \varphi) \mapsto (S(x), f_x(\varphi)), \qquad x \in \Lambda, \quad \varphi \in S^1.$$

Let us describe the results obtained by the first author on smooth perturbations of the diffeomorphism H and the related skew products. These results apply to the hyperbolic theory and will be dealt with in the next paper. Here, we present these results to motivate the constraints imposed on the skew products considered below.

Under certain constraints imposed on the derivatives of the mappings f_x with respect to x and φ, a small smooth perturbation $\mathcal{H}$ of the diffeomorphism H (see (4.1)) has a maximal attractor $\Delta_\mathcal{H}$ homeomorphic to Δ. The homeomorphism $h\colon \Delta \to \Delta_\mathcal{H}$ can be chosen so that

h preserves the coordinate φ, and

h conjugates the restriction $\mathcal{H}|_{\Delta_\mathcal{H}}$ and the mapping

$$\tilde{G}\colon\ \Lambda \times S^1 \to \Lambda \times S^1, \quad (x, \varphi) \mapsto (S(x), \tilde{f}_x(\varphi)), \qquad x \in \Lambda, \quad \varphi \in S^1. \tag{4.2}$$

These facts are related to the known results of the hyperbolic theory. A new result is that the mapping $\tilde{f}_x$ *Hölder continuously depends on* x. This is explained as follows.

The image of the fiber $\{x\} \times S^1$ under the action of the homeomorphism h is an embedded circle and can be defined as the graph of the mapping

$$\varphi_x\colon\ S^1 \to T, \quad \varphi \mapsto (\psi_x(\varphi), \varphi).$$

It turns out that *the mapping ψ_x Hölder continuously depends on x as an element of the space C^1 and has the second derivatives that are uniformly bounded with respect to $x \in \Lambda$.* In particular, there exist positive C_1 and α_1 such that

$$d_{C^1(S^1,T)}(\psi_x, \psi_y) \le C_1(d_T(x,y))^{\alpha_1}, \qquad x, y \in \Lambda.$$

Let $\pi\colon \Delta = \Lambda \times S^1 \to S^1$ be a projection onto the fiber along the base Λ. Then,

$$\tilde{f}_x(\varphi) = \pi \circ \mathcal{H}(\psi_x(\varphi), \varphi).$$

Now, the smoothness of $\mathcal{H}$ implies

$$d_{C^1(S^1,S^1)}(\tilde{f}_x, \tilde{f}_y) \le C_2(d_T(x,y))^{\alpha_1}, \quad x, y \in \Lambda; \qquad |D^2_\varphi f_x| < C_3. \tag{4.3}$$

4.2. Periodic points of different indices: The main result. Consider a space $\mathcal{G}$ of mappings (4.2) satisfying condition (4.3) with the constants C_2, C_3, and α that are common to all mappings. Introduce the metric

$$d(G, \tilde{G}) = \max_{x \in \Lambda} d_{C^1(S^1,S^1)}(f_x, \tilde{f}_x)$$

in this space. The space thus defined depends on the choice of C_2, C_3, and α; however, the results formulated below are independent of these constants.

Theorem 3. *In the space $\mathcal{G}$ of mappings* (4.2) *equipped with the metric introduced above, there exists an open set such that, for all mappings from this set, the periodic orbits with the multiplier along the circle greater than one are dense in $\Lambda \times S^1$. The same is valid for the periodic orbits with the multiplier along the circle less than one.*

This theorem is proved in the remaining part of the section.

4.3. Almost step skew products over a solenoid and their perturbations. Let g_1 be the same mapping as in Subsection 2.1. Denote by g_2 the rotation $x \to x + b$. The constants $\ln a$ and b are chosen as small as it is required in what follows. Consider the mapping G^0 of the form (4.2) such that

$$\begin{aligned} \tilde{f}_x &= g_1 && \text{for } \ x = (s,z) \in \Lambda_1 := \left\{ (s,z) \in \Lambda \;\Big|\; \frac{5}{8} \le s < 1 \right\}, \\ \tilde{f}_x &= g_0 = g_1^{-1} \circ g_2 && \text{for } \ x = (s,z) \in \Lambda_0 := \left\{ (s,z) \in \Lambda \;\Big|\; \frac{1}{8} \le s < \frac{1}{2} \right\}. \end{aligned} \tag{4.4}$$

Suppose that, for other $x \in T$, the mapping $\tilde{f}_x$ is a smooth function of x and such that $|D\tilde{f}_x| \le a$ and $|D\tilde{f}_x^{-1}| \le a$.

The mapping G^0 thus constructed (there is a large freedom in the choice of this mapping; we fix just one mapping) is called "almost step." It is impossible to modify it so that the mapping $\tilde{f}_x$ would coincide either with g_1 or with g_0 on the whole solenoid: the mapping $\tilde{f}_x$ is a continuous function of x, while the solenoid is connected. This brings up additional difficulty in the analysis of skew products over a solenoid.

The open set mentioned in Theorem 3 can be chosen as

$$\mathcal{U} = \{G \in \mathcal{G} \mid d_{C^2}(G, G^0) \le \lambda\},$$

where λ, which is small as compared with $\ln a$ and b, is chosen below. It is these mappings G for which we will find the dense sets of periodic points that are attracting and repelling on a fiber.

Remark. The mapping G^0 can be chosen so that it is smoothly continued to the mapping H of the form (4.1) satisfying the requirement (4.3). If the constants $\ln a$ and b are sufficiently small, then the mapping H can be constructed so that its small perturbations will have invariant sets described in Subsection 4.1. The Hölder exponent in inequality (4.3) increases as $\ln a$ and b decrease. The restrictions of the perturbed mappings onto their attractors are the mappings from the set $\mathcal{U}$. Thus, the proof of Theorem 3 for this set gives information about the robust properties of attractors of smooth mappings. The properties described were obtained in [4] by different methods.

4.4. "Solenoidal" skew products over the Bernoulli shift. As is known, the mapping of a solenoid is semiconjugate to the Bernoulli shift. Namely, denote by Σ_0^2 the set of all sequences from Σ^2 in which 0 is encountered to the right of any element. There exists a Hölder one-to-one mapping $\Psi\colon \Sigma_0^2 \to \Lambda$ such that $S \circ \Psi = \Psi \circ \sigma$. The mapping Ψ is inverse to the "fate map" $\Phi\colon \Lambda \to \Sigma_0^2$ (which is not Hölder!):

$$\Phi(x) = \omega, \qquad \omega_n = j \quad \text{for} \quad S^n(x) \in \left\{\frac{j}{2} \le s(x) < \frac{j}{2} + \frac{1}{2}\right\}.$$

Set $\Phi(\Lambda_j) = \Omega_j$, where

$$\Omega_0 = \{\omega \in \Sigma_0^2 \mid 001 \le \omega_0\omega_1\omega_2 \le 011\}, \qquad \Omega_1 = \{\omega \in \Sigma_0^2 \mid 101 \le \omega_0\omega_1\omega_2 \le 111\} \tag{4.5}$$

(the ordering is lexicographic).

Denote by $\mathcal{W}$ the class of mappings

$$F\colon\ \Sigma_0^2 \times S^1 \to \Sigma_0^2 \times S^1, \qquad F(\omega, \varphi) = (\sigma\omega, f_\omega(\varphi)), \quad \omega \in \Sigma_0^2, \quad \varphi \in S^1, \tag{4.6}$$

into which the mappings of the class $\mathcal{U}$ go over under the mapping $(\omega, \varphi) \to (\Psi(\omega), \varphi)$. If the mapping $\tilde{G} \in \mathcal{U}$ has the form (4.2), then the corresponding mapping F has the form (4.6); in addition,

$$f_\omega = \tilde{f}_x, \qquad \text{where} \ \ x = \Psi(\omega).$$

Hence, we obtain

$$d_{C^1}(f_\omega, f_{\omega'}) \le C(d_{\Sigma^2}(\omega, \omega'))^\alpha, \qquad |D_\varphi^2 f_\omega| < C \tag{4.7}$$

for certain positive C and α and

$$d_{C^1}(f_\omega, g_j) \le \lambda \qquad \text{for} \ \ \omega \in \Omega_j. \tag{4.8}$$

Now, let, by analogy with (3.3),

$$L = \max_\Lambda\bigl(\|Df_\omega(\varphi)\|, \|Df_\omega^{-1}(\varphi)\|\bigr). \tag{4.9}$$

Choosing $\ln a$ and b sufficiently small, one can make all mappings $\tilde{f}_x$ for G^0, and hence for all $\tilde{G} \in \mathcal{U}$, C^1-close to the identity mapping. Namely, L in (4.9) will be so close to 1 that

$$L \cdot 2^{-\alpha} < 1, \tag{4.10}$$

where α is the Hölder exponent from inequality (4.7). Let us fix δ small as compared with $\ln a$ and b; the requirements of smallness of δ are imposed below. The number λ in the definition of the set $\mathcal{U}$ can be chosen so small as compared with δ that the mappings f_ω satisfy the requirement

$$d_{C^1}(f_\omega \circ f_{\omega'}, g_2) \le \delta \qquad \text{for} \quad \omega \in \Omega_1, \quad \omega' \in \Omega_0. \tag{4.11}$$

Remark. For $\lambda \ll \delta$, (4.8) implies that

$$d_{C^1}(f_\omega, f_{\omega'}) < \delta \qquad \text{for} \quad (\omega, \omega') \in \Omega_0 \times \Omega_0 \cup \Omega_1 \times \Omega_1. \tag{4.12}$$

Sometimes, we will use this relaxation of inequality (4.8) instead of (4.8).

Now, we can formulate the strengthened variant of Theorem 3.

Theorem 4. *Suppose that mapping* (4.6) *satisfies conditions* (4.7)–(4.11)*. Then, this mapping has a dense set of periodic points with the multiplier along the circle greater than* 1*. The same is valid for the periodic points with the multiplier less than* 1*.*

Theorem 4 strengthens Theorem 3 because the class of mappings considered here is wider than the set of mappings that are induced from the class $\mathcal{U}$. Subsections 4.5–4.8 are devoted to the proof of Theorem 4.

4.5. The second lemma on error.

Definition 4.1. A segment $\bar\beta = \beta_{-n} \dots \beta_{n-1}$ of the sequence is called *tame* if $\beta_{n-1} = 1$ and there are no two zeros in succession sufficiently far away from the zero position; more precisely,

$$\beta_i \beta_{i+1} \ne 00 \qquad \text{for} \quad i > \frac{n}{2}, \quad i+1 < -\frac{n}{2}.$$

For any segment $\bar\beta = \beta_{-m} \dots \beta_{m-1}$, we set

$$\Gamma_{\bar\beta} = C_{\bar\beta 01} \cup C_{\bar\beta 10} \cup C_{\bar\beta 11}, \tag{4.13}$$

where

$$\bar\beta ij = \beta_{-m} \dots \beta_{m-1} ij.$$

Equivalent definition. A segment $\bar\beta = \beta_{-m} \dots \beta_{m-1}$ is tame if, for any sequence $\omega \in \Gamma_{\bar\beta}$,

$$\sigma^i \omega \in \Omega_{\omega_i} \qquad \text{for} \quad i > \frac{n}{2}, \quad i < -\frac{n}{2},$$

where Ω_j are defined in (4.5).

For the tame segment $\bar\beta$ and any sequence $\omega \in \Gamma_{\bar\beta}$, the mappings $f_{\sigma^i\omega}$ are well controlled for $i > \frac{n}{2}$ and $i < -\frac{n}{2}$: they are close to the mappings g_{ω_i}. For $|i| < \frac{n}{2}$, the mappings $f_{\sigma^i\omega}$ are virtually unknown. However, for any two sequences $\omega, \omega' \in \Gamma_{\bar\beta}$, the sequences $\sigma^i\omega$ and $\sigma^i\omega'$ are close for $|i| < \frac{n}{2}$. Therefore, the mappings $f_{\sigma^i\omega}$ and $f_{\sigma^i\omega'}$ are close. Hence, as in the case of the systems over a horseshoe, the contributions of these mappings to the superpositions $\bar f_{\pm m}[\omega]$ and $\bar f_{\pm m}[\omega']$ differ so insignificantly that, in a sense, this difference can be neglected.

These considerations are formalized in the following three lemmas of which the first one is the main.

Lemma 4.1. *Consider mapping* (4.6) *satisfying conditions* (4.7), (4.9), (4.10), *and* (4.12)*. Let $\bar\beta = \beta_{-m} \dots \beta_{m-1}$ be a tame segment with $m > m_0(\alpha, L, C)$. Then, for any two sequences $\omega, \omega' \in \Gamma_{\bar\beta}$, the inequality*

$$d_{C^0}(\bar f_{\pm m}[\omega], \bar f_{\pm m}[\omega']) \le K\delta^\beta$$

holds. Here, $\beta = 1 - \frac{\ln L}{\ln 2^\alpha}$, and $K = K(\alpha, L, C)$ is a constant independent of δ.

Proof. Consider the estimate for $\bar{f}_m$. The estimate for $\bar{f}_{-m}$ is proved analogously but simpler. The lemma is proved along the same lines (almost word for word) as Lemma 3.1. An additional difficulty consists in the fact that relation (3.2), which reflects the closeness of the system to the step one, is replaced by the weaker inequality (4.8).

Note that inequality (3.6) follows from the fact that $\omega, \omega' \in \Gamma_{\bar{\beta}}$.

Step 1 of the proof of Lemma 3.1 is repeated almost word for word since the requirements (3.3), (3.4), and (3.5) coincide with (4.9), (4.7), and (4.10), respectively.

Step 2. Let, as in the proof of Lemma 3.1,

$$\delta_l = d_{C^0}(\bar{f}_m[\omega], \bar{f}_m[\omega']),$$

$\omega, \omega' \in \Gamma_{\bar{\beta}}$. To estimate the terms of the sequence δ_i for $i > k$, we use the fact that the segment $\bar{\beta}$ is tame and m is large.

The mappings $f = f_{\sigma^i\omega}$ and $g = f_{\sigma^i\omega'}$ play the role of f and g in inequality (3.9). Choose m so large that the inequality

$$\delta \leq C \cdot 2^{(-m+k+1)\alpha}$$

implies that $k > m/2$. To this end, it suffices to take

$$m > 2\frac{\ln C - \ln \delta}{\alpha \ln 2} + 2.$$

It follows from the definition of the tame segment that now f and g simultaneously belong either to Ω_0 or to Ω_1. Now, inequality (3.9) for $i > k$ follows from (4.12). The further proof of Lemma 4.1 is a word-for-word repetition of Step 2 in the proof of Lemma 3.1.

4.6. The second lemma on separation of points. Let $\Gamma_{\bar{\beta}}$ be a set of sequences defined by formula (4.13). Set

$$V_{\pm m}[\bar{\beta}](\varphi) = \{\bar{f}_{\pm m}[\omega](\varphi) \mid \omega \in \Gamma_{\bar{\beta}}\}. \tag{4.14}$$

Lemma 4.2. *Consider mapping* (4.1) *satisfying conditions* (4.7)–(4.12). *Suppose that* δ *is small as compared with* $\ln a$ *and* b. *The requirements that* δ *is small are specified below. Let* $\bar{\alpha}$ *be an arbitrary segment and* $\varphi_1, \varphi_2 \in S^1$ *be arbitrary points. Then, there exists a segment* $\bar{\beta} = \beta_{-m} \ldots \beta_{m-1}$, $C_{\bar{\beta}} \subset C_{\bar{\alpha}}$, *such that*

$$d_{S^1}(V_{\pm m}[\bar{\beta}](\varphi_1), V_{\pm m}[\bar{\beta}](\varphi_2)) \geq 2b. \tag{4.15}$$

Let us add ones to both ends of $\bar{\alpha}$ so that the new segment is tame. Denote it by $\bar{\beta}_n = \beta_{-n} \cdots \beta_{n-1}$.

We will construct the sequence $\bar{\beta}_l$ inductively, starting from $\bar{\beta}_n$ and adding two elements to each end. More precisely, we accomplish the following algorithm. Suppose that the segment $\bar{\beta}_l$ has already been constructed. Set

$$\Gamma_l = \Gamma_{\bar{\beta}_l}, \qquad M_{\pm l} = \min_{\omega \in \Gamma_l} d_{S^1}(\bar{f}_{\pm l}[\omega](\varphi_1), \bar{f}_{\pm l}[\omega](\varphi_2)).$$

Step 1. Check the following conditions:

$$M_l > 3b, \quad \text{(A1)} \qquad\qquad M_{-l} > 3b. \quad \text{(A2)}$$

If both conditions hold, then algorithm stops. Otherwise, go to Step 2.

Step 2. Set

$$W_{\pm l} = V_{\pm l}[\bar\beta_l](\varphi_1) \cup V_{\pm l}[\bar\beta_l](\varphi_2).$$

Check the following conditions:

$$W_l \subset W^+, \qquad \text{(B1)} \qquad\qquad W_l \subset W^-. \qquad \text{(B2)}$$

The addition of digits to the segment $\bar\beta_l$ is performed as follows: if condition (B1) is fulfilled, then $\beta_l = \beta_{l+1} = 1$; otherwise, $\beta_l = 0$ and $\beta_{l+1} = 1$; if condition (B2) is fulfilled, then $\beta_{-(l+1)} = \beta_{-(l+2)} = 1$; otherwise, $\beta_{-(l+1)} = 1$ and $\beta_{-(l+2)} = 0$.

Then, we pass to Step 1. Note that, at each step of the algorithm, a tame segment transforms into a tame segment. The algorithm described stops in a finite number of steps. We derive from here the lemma on separation.

If both conditions (A1) and (A2) are fulfilled at step l, then one can choose a point in each set $V_{\pm l}[\bar\beta_l](\varphi_j)$ such that the distance between these points is greater than $3b$. On the other hand, as we noted above, the segment $\bar\beta_l$ is tame. Therefore, one can apply the second lemma on error to the sequences $\omega, \omega' \in \Gamma_l$. Hence, the diameters of the sets $V_{\pm l}[\bar\beta_l](\varphi_j)$ are small as compared with b. This implies (4.15).

Now, note that, if the segment $\bar\beta_l$ ends with the symbols 01 (respectively, 11), then, for any sequence $\omega \in \Gamma_l$, the mapping $f_{\sigma^{l-1}\omega} f_{\sigma^{l-2}\omega}$ is close to $g_1 \circ g_0 = g_2$—"almost full rotation" (respectively, close to $g_1 \circ g_1$).

The proof of the termination of the algorithm described is performed as in Subsection 3.3, and we do not present it to avoid repetition.

4.7. Once again on the distortion of arcs.

Lemma 4.3. *For any segment $\bar\alpha$ and any arc $J \subset S^1$, there exists a segment $\bar\beta$ that continues α such that, for any sequence $\omega \in \Gamma_{\bar\beta}$,*

$$\bar f_m J \subset W^- \subset \bar f_{-m} J, \tag{4.16}$$

$$|\bar f'_m|_J| < 1, \qquad |\bar f'_{-m}|_{\bar f_{-m}^{-1} W^-}| > 1, \tag{4.17}$$

where $\bar f_{\pm m} = \bar f_{\pm m}[\omega]$.

Proof almost word for word repeats the proof of Lemma 3.4 with the modifications similar to those made in Subsection 4.6. Namely, let φ_1 and φ_2 be the ends of the arc J. Using Lemma 4.2, we construct the tame segment $\bar\beta_{k_1}$ that continues α and is such that inequality (4.15) holds for $\bar\beta = \bar\beta_{k_1}$.

In the algorithm of Subsection 3.4, the addition of zero to the right and left ends of the segment is replaced by the addition of 01, while the addition of 1 is replaced by the addition of 11. All the segments $\bar\beta_l$, $l \geq k_1$, thus obtained will be tame, and the mappings $f_{\sigma^i\omega}$ for $\omega \in \Gamma_l$ and $i > k_1$ will be controlled, i.e., close to g_{ω_i}. After that, the proof of Lemma 3.4 is repeated virtually word for word.

4.8. Proof of the theorem on periodic points. Now, the proof of Theorem 4 reproduces the proof of assertion (i) of Theorems 1 and 2 and is presented without details. Let $C_{\bar\alpha} \times J$ be an open set from the base of topology in $\Sigma_0^2 \times S^1$. Let us find a periodic point in $C_{\bar\alpha} \times J$ whose orbit is stable over the fiber. Apply Lemma 4.3 to the segment $\bar\alpha$ and the arc J. Suppose that $\bar\beta$ is a segment that extends $\bar\alpha$ such that relations (4.16) and (4.17) hold for any sequence $\omega \in \Gamma_{\bar\beta}$. Consider a sequence ω^0 that is periodic with period $\bar\beta$. Then, $\omega^0 \in \Gamma_{\bar\beta}$ since the two leftmost symbols ij

of the segment $\bar{\beta}$ that are added to the sequence ω^0 after the rightmost symbol of this segment are equal to 01 or 11. Hence, relations (4.16) and (4.17) hold for the sequence ω^0. Therefore, the mapping $f = \bar{f}_{-m}^{-1} \circ \bar{f}_m$ has a fixed point $y \in J$ such that $f'(y) < 1$; here, $\bar{f}_{\pm m} = \bar{f}_{\pm m}[\omega^0]$. The point (ω^0, y) is the required periodic point for mapping (4.1). Theorem 4 and therewith Theorem 3 are proved.

Analogues of assertions (ii) and (iii) of Theorem 2 are also valid for the skew products over a solenoid; however, we do not consider them here.

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