Math 199A—Fall 2016
Final Exam Exercises
Part 1, #1-10; Part 2, #11-20; Part 3, #21-23
Your choice of any 10 of the 23 problems will be due on December 9, 2016
(Extensions of this deadline are possible)

Part 1 #1-10

1. (Transfer Seminar #9) Fix two matrices a, b in $M_n(\mathbf{C})$ and define $\delta_{a,b}(x) = [[a, b], x]$. Then $\delta_{a,b}$ is a derivation with respect to triple bracket multiplication. (Use the notation $[abc]$ or $[a, b, c]$ for $[[a, b], c]$)

   Discussion: Let δ denote $\delta_{a,b}$. It must be shown that

$$\delta[xyz] \stackrel{?}{=} [\delta x, y, z] + [x, \delta y, z] + [x, y, \delta z]$$

   Write this as $LHS \stackrel{?}{=} RHS_1 + RHS_2 + RHS_3$. Then

$$\begin{aligned} LHS &= \delta[[x, y], z] = [[a, b], [[x, y], z]] = [ab - ba, [xy - yx, z]] \\ &= \dots \\ &= abxyz - abyxz - abzyx + abzyx + \text{ 12 other terms} \end{aligned}$$

$$\begin{aligned} RHS_1 &= [[[a, b], x], y], z] = [[(ab - ba)x - x(ab - ba), y], z] \\ &= \dots \\ &= (ab - ba)xyz - x(ab - ba)yz + \text{ 12 other terms} \\ &\quad (\text{ we consider } (ab - ba)xyz \text{ as two terms}) \end{aligned}$$

   etc.

2. (Transfer Seminar #10) Fix two matrices a, b in $M_{m,n}(\mathbf{C})$ and define $\delta_{a,b}(x) = \{abx\} - \{bax\}$. Then $\delta_{a,b}$ is a derivation with respect to triple circle multiplication. ($\{abc\}$ denotes $(ab^*c + cb^*a)/2$, but for purposes of this problem it is sufficient to let $\{abc\}$ denote $ab^*c + cb^*a$.)

   Discussion: Let δ denote $\delta_{a,b}$. It must be shown that

$$\delta\{xyz\} \stackrel{?}{=} \{\delta x, y, z\} + \{x, \delta y, z\} + \{x, y, \delta z\}$$

   Write this as $LHS \stackrel{?}{=} RHS_1 + RHS_2 + RHS_3$. Then

$$\begin{aligned} LHS &= \delta\{xyz\} = \{ab\{xyz\}\} - \{ba\{xyz\}\} \\ &= ab^*(xy^*z + zy^*x) + (xy^*z + zy^*x)b^*a + \text{ 4 other terms} \\ &\quad (\text{ we consider } ab^*(xy^*z + zy^*x) \text{ as two terms}) \end{aligned}$$

$$\begin{aligned} RHS_1 &= \{\delta x, y, z\} = \{\{abx\}yz\} - \{\{bax\}yz\} \\ &= (ab^*x + xb^*a)y^*z + zy^*(ab^*x + xb^*a) + \text{ 4 other terms} \end{aligned}$$

   etc.

3. (Transfer Seminar #11) Show that $M_n(\mathbf{R})$ is a Lie triple system with respect to triple bracket multiplication. In other words, show that the three axioms for Lie triple systems are satisfied if abc denotes $[[a, b], c] = (ab - ba)c - c(ab - ba)$ (a, b and c denote matrices). (Use the notation $[abc]$ for $[[a, b], c]$) The axioms for a Lie triple system are

   - $[aab] = 0$

- $[abc] + [bca] + [cab] = 0$
- $[de[abc]] = [[dea]bc] + [a[deb]c] + [ab[dec]]$

Discussion: Note that the third axiom holds by quoting Problem 1.

$[aab] = [[a, a], b] = [a^2 - a^2, b] = [0, b] = 0$

$[abc] = [[a, b], c] = [ab - ba, c] = abc - bac - cab + cba$

etc.

4. (Transfer Seminar #12) Show that $M_{m,n}(\mathbf{R})$ is a Jordan triple system with respect to triple circle multiplication. In other words, show that the two axioms for Jordan triple systems are satisfied if abc denotes $(ab^*c + cb^*a)/2$ (a, b and c denote matrices). (Use the notation $\{abc\}$ for $(ab^*c + cb^*a)/2$. As in Problem 2, for purposes of this problem it is sufficient to let $\{abc\}$ denote $ab^*c + cb^*a$.) The axioms for a Jordan triple system are

   - $\{abc\} = \{cba\}$
   - $\{de\{abc\}\} = \{\{dea\}bc\} - \{a\{edb\}c\} + \{ab\{dec\}\}$

   Discussion: The first axiom is trivial: $\{abc\} = ab^*c + cb^*a = cb^*a + ab^*c = \{cba\}$.

   Write the second axiom as $LHS \stackrel{?}{=} RHS_1 + RHS_2 + RHS_3$. Then

   $$LHS = \{de\{abc\}\} = de^*(ab^*c + cb^*a) + (ab^*c + cb^*a)e^*d$$

   $$RHS_1 = \{\{dea\}bc\} = (de^*a + ae^*d)b^*c + cb^*(de^*a + ae^*d)$$

   etc. (remember that $(xy^*z)^* = z^*yx^*$)

5. (Transfer Seminar #14) Let us write $\delta_{a,b}$ for the linear process $\delta_{a,b}(x) = \{abx\} - \{bax\}$ in a Jordan triple system. Show that $\delta_{a,b}$ is a derivation of the Jordan triple system by using the axioms for Jordan triple systems. The axioms for a Jordan triple system are

   - $\{abc\} = \{cba\}$
   - $\{de\{abc\}\} = \{\{dea\}bc\} - \{a\{edb\}c\} + \{ab\{dec\}\}$

   Discussion: Let δ denote $\delta_{a,b}$. It must be shown that

   $$\delta\{xyz\} \stackrel{?}{=} \{\delta x, y, z\} + \{x, \delta y, z\} + \{x, y, \delta z\}$$

   Write this as $LHS \stackrel{?}{=} RHS_1 + RHS_2 + RHS_3$. Then

   $$\begin{aligned} LHS &= \delta\{xyz\} = \{ab\{xyz\}\} - \{ba\{xyz\}\} \\ &= \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\} \\ &\quad -\{\{bax\}yz\} + \{x\{aby\}z\} - \{xy\{baz\}\} \end{aligned}$$

   $$RHS_1 = \{\delta x, y, z\} = \{\{abx\}yz\} - \{\{bax\}yz\}$$

   etc.

6. (Transfer Seminar #15) On the Jordan algebra $M_n(\mathbf{R})$ with the circle product $a \circ b = (ab + ba)/2$, define a triple product $\langle abc\rangle = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$. Show that $M_n(\mathbf{R})$ is a Jordan triple system with this triple product.

   Discussion: Show that $\langle abc\rangle = (ab^*c + cb^*a)/2$ and then quote Problem 4.

   $$\langle abc\rangle = \frac{\left(\frac{ab^*+b^*a}{2}\right)c + c\left(\frac{ab^*+b^*a}{2}\right)}{2} + \text{two other terms}$$

7. (Transfer Seminar #18) In an associative triple system F (of the second kind) with triple product denoted $\langle abc\rangle$, define a binary product ab to be $\langle aub\rangle$, where u is a fixed element. Show that the triple system F becomes an associative algebra with this product. Suppose further that $\langle uua\rangle = \langle auu\rangle = a$ for all a. Show that we get a unital involutive algebra with unit u and involution $a^\sharp = \langle uau\rangle$.

   Discussion: The product $ab = \langle aub\rangle$ is linear in each variable, so F, with this product is an algebra. The axioms for an associative triple system of a second kind are

$$\langle\langle xyz\rangle uv\rangle = \langle x\langle uzy\rangle v\rangle = \langle xy\langle zuv\rangle\rangle.$$

   Use the associative triple system axioms to show that

   - $(ab)c = a(bc)$
     If u is an element of F which satisfies $\langle uua\rangle = \langle auu\rangle = a$ for all a, then
   - $ua = au = a$
   - $(ab)^\sharp = b^\sharp a^\sharp$

   $(ab)^\sharp = \langle u\langle aub\rangle u\rangle = \langle\langle ubu\rangle au\rangle = \langle ub\langle uau\rangle\rangle = \langle b^\sharp au\rangle = \langle uba^\sharp\rangle$, and
   $b^\sharp a^\sharp = \langle\langle ubu\rangle u\langle uau\rangle\rangle$. (You're almost there!)

8. (Transfer Seminar #19) In a Lie algebra with product denoted by $[a, b]$, define a triple product $[abc]$ to be $[[a, b], c]$. Show that the Lie algebra becomes a Lie triple system with this triple product. (Meyberg Lectures, chapter 6, example 1, page 43)

   Discussion: The axioms for a Lie triple system are

   (i) $[aab] = 0$
   (ii) $[abc] + [bca] + [cab] = 0$
   (iii) $[de[abc]] = [[dea]bc] + [a[deb]c] + [ab[dec]]$
   The axioms for a Lie algebra are
   (iv) $[a, a] = 0$
   (v) $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$

   Let us rewrite (v) as

$$[c, [a, b]] = [a, [c, b]] + [[c, a], b] \tag{1}$$

   which makes it easier to remember: for a fixed element c, the linear transformation $x \mapsto [c, x]$ is a derivation of the Lie algebra.

   You need to prove that (iv) and (v) imply (i), (ii) and (iii). Check that (i) is immediate from (iv) and (ii) is immediate from (v). You need to show that (1) can be used to prove (iii). Write (iii) as

$$LHS \stackrel{?}{=} RHS_1 + RHS_2 + RHS_3.$$

   Then

$$\begin{aligned} LHS &= [de[abc]] = [\underbrace{[d, e]}_{x}, [\underbrace{[a, b]}_{y}, \underbrace{c}_{z}]] = [[x, y], z] + [y, [x, z]] \\ &= [[[d, e], [a, b]], c] + \underbrace{[[a, b], [d, e], c]]}_{RHS_3} \text{ (a good start!)} \end{aligned}$$

   Now look at $[\underbrace{[d, e]}_{x}, [\underbrace{a}_{y}, \underbrace{b}_{z}]] = [[x, y], z] + [y, [x, z]]$, etc. and

   $[[[d, e], [a, b]], c] = -[\underbrace{c}_{x}, [\underbrace{[d, e]}_{y}, \underbrace{[a, b]}_{z}]] = -[[x, y], z] - [y, [x, z]]$, etc. to finish the proof.

9. (Transfer Seminar #20) Let A be an algebra (associative, Lie, or Jordan; it doesn't matter). Show that the set $\mathcal{D} := \mathrm{Der}\,(A)$ of all derivations of A is a Lie subalgebra of $\mathrm{End}\,(A)^-$. That is, $\mathcal{D}$ is a linear subspace of the vector space of linear transformations on A, and if $D_1, D_2 \in \mathcal{D}$, then $D_1D_2 - D_2D_1 \in \mathcal{D}$.

   Discussion: Let $T = D_1D_2 - D_2D_1$. T is obviously a linear transformation. Show that $T(ab) = aT(b) + T(a)b$.

10. (Meyberg Chapter 2 #6, page 17) For any associative algebra A, consider the Peirce decomposition $A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$ with respect to an idempotent c. (Later, we will consider a Peirce decomposition in Jordan algebras (which are not associative.) Prove that $A_{ii}A_{ii} \subset A_{ii}$ for $i = 1, 0$, $A_{11}A_{10} \subset A_{10}$, $A_{11}A_{01} = \{0\}$, and so on, that is, $A_{ij}A_{kl} = \{0\}$ if $j \neq k$ and $A_{ij}A_{jk} \subset A_{ik}$.

    Discussion: It is required to prove that

    - $A_{i1}A_{1j} \subset A_{ij}$ for $i, j = 0, 1$

i	j	$A_{i1}A_{1j} \subset A_{ij}$
1	1	$A_{11}A_{11} = \{(cxc)cyc : x, y \in A\} \subset A_{11}$
1	0	$A_{11}A_{10} = \{(cxc)cy(1-c) : x, y \in A\} \subset A_{10}$
0	1	
0	0	

    - $A_{i0}A_{0j} \subset A_{ij}$ for $i, j = 0, 1$

i	j	$A_{i0}A_{0j} \subset A_{ij}$
1	1	
1	0	
0	1	
0	0	

    - $A_{i1}A_{0j} = \{0\}$ for $i, j = 0, 1$

i	j	$A_{i1}A_{0j} = \{0\}$
1	1	$A_{11}A_{01} = \{(cxc)(1-c)yc : x, y \in A\} = \{0\}$
1	0	$A_{11}A_{00} = \{(cxc)(1-c)y(1-c) : x, y \in A\} = \{0\}$
0	1	
0	0	

    - $A_{i0}A_{1j} = \{0\}$ for $i, j = 0, 1$

i	j	$A_{i0}A_{1j} = \{0\}$
1	1	
1	0	
0	1	
0	0	

Part 2 #11-20

11. (Meyberg Chapter 2 #1, page 12) If x is an element of an associative algebra A such that $\mathrm{Id} - L(x)$ is invertible, and $y := (\mathrm{Id} - L(x))^{-1}x$, then $xy = yx$.

    Discussion: The suggestion in Meyberg's notes on page 12 does not seem helpful. Instead let us take our cue from Lemma 2 on page 11, which states that a nilpotent element in a unital algebra is quasi-invertible. (See Remark 1 on page 12.) Suppose first, for purposes of motivation, that x is nilpotent, that is $x^n = 0$ for some $n > 1$. Then $L(x)^n = L(x^n) = 0$ also, and so

    $$(\mathrm{Id} - L(x))^{-1} = \mathrm{Id} + L(x) + L(x)^2 + \cdots + L(x)^{n-1},$$

    $$y = (\mathrm{Id} - L(x))^{-1}x = (\mathrm{Id} + L(x) + L(x)^2 + \cdots + L(x)^{n-1})x = x + x^2 + \cdots + x^n$$

    and it follows that $xy = yx$. To prove this for arbitrary x (not necessarily nilpotent) you need to show, using advanced calculus, that

    $$\begin{aligned} y = (\mathrm{Id} - L(x))^{-1}x &= (\mathrm{Id} + L(x) + L(x)^2 + \cdots + L(x)^n + \cdots)x \\ &= x + x^2 + \cdots + x^n + \cdots \end{aligned}$$

    and the infinite series converges. To do this, note that A is a finite dimensional vector space so is isomorphic to Φ^n (=$\mathbf{R}^n$ or $\mathbf{C}^n$) and the series converges if $|x| < 1$, where $|x|$ is the Euclidean length of x, namely, if $x = (x_1, \ldots, x_n)$, then

    $$|x| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}.$$

12. (Meyberg Chapter 2 #3 and #4, page 15) In an associative algebra A, let $B(x, y) := \mathrm{Id} - L(xy)$
    (i) Prove that for $x, y, u \in A$, $B(u - x - uyx, -y) = B(u, -y)B(x, y)$, and therefore if $u - x - xyu \in \mathrm{Rad}(A)$, then $B(x, y)$ is left invertible.
    (ii) suppose $x, y, u \in A$ are such that $u - x - xyu \in \mathrm{Rad}(A)$. Prove $B(x, y)$ is right invertible.

    Discussion: I will give a complete proof of (i): Lemma 4 on page 12-13 of Meyberg's notes states that x is quasi-invertible in A_y if and only if $B(x, y)$ is invertible in $\mathrm{End}\,(A)$. By Lemma 3 on page 11 of Meyberg this is equivalent to the existence of $u \in A$ such that $u - x = xyu = uyx$. Let z denote $u - x - xyu$. By the definition of $\mathrm{Rad}\,A$, z is quasi-invertible in A_y for every $y \in A$ and therefore $B(z, y)$ is invertible for every $y \in A$, so

    $$\mathrm{Id}_A = B(z, -y)^{-1}B(z, -y) = \underbrace{B(z, -y)^{-1}B(u, -y)}_{\text{left inverse of } B(x,y)} B(x, y).$$

13. (Meyberg Chapter 3 #1, page 22) Prove Theorem 1 on page 22:

    Theorem 1 Let F be a triple system. Then
    (i) A subset $U \subset F$ is an ideal if and only if U is the kernel of some homomorphism.
    (ii) If $f : F \to F'$ is a homomorphism, then $f(F) \simeq F/\ker f$
    (iii) If U and V are ideals of F, then $(U + V)/U \simeq U/(U \cap V)$

    Discussion: (i) Assume that U is an ideal. Let $F' = F/U$, denote $x + U$ by $\overline{x}$, and define a triple product

    $$\langle \overline{x}, \overline{y}, \overline{z} \rangle = \langle x + U, y + U, z + U \rangle := \langle xyz \rangle + U.$$

    Show that this triple product is well defined, that is, if $\overline{x} = \overline{x'}$, $\overline{y} = \overline{y'}$, $\overline{z} = \overline{z'}$, then $\langle xyz \rangle + U = \langle x'y'z' \rangle + U$. Recall that $\overline{x} = \overline{x'}$ means there is $u \in U$ such that $x = x' + u$.

    Show that the triple product is trilinear, that is, $\langle \overline{x} + \overline{x'}, \overline{y}, \overline{z} \rangle = \langle \overline{x}, \overline{y}, \overline{z} \rangle + \langle \overline{x'}, \overline{y}, \overline{z} \rangle$, etc.

Let $f : F \to F'$ be the quotient map $f(x) = \overline{x} = x + U$. Show that f is a triple homomorphism with kernel U.

Conversely, prove that if $f : F \to F'$ is a triple homomorphism, then $\ker f$ is an ideal.

(ii) Define $\varphi : f(F) \to F/\ker f$ by $\varphi(f(x) = \overline{x} = x + \ker f$. Show that φ is well-defined, one-to-one, onto, linear, and multiplicative.

(iii) Define $\psi : (U + V)/V \to U/(U \cap V)$ by $\varphi(u + v + V) = u + U \cap V$. Show that ψ is well-defined, one-to-one, onto, linear, and multiplicative.

14. (Meyberg Chapter 3 #4, page 24) If A is an algebra which is made into a triple system by the definition $\langle abc\rangle = (ab)c$, prove that every derivation of A as an algebra is a derivation of A as a triple system, and every homomorphism of A as an algebra (into another algebra B) is a homomorphism of the triple system $(A, \langle abc\rangle = (ab)c)$ into the triple system $(B, \langle xyz\rangle = (xy)z)$.

    Discussion: A derivation D of A satisfies $D(ab) = aD(b) + D(a)b$ and a homomorphism φ of A satisfies $\varphi(ab) = \varphi(a)\varphi(b)$. A triple derivation δ of A satisfies $\delta(\langle xyz\rangle) = \langle \delta(x), y, z\rangle + \langle x, \delta(y), z\rangle + \langle x, y, \delta(z)\rangle$, and a triple homomorphism ψ of A satisfies $\varphi(\langle xyz\rangle) = \langle \psi(x), \psi(y), \psi(z)\rangle$.

15. (Meyberg Chapter 3 #2-part 1, page 23)

    (i) Subtriples and homomorphic images of solvable triple systems are solvable
    (ii) If U is an ideal in F, then F is solvable if and only if U and F/U are solvable.
    (iii) If U and V are solvable ideals in a triple system F, then $U + V$ is a solvable ideal.

    Discussion: A triple system F is *solvable* if $F^{(n)} = 0$ for some $n \geq 1$, where $F^{(0)} := F$ and $F^{(n+1)} := \langle F^{(n)}, F^{(n)}, F^{(n)}\rangle$.

    (i) If G is a subtriple of F, show by induction that $G^{(n)} \subset F^{(n)}$. If $f : F \to F'$ is a triple homomorphism, then with $G := f(F) \subset F'$, show by induction that $G^{(n)} = f(F^{(n)})$.

    (ii) If F is solvable, then U and F/U are solvable by (i). To show the converse, show first by induction that $(F/U)^{(k)} = (F^{(k)} + U)/U$ as vector spaces. Then show that F/U being solvable implies $F^{(n)} \subset U$ for some $n \geq 1$. Now use the assumption that U is solvable.

    (iii) Show first that $U + V$ is an ideal. Then use the (i), (ii), and Problem 13(iii).

16. (Meyberg Chapter 3 #2-part 2, page 23)

    (i) Let I be an ideal in the quotient triple F/V. Show that $I = F/V$ where $V \subset U \subset F$, U is an ideal in F, and V is an ideal in U.
    (ii) If F is Noetherian (every non empty set of ideals has a maximal element), then F has a unique maximal solvable ideal $R(F)$ which contains all other solvable ideals, and R(F/R(F))={0}.
    (iii) If U is an ideal in F, and $R(F/U) = \{0\}$, then $R(F) \subset U$.

    Discussion: (i) Let $U = \{a \in F : a + V \in I\}$, verify that U is an ideal in F and V is an ideal in U. Then show that $I = (U + V)/V$ as sets and use Problem 13(iii).

    (ii) Let $R(F)$ be a maximal element of the set of solvable ideals of F. If R' is a solvable ideal, then $R(F) + R'$ is a solvable ideal containing $R(F)$,...

    A solvable ideal of $F/R(F)$ is of the form $G/R(F)$, where G is an ideal in F, and $R(F)$ is an ideal in G. Show that G is solvable and then use Problem 13(iii) to show that $R(F/R(F)) = 0$.

    (iii) If $\varphi : F \to F/U$ is the quotient homomorphism, consider $\varphi(R(F))$.

17. (Meyberg Chapter 3 #3, page 23)

    (i) If F is a triple system which satisfies $(a^{2n+1})^{2m+1} = a^{(2n+1)(2m+1)}$ for all $m, n > 0$ and $a \in F$, then the sum of two nil ideals is a nil ideal.
    (ii) In any triple system F, there is a maximal nil ideal $N(F)$.

(iii) If a triple system F satisfies $(a^{2n+1})^{2m+1} = a^{(2n+1)(2m+1)}$ for all $m, n > 0$ and $a \in F$, then F has a unique maximal nil ideal (called the nil radical).

Discussion: Powers in a triple system are defined inductively by $a^1 = a, a^3 = \langle aaa\rangle, a^5 = \langle a^3aa\rangle, \ldots, a^{2(n+1)+1} = \langle a^{2n+1}aa\rangle$, and a subsystem U of F is nil if every one of its elements is nilpotent, that is, for each $a \in U$, $a^{2n+1} = 0$ for some $n > 0$ which depends on a.

(i) If G and H are ideals in F, show by induction that if $g \in G$ and $h \in H$, then $(g+h)^{2n+1} = g^{2n+1} + d$ where $d \in H$. Now use the fact that G and H are nil together with the assumption. (You can also imitate Meyberg pp 6-7 and page 2 of my informal notes for Meyberg pp. 6-7)

(ii) and (iii) Use Zorn's lemma; (Imitate Meyberg pp 6-7 and pp. 3-4 of my informal notes for Meyberg pp. 6-7)

18. (Meyberg Chapter 4 #2, page 28) Prove the two identities for an associative triple system M.

(4.12) $\ell(x,y)\ell(u,v) = \ell(\langle xyu\rangle, v) = \ell(x, \langle vuy\rangle)$
(4.13) $r(x,y)r(u,v) = r(x, \langle yuv\rangle, v) = r(\langle uyx\rangle, v)$,

where $\ell(x,y)$ and $r(x,y)$ are defined on page 28 as follows: $L(x,y)z = \langle xyz\rangle = R(z,y)x$,

$$\ell(x,y) = (L(x,y), L(y,x)) \in \mathrm{End}\, M \oplus (\mathrm{End}\, M)^{op},$$

$$r(x,y) = (R(y,x), R(x,y)) \in (\mathrm{End}\, M \oplus (\mathrm{End}\, M)^{op})^{op}$$

Discussion: I will give a complete proof of (4.12):

$\ell(x,y)\ell(u,v) = (L(x,y), L(y,x))(L(u,v), L(v,u)) = (L(x,y)L(u,v), L(v,u)L(y,x))$
(remember, the product in $(\mathrm{End}\, M)^{op}$ is reversed).

Now use (4.9): $L(x,y)L(z,u) = L(\langle xyz\rangle, u\rangle = L(x, \langle uzy\rangle)$ to obtain

$$L(x,y)L(u,v) = L(\langle xyu\rangle, v) = L(x, \langle vuy\rangle),$$

$$L(v,u)L(y,x) = L(\langle vuy\rangle, x) = L(v, \langle xyu\rangle).$$

By definition,

$$\ell(\langle xyu\rangle, v) = (L(\langle xyu\rangle, v), L(v, \langle xyu\rangle)),$$
$$\ell(x, \langle vuy\rangle) = (L(x, \langle vuy\rangle), L(\langle vuy\rangle, x)).$$

$$\begin{array}{rrcl} \text{So} & \ell(x,y)\ell(u,v) & = & (L(x,y)L(u,v), L(v,u)L(y,x)) \\ & & = & (L(x, \langle vuy\rangle), L(\langle vuy\rangle, x)) \\ & & = & \ell(x, \langle vuy\rangle). \\ \text{Also} & \ell(x,y)\ell(u,v) & = & (L(\langle xyu\rangle, v), L(v, \langle xyu\rangle)) \\ & & = & \ell(\langle xyu\rangle, v). \end{array}$$

To prove (4.13), remember that the product $r(x,y)r(u,v)$ is taken in $(\mathrm{End}\, M \oplus (\mathrm{End}\, M)^{op})^{op}$, that is,

$$\begin{array}{rcl} r(x,y)r(u,v) & = & (R(y,x), R(x,y)) \circ (R(v,u), R(u,v)) \\ & = & (R(v,u), R(u,v)(R(y,x), R(x,y)) \\ & = & (R(v,u)R(y,x), R(x,y)R(u,v)) \\ & & \text{(remember, the product in } (\mathrm{End}\, M)^{op} \text{ is reversed).} \end{array}$$

Now use (4.10): $R(v,u)R(z,y) = R(\langle zuv\rangle, y) = R(v, \langle uzy\rangle)$.

19. (Meyberg Chapter 4 #1-part 1, page 27) Prove the statements (i)-(iii) in the Discussion below, which are used in Problem #20. (Problems 19 and 20 were discussed in class on November 18 and some details of their solutions has been posted on the website.)

    Discussion: The axiom for an associative triple system of the first kind is

    $$((4.1) \text{ on page } 25) \qquad \langle xy\langle uvw\rangle\rangle = \langle\langle xyu\rangle vw\rangle = \langle x\langle yuv\rangle w\rangle.$$

    We let $\tilde{E} = \operatorname{End} F \oplus (\operatorname{End} F)^{op}$, which is an algebra with unit $E = (\mathrm{Id}_F, \mathrm{Id}_F)$. Recall that the product in $\tilde{E}$ is, for $A = (A_1, A_2), B = (B_1, B_2) \in \tilde{E}$, given by

    $$AB = (A_1B_1, B_2A_2).$$

    Recall that if A is an associative algebra and V is a vector space, then we say that V is a left-A-module if there is a bilinear map $A \times V \ni (T, v) \mapsto T \cdot v \in V$ satisfying the axiom $(T_1T_2) \cdot v = T_1 \cdot (T_2v)$. We say that V is a right-A-module if there is a bilinear map $A \times V \ni (T, v) \mapsto v \cdot T \in V$ satisfying the axiom $v \cdot (T_1T_2) = (v \cdot T_1) \cdot T_2$. A vector space V which is both a left A-module and a right A-module is said to be an A-bimodule if in addition $(T_1 \cdot v) \cdot T_2 = T_1 \cdot (v \cdot T_2)$.

    The vector space F is a left-$\tilde{E}$-module and a right-$\tilde{E}$-module as follows: if $A = (A_1, A_2) \in \tilde{E}$, and $x \in F$,

    $$A \cdot x = A_1x \text{ (left module action)} \quad \text{and} \quad x \cdot A = A_2x \quad \text{(right module action)}.$$

    (i) Check that $A \cdot (B \cdot x) = (AB) \cdot x$ (left module axiom)
    (ii) Check that $(x \cdot A) \cdot B = x \cdot (AB)$ (right module axiom)
    (iii) Check that $(A \cdot x) \cdot B = A \cdot (x \cdot B)$ (bimodule axiom)

20. (Meyberg Chapter 4 #1-part 2, page 27) Prove the statements (i)-(iii) in the Discussion below, which constitues a proof of Theorem 1 on page 27.

    Theorem 1 Let F be an associative triple system of the first kind. Then $\tilde{A} := L \oplus F$ (with L defined below and on page 26) is an associative algebra with unit element (with multiplication defined below and on page 27) and there is a linear isomorphism f of F into A, such that $f(\langle xyz\rangle) = (f(x)f(y))f(z)$.

    Discussion: Let

    $$L_0 = \operatorname{span}\{\lambda(x, y) : x, y \in F\} \subset \tilde{E}$$

    where $\lambda(x, y) = (L(x, y), R(y, x)) \in \tilde{E}$. We have

    $$((4.5) \text{ on page } 25) \qquad \lambda(x, y)\lambda(u, v) = \lambda(x, \langle yuv\rangle) = \lambda(\langle xyu\rangle, v).$$

    Therefore, L_0 is a subalgebra of $\tilde{E}$ and we define a subalgebra L of $\tilde{E}$ with unit $E_1 = (\mathrm{Id}_M, \mathrm{Id}_M)$ by $L = \Phi E_1 + L_0$. We define the multiplication in $\tilde{A} = L \oplus F$ by

    $$(A, x)(B, y) = (AB + \lambda(x, y), A \cdot y + x \cdot B).$$

    (i) Show that this product is bilinear, that is

    $$((A, x) + (A', x'))(B, y) = (A, x)(B, y) + (A', x')(B, y)$$

    and

    $$(A, x)((B, y) + (B', y')) = (A, x)(B, y) + (A, x)(B', y').$$

    (ii) Prove that the product is associative, that is,

    $$((A, x)(B, y))(C, z) = (A, x)((B, y)(C, z)).$$

    and therefore $\tilde{A}$ becomes an associative triple system of the first kind with the triple product $\langle(A, x), (B, y), (C, z)\rangle = (A, x)(B, y)(C, z)$.

    (iii) Show that the function $f : F \to \tilde{A}$ defined by $f(x) = (0, x)$ is a triple homomorphism of F into the triple system $\tilde{A}$, that is, $f(x)f(y)f(z) = f(\langle xyz\rangle)$.

Part 3 #21-23

21. (Meyberg Chapter 4 #3, page 29) Prove the statements (i)-(ii) in the Discussions below, which are used in Problem 22.

Discussion: Let M be an associative triple system of the second kind. It is stated in Lemma 3(i) on page 29 that M is a left L-module and a right R-module. L and R are defined in the next paragraphs. (**Warning!** This L is different from the L defined in Problem 20, which is anyway concerned with associative triple systems of the first kind.)

As in Problem 19, we let $\tilde{E} = \operatorname{End} M \oplus (\operatorname{End} M)^{op}$, which is an algebra with unit $E = (\mathrm{Id}_M, \mathrm{Id}_M)$. Recall that the product in $\tilde{E}$ is, for $A = (A_1, A_2), B = (B_1, B_2) \in \tilde{E}$, given by

$$AB = (A_1B_1, B_2A_2).$$

L is defined on page 29 of Meyberg, as follows. Let $L_0 = \operatorname{span}\{\ell(x,y) : x, y \in M\} \subset \tilde{E}$ where $\ell(x,y) = (L(x,y), L(y,x)) \in \tilde{E}$. By (4.12) in Problem 18, L_0 is a subalgebra of $\tilde{E}$ and we define a subalgebra L of $\tilde{E}$ with unit $E_1 = (\mathrm{Id}_M, \mathrm{Id}_M)$ by $L = \Phi E_1 + L_0 \subset \tilde{E}$.

R is defined on page 29 of Meyberg, as follows. Let $R_0 = \operatorname{span}\{r(x,y) : x, y \in M\} \subset (\tilde{E})^{op}$ where $r(x,y) = (R(y,x), R(x,y)) \in (\tilde{E})^{op}$. By (4.13) in Problem 18, R_0 is a subalgebra of $(\tilde{E})^{op}$ and we define a subalgebra R of $\tilde{E}$ with unit $E_2 = (\mathrm{Id}_M, \mathrm{Id}_M)$ by $R = \Phi E_2 + R_0 \subset (\tilde{E})^{op}$.

(Note that $\tilde{E}$ has a natural involution: $(A,B) \mapsto \overline{(A,B)} := (B,A)$. This fact will be used in Problem 23.)

Recall the module actions: if $A = (A_1, A_2) \in L \subset \tilde{E}$, and $x \in M$, $A \cdot x = A_1 x$. As for the right R-module action, if $B = (B_1, B_2) \in R \subset (\tilde{E})^{op}$ and $x \in M$, $x \cdot B = B_1 x$.

(i) Prove Lemma 3(i) on page 29, that is, show that M is a left L-module, a right R-module, and an (L,R)-bimodule. (You will need to use (4.11) on page 28 of Meyberg.)

More discussion: It is stated in Lemma 3(ii) on page 29 that $\overline{M}$ is a right L-module and a left R-module ($\overline{M}$ is an isomorphic copy of M). Here are the module actions:
If $A = (A_1, A_2) \in L \subset \tilde{E}$, and $\overline{x} \in \overline{M}$,

$$\overline{x} \cdot A = \overline{A_2 x} \text{ (=a right module action of } L \subset \tilde{E} \text{ on } \overline{M}).$$

If $B = (B_1, B_2) \in R \subset (\tilde{E})^{op}$, and $\overline{x} \in \overline{M}$,

$$B \cdot \overline{x} = \overline{B_2 x} \text{ (=a left module action of } R \subset (\tilde{E})^{op} \text{ on } \overline{M}).$$

(ii) Prove Lemma 3(ii) on page 29, that is, show that $\overline{M}$ is a left R-module, a right L-module, and an (R,L)-bimodule. (You will need to use (4.11) on page 28 of Meyberg.)

22. (Meyberg Chapter 4 #4-part 1, page 31) Let M be an associative triple system of the second kind. Let $\tilde{A} := L \oplus M \oplus \overline{M} \oplus R$ (L and R were defined in Problem 21 and on page 29 of Meyberg, and $\overline{M}$ is an isomorphic copy of M). Prove the statements (i)-(ii).

(i) $\tilde{A}$ is an associative algebra with unit element $E = (E_1, 0, 0, E_2)$, with multiplication defined on page 30 as follows:

$$(A, x, \overline{y}, B)(A', x', \overline{y'}.B') = (\underbrace{AA' + \ell(x,y')}_{\in L}, \underbrace{A \cdot x' + x \cdot B'}_{\in M}, \underbrace{\overline{y} \cdot A' + B \cdot \overline{y'}}_{\in \overline{M}}, \underbrace{r(y,x') + BB'}_{\in R}). \quad (2)$$

(ii) $\tilde{A}_0 := L_0 \oplus M \oplus \overline{M} \oplus R_0$ (with L_0 and R_0 defined in Problem 21 and on page 28) is an ideal in $\tilde{A}$.

To help remember the formula (2), think of it as "matrix multiplication":

$$\begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix} \begin{bmatrix} A' & x' \\ \overline{y'} & B' \end{bmatrix} = \begin{bmatrix} AA' + \ell(x,y') & A \cdot x' + x \cdot B' \\ \overline{y} \cdot A' + B \cdot \overline{y'} & r(y,x') + BB' \end{bmatrix}.$$

To prove (i), you need to show bilinearity, for example:

$$\begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix}\left(\begin{bmatrix} A' & x' \\ \overline{y'} & B' \end{bmatrix}+\begin{bmatrix} A'' & x'' \\ \overline{y''} & B'' \end{bmatrix}\right)=\begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix}\begin{bmatrix} A' & x' \\ \overline{y'} & B' \end{bmatrix}+\begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix}\begin{bmatrix} A'' & x'' \\ \overline{y''} & B'' \end{bmatrix},$$

and associativity:

$$\left(\begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix}\begin{bmatrix} A' & x' \\ \overline{y'} & B' \end{bmatrix}\right)\begin{bmatrix} A'' & x'' \\ \overline{y''} & B'' \end{bmatrix}=\begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix}\left(\begin{bmatrix} A' & x' \\ \overline{y'} & B' \end{bmatrix}\begin{bmatrix} A'' & x'' \\ \overline{y''} & B'' \end{bmatrix}\right).$$

To prove (ii), you need to show:

$$\begin{bmatrix} \ell(x,y) & z \\ \overline{w} & r(u,v) \end{bmatrix}\begin{bmatrix} A' & x' \\ \overline{y'} & B' \end{bmatrix}\in \tilde{A}_0 \text{ and } \begin{bmatrix} A' & x' \\ \overline{y'} & B' \end{bmatrix}\begin{bmatrix} \ell(x,y) & z \\ \overline{w} & r(u,v) \end{bmatrix}\in \tilde{A}_0,$$

23. (Meyberg Chapter 4 #4-part 2, page 31) With the notation of Problem 22, prove (iii)-(v).

(iii) There is an involution $u \mapsto \overline{u}$ on A, defined in the statement of Theorem 2 on page 30 and as follows:

$$j(u) = j\left(\begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix}\right) = \overline{u} = \begin{bmatrix} \overline{A} & y \\ \overline{x} & \overline{B} \end{bmatrix}.$$

Recall from page 29 of Meyberg that if $A = (A_1, A_2) \in \tilde{E}$, then $\overline{A} = (A_2, A_1)$, and from page 8 of Meyberg that an involution of $\tilde{A}$ is a linear map $j : \tilde{A} \to \tilde{A}$ such that $j(uv) = j(v)j(u)$ and $j(j(u)) = u$ for $u, v \in \tilde{A}$.

Since it is obvious that $j : u \mapsto \overline{u}$ defined above is linear and satisfies $j(j(u)) = u$, to prove (iii) you need to show that

$$j\left(\left(\begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix}\begin{bmatrix} A' & x' \\ \overline{y'} & B' \end{bmatrix}\right)\right)=\begin{bmatrix} \overline{A'} & y' \\ \overline{x'} & \overline{B'} \end{bmatrix}\begin{bmatrix} \overline{A} & y \\ \overline{x} & \overline{B} \end{bmatrix}.$$

(iv) there is a linear isomorphism f of M into $\tilde{A}$, such that $f(\langle xyz\rangle) = f(x)\overline{f(y)}f(z)$.

To prove (iv), let $f(x) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$, and calculate $f(x)\overline{f(y)}f(z)$.

(v) The Peirce components of $\tilde{A}$ with respect to the idempotent $\tilde{E}_1 = \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}$ are

$$\tilde{A}_{11} = L \quad , \quad \tilde{A}_{10} = M \quad , \quad \tilde{A}_{01} = \overline{M} \quad , \quad \tilde{A}_{00} = R.$$

For examples,

$$\begin{aligned}
\tilde{A}_{11} &= \tilde{E}_1\tilde{A}\tilde{E}_1 = \{\begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix}\begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix} \in \tilde{A}\} \\
&= \{\begin{bmatrix} E_1AE_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix} \in \tilde{A}\} = \{\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix} \in \tilde{A}\} = L;
\end{aligned}$$

and

$$\begin{aligned}
\tilde{A}_{10} &= \tilde{E}_1\tilde{A}(1-\tilde{E}_1) \\
&= \{\begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix} - \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix}\begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix} \in \tilde{A}\} \\
&= \{\begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix} \in \tilde{A}\} \\
&= \{\begin{bmatrix} A & x \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix} \in \tilde{A}\} \\
&= \{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A & x \\ \overline{y} & B \end{bmatrix} \in \tilde{A}\} = M. \quad \text{(The rest is up to you)}
\end{aligned}$$