Math 199B—Winter 2017
Final Exam Exercises
Problems will be due on March 24, 2017
(Extensions of this deadline are possible)

Part 1 Chapter 7 #1-5

1. (Chapter 7, Exercise 1 page 66)
   If $\mathcal{F}$ is a Jordan algebra over the field Φ (=**R** or **C**), then the unital algebra $\widehat{\mathcal{F}} = \Phi \oplus \mathcal{F}$ is a Jordan algebra.

2. (Chapter 7, Exercise 2 page 67)
   In a Jordan algebra $\mathcal{F}$, the mappings $[L(x), L(y)]$, $x, y \in \mathcal{F}$ are derivations of $\mathcal{F}$.

3. (Chapter 7, Exercise 3 page 69)
   In a Jordan algebra, $y\{uvw\} = \{(yu)vw\} - \{u(yv)w\} + \{uv(yw)\}$.

4. (Chapter 7, Exercise 4 page 69)
   In a Jordan algebra, if D is a derivation, then $D\{uvw\} = \{(Du)vw\}+\{u(D(v))w\}+\{uv(D(w))\}$.

5. (Chapter 7, Exercise 5 page 70)
   Fill in the details of the proof of (7.14): $P(P(u)v) = P(u)P(v)P(u)$.

Part 2 Chapter 8 #1-3

6. (Chapter 8 , Exercise 1 page 73)
   If V is a vector space over F (=**R** or **C**), and $q : V \to F$ is a quadratic form, then $D = F \oplus V$ is a Jordan algebra if we define
   $$(\alpha, x)(\beta, y) = (\alpha\beta + q(x, y), \alpha y + \beta x).$$

7. (Chapter 8 , Exercise 2 page 73)
   The space D in the previous problem is a quadratic extension of F, or if the dimension of V is 1 and q is non degenerate, then D is isomorphic to $F \oplus F$.

8. (Chapter 8 , Exercise 3 page 73)
   Is the space D in the previous two problems a field?

   (See the webpage: Fifth meeting, February 10, 2017. Simple Jordan algebras (informal notes: Meyberg pages 73-74, Exercise 3 added)—see page 7)

Part 3 Chapter 5 #1-2

9. (Chapter 5 , Exercise 1 page 41)
   In a Lie algebra $\mathcal{L}$, (a) $[\mathcal{L}^m, \mathcal{L}^n] \subset \mathcal{L}^{m+n}$ and (b) $\mathcal{L}^{(n)} \subset \mathcal{L}^{2^n}$.

10. (Chapter 5 , Exercise 2 page 41)
    In a Lie algebra, the nil radical is contained in the radical.

Part 4 Chapter 6 #1-6

11. (Chapter 6 , Exercise 1 page 43)
    If $(\mathcal{F}, < xyz >)$ is an associative triple system, then $(\mathcal{F}, [xyz])$ is a Lie triple system, where
    $$[xyz] :=< xyz > - < yxz > - < zxy > + < zyx > .$$

12. (Chapter 6 , Exercise 2 page 43)
Let A be a commutative algebra over our field Φ (=$\mathbf{R}$ or $\mathbf{C}$) with multiplication $xy = L(x)y$. Set $D(x,y) = [L(x), L(y)]$ and assume for all $x, y, u, v \in A$ that

$$[D(x,y), D(u,v)] = D(D(x,y)u, v) + D(u, D(x,y)v).$$

(For example, A could be a Jordan algebra.)

Define $[xyz] := D(x,y)z$ and suppose $\mathcal{F}$ is a subspace of A closed under $[xyz]$. Then $(\mathcal{F}, [xyz])$ is a Lie triple system.

13. (Chapter 6 , Exercise 3 page 46)
Let F be a field, and $\mathcal{F} = F^n$ the Lie triple system of column vectors over F under the triple product $[xyz] := yx^tz - xy^tz$. The mapping $\mathcal{H} \oplus \mathcal{F} \ni (A, x) \mapsto \begin{bmatrix} A & x \\ -x^t & 0 \end{bmatrix} \in A_{n+1}(F)$ (= the Lie algebra of all $n+1$ by $n+1$ skew symmetric matrices over F) is a Lie algebra isomorphism.

14. (Chapter 6 , Exercise 4 page 47)
Let $\mathcal{F}_i$ $(i = 1, 2)$ be Lie triple systems with standard imbeddings $\mathcal{L}_i = \mathcal{H}_i \oplus \mathcal{F}_i$. If $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a Lie algebra homomorphism and $\eta : \mathcal{F}_1 \to \mathcal{F}_2$ is a linear map such that
(i) $\phi L_1(x,y) = L_2(\eta x, \eta y)$ and
(ii) $\eta H = \phi(H)\eta$, then
$\lambda : \mathcal{L}_1 \to \mathcal{L}_2$ defined by $\lambda(H, x) = (\phi(x), \eta(x))$ is a Lie algebra homomorphism.
(L_i is the left multiplication of $\mathcal{F}_i$.)

15. (Chapter 6 , Exercise 5 page 47)
(i) Let $\eta : \mathcal{F}_1 \to \mathcal{F}_2$ be a Lie triple isomorphism and define $\Lambda(H, x) = (\eta H \eta^{-1}, \eta x)$. Then Λ is an isomorphism of $\mathcal{L}_1$ onto $\mathcal{L}_2$ which commutes with the main involutions, that is, $\Lambda\theta_1 = \theta_2\Lambda$.

(ii) Conversely, if $\Lambda : \mathcal{L}_1 \to \mathcal{L}_2$ is an isomorphism such that $\Lambda\theta_1 = \theta_2\Lambda$, then the restriction of Λ to $\mathcal{F}_1$ maps onto $\mathcal{F}_2$ and is a Lie triple isomorphism.

16. (Chapter 6 , Exercise 6 page 57)
If $\mathcal{F}$ is a Lie triple system with standard imbedding $\mathcal{L} = \mathcal{H} \oplus \mathcal{F}$, and D is a derivation of $\mathcal{F}$, then $\delta : \mathcal{L} \to \mathcal{L}$, defined by $\delta(H, a) = ([D, H], Da)$ is a derivation of the Lie algebra $\mathcal{L}$.

(See the webpage: Ninth meeting, March 10, 2017. Derivations on finite dimensional semisimple Lie Triple Systems are Inner (Theorem 10, page 57, informal notes and review of radicals, Exercise 6 added)—see page 3)