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ON SOME CLASSES OF NILPOTENT LEIBNIZ ALGEBRAS

Sh. A. Ayupov and B. A. Omirov

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This article is devoted to studying Leibniz algebras that were introduced in Loday's articles [1, 2] as a "noncommutative" analog of Lie algebras.

We define null-filiform algebras and study their properties. For Lie algebras, the notion of p-filiform algebra makes sense for $p \geq 1$ [3] and looses sense for $p = 0$, since a Lie algebra has at least two generators. In the case of Leibniz algebras, this notion is meaningful for $p = 0$; so the introduction of null-filiform algebra is quite justified.

We study complex non-Lie filiform Leibniz algebras. In particular, we give some equivalent conditions for a Leibniz algebra to be filiform and describe naturally graded complex Leibniz algebras.

§ 1. Description for the Irreducible Component of the Set of Nilpotent Leibniz Algebras Containing an Algebra of Maximal Nilindex

Definition 1. An algebra L over a field F is a *Leibniz algebra* if the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds for all $x, y, z \in L$. Here $[\ ,]$ is the multiplication in L.

Observe that if the identity $[x, x] = 0$ holds in L then the Leibniz identity coincides with the Jacobi identity. Thus, a Leibniz algebra is a "noncommutative" analog of a Lie algebra.

Given an arbitrary algebra L, define its lower central series

$$L^{\langle 1 \rangle} = L, \quad L^{\langle n+1 \rangle} = [L^{\langle n \rangle}, L].$$

Definition 2. An algebra L is *nilpotent* if $L^{\langle n \rangle} = 0$ for some $n \in N$.

It is easy to see that the nilpotency class of an arbitrary n-dimensional nilpotent algebra is at most $n + 1$.

Definition 3. A Leibniz algebra L of dimension n is a *null-filiform algebra* if $\dim L^i = (n+1)-i$, $1 \leq i \leq n + 1$.

Clearly, the definition of a null-filiform algebra L amounts to requiring that L has a maximal nilpotency class.

Lemma 1. *In every null-filiform Leibniz algebra of dimension n, there is a basis with the following multiplications:*

$$[x_i, x_1] = x_{i+1} \text{ for } 1 \leq i \leq n - 1, \quad [x_i, x_j] = 0 \text{ for } j \geq 2. \tag{1}$$

Proof. Let L be a null-filiform Leibniz algebra of dimension n and let $\{e_1, e_2, \dots, e_n\}$ be a basis for L such that $e_1 \in L^1 \setminus L^2$, $e_2 \in L^2 \setminus L^3, \dots, e_n \in L^n$ (such a basis can be chosen always). Since $e_2 \in L^2$, for some elements a_{2p}, b_{2p} of L we have

$$e_2 = \sum [a_{2p}, b_{2p}] = \sum \alpha_{ij}^2 [e_i, e_j] = \alpha_{11}^2 [e_1, e_1] + (^*),$$

where $(^*) \in L^3$; i.e., $e_2 = \alpha_{11}^2 [e_1, e_1] + (^*)$. Notice that $\alpha_{11}^2 [e_1, e_1] \neq 0$ (otherwise $e_2 \in L^3$). Similarly, obtain

$$e_3 = \sum [[a_{3p}, b_{3p}], c_{3s}] = \sum \alpha_{ijk}^3 [[e_i, e_j], e_k] = \alpha_{111}^3 [[e_1, e_1], e_1] + (^{**}),$$

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where (**) $\in L^4$; i.e., $e_3 = \alpha^3_{111}[[e_1, e_1], e_1] + (**)$. Notice that $\alpha^3_{111}[[e_1, e_1], e_1] \neq 0$ (otherwise $e_3 \in L^4$). Continuing likewise, we conclude that the elements

$$x_1 := e_1,\ x_2 := [e_1, e_1],\ x_3 := [[e_1, e_1], e_1], \dots, x_n := [[[e_1, e_1], e_1], \dots, e_1]$$

differ from zero. It is easy to check that these elements are linearly independent. Hence, they constitute a basis for L. Thus, $[x_i, x_1] = x_{i+1}$ for $1 \leq i \leq n-1$; moreover, $[x_i, x_j] = 0$ for $j \geq 2$. Indeed, if $j = 2$ then

$$[x_i, x_2] = [x_i, [x_1, x_1]] = [[x_i, x_1], x_1] - [[x_i, x_1], x_1] = 0.$$

Assume this proven for $j > 2$. Validity for $j + 1$ follows then from the inductive hypothesis and the equality

$$[x_i, x_{j+1}] = [x_i, [x_j, x_1]] = [[x_i, x_j], x_1] - [[x_i, x_1], x_j] = 0.$$

The proof of the lemma is over.

Henceforth we denote the algebra with multiplication (1) by L_0.

Take $x \in L \setminus [L, L]$. For the nilpotent operator R_x of right multiplication, define the decreasing sequence $C(x) = (n_1, n_2, \dots, n_k)$ that consists of the dimensions of the Jordan blocks of R_x. Endow the set of these sequences with the lexicographic order; i.e. $C(x) = (n_1, n_2, \dots, n_k) \leq C(y) = (m_1, m_2, \dots, m_s)$ means that there is an $i \in N$ such that $n_j = m_j$ for all $j < i$ and $n_i < m_i$.

DEFINITION 4. The sequence $C(L) = \max_{x \in L \setminus [L,L]} C(x)$ is defined to be the *characteristic sequence* of the algebra L.

DEFINITION 5. The set $Z(L) = \{x \in L : [y, x] = 0\ \forall y \in L\}$ is the *right annihilator* of L.

EXAMPLE 1. Let L be an arbitrary algebra and $C(L) = (1, 1, \dots, 1)$. Then L is abelian.

EXAMPLE 2. Let L be an n-dimensional Leibniz algebra. By Lemma 1, L is a null-filiform algebra if and only if $C(L) = (n, 0)$.

Consider an arbitrary algebra L in the set of n-dimensional Leibniz algebras over a field F. Let $\{e_1, e_2, \dots, e_n\}$ be a basis for L. Then L is determined, up to isomorphism, by the multiplication rule for the basis elements; namely,

$$[e_i, e_j] = \sum_{k=1}^{n} \gamma^k_{ij} e_k,$$

where γ^k_{ij} are the structure constants. Therefore, fixing a basis, we can regard each algebra of dimension n over a field F as a point in the n^3-dimensional space of structure constants endowed with the Zariski topology. A change of the basis corresponds to a natural action of the group $GL_n(F)$ over F; the orbit of a point under this action is the set of all isomorphic algebras.

Let $\mathfrak{I}_n(F)$ be the set of structure constants of all n-dimensional Leibniz algebras over a field F and let N_n be the subset of $\mathfrak{I}_n(F)$ consisting of the structure constants of all nilpotent n-dimensional Leibniz algebras over F.

The Leibniz identity implies the polynomial identities

$$\sum_{l=1}^{n} \left(\gamma^l_{jk} \gamma^m_{il} - \gamma^l_{ij} \gamma^m_{lk} + \gamma^l_{ik} \gamma^m_{lj} \right) = 0$$

for structure constants. Hence, the set $\mathfrak{I}_n(F)$ in F^{n^3} is an affine variety.

DEFINITION 6. Define the action of the group $GL_n(F)$ on the set $\mathfrak{I}_n(F)$ as follows: $[x, y]_g := g[g^{-1}x, g^{-1}y]$, where $g \in GL_n(F)$ and $x, y \in L$. Denote by $\mathrm{Orb}_n(L)$ the orbit $GL^*_n L$ of an algebra L.

Clearly, $\mathrm{Orb}_n(L)$ consists of all algebras isomorphic to L (the stabilizer of L is the group $\mathrm{Aut}(L) \Rightarrow \mathrm{Orb}_n(L) = GL_n(F)/\mathrm{Aut}(L)$). In the case of an arbitrary field F the closure $\overline{\mathrm{Orb}_n(L)}$ of the orbit $\mathrm{Orb}_n(L)$ is understood to be taken with respect to the Zariski topology; for $F = C$ it coincides with closure with respect to the Euclidean topology.

It is easy to see that the scalar matrices of $GL_n(F)$ act on $\mathfrak{I}_n(F)$ scalarly; therefore, the orbits $\mathrm{Orb}_n(L)$ are cones with the deleted vertex $\{0\}$ that corresponds to the abelian algebra a_n. Thus, a_n belongs to $\overline{\mathrm{Orb}_n(L)}$ for all $L \in \mathfrak{I}_n(F)$. In particular, among the orbits $\mathrm{Orb}_n(L)$ only one is closed, the orbit of a_n (a_n is abelian).

By [4] the set $\{L \in \mathfrak{I}_n(F) : \dim Z(L) \geq n-1\}$ is closed in the Zariski topology. Therefore,

$$\overline{\mathrm{Orb}_n(L_0)} \subseteq N_n \cap \{L \in \mathfrak{I}_n(F) : \dim Z(L) \geq n-1\}.$$

For convenience, we introduce the notation

$$N_nZ := N_n \cap \{L \in \mathfrak{I}_n(F) : \dim Z(L) = n-1\}.$$

The case in which $\dim Z(L) = n$ is not interesting, since L is in this case abelian.

Lemma 2. *Let L be an algebra in N_nZ with a characteristic sequence $C(L) = (m, n-m)$. Then for $m = n/2$ L is isomorphic to the algebra*

$$[e_1, e_n] = 0,\ [e_2, e_n] = e_1, \dots, [e_m, e_n] = e_{m-1},\ [e_{m+1}, e_n] = 0, [e_{m+2}, e_n] = e_{m+1},$$
$$[e_{m+3}, e_n] = e_{m+2}, \dots, [e_n, e_n] = e_{n-1},$$

and for $m > \frac{n}{2}$ it is isomorphic to one of the two nonisomorphic algebras:

$$[e_1, e_m] = 0,\ [e_2, e_m] = e_1, \dots, [e_m, e_m] = e_{m-1},$$
$$[e_{m+1}, e_m] = 0,\ [e_{m+2}, e_m] = e_{m+1},\ [e_{m+3}, e_m] = e_{m+2}, \dots, [e_n, e_m] = e_{n-1},$$

$$[e_1, e_n] = 0,\ [e_2, e_n] = e_1, \dots, [e_m, e_n] = e_{m-1},\ [e_{m+1}, e_n] = 0,$$
$$[e_{m+2}, e_n] = e_{m+1},\ [e_{m+3}, e_n] = e_{m+2}, \dots, [e_n, e_n] = e_{n-1}.$$

PROOF. Let $\{e_1, \dots, e_n\}$ be a basis for L, $L \in N_nZ$, and $C(L) = (m, n-m)$. Then there is $x \in L \setminus [L, L]$ such that

$$R_x = \begin{pmatrix} J_m & 0 \\ 0 & J_{n-m} \end{pmatrix};$$

i.e.,

$$[e_1, x] = 0,\ [e_2, x] = e_1, \dots, [e_m, x] = e_{m-1},\ [e_{m+1}, x] = 0,$$
$$[e_{m+2}, x] = e_{m+1},\ [e_{m+3}, x] = e_{m+2}, \dots, [e_n, x] = e_{n-1}.$$

For convenience, assume x to be a basis element (which is possible due to $\dim Z(L) = n-1$). Since $\dim Z(L) = n-1$, it follows that $[L, L] \subseteq Z(L)$ and so x does not belong to the linear span of the vectors $\{e_1, \dots, e_{m-1}, e_{m+1}, \dots, e_{n-1}\} \subseteq Z(L)$. Hence, $x = e_m$ or $x = e_n$. For $m = n/2$, changing the basis to

$$\bar{e}_1 = e_{m+1},\ \bar{e}_2 = e_{m+2}, \dots, \bar{e}_m = e_n,\ \bar{e}_{m+1} = e_1,\ \bar{e}_{m+2} = e_2, \dots, \bar{e}_n = e_m,$$

we may assume that the algebras

$$[e_1, e_m] = 0,\ [e_2, e_m] = e_1, \dots, [e_m, e_m] = e_{m-1},\ [e_{m+1}, e_m] = 0,$$
$$[e_{m+2}, e_m] = e_{m+1},\ [e_{m+3}, e_m] = e_{m+2}, \dots, [e_n, e_m] = e_{n-1},$$

$$[e_1, e_n] = 0,\ [e_2, e_n] = e_1, \dots, [e_m, e_n] = e_{m-1},\ [e_{m+1}, e_n] = 0,$$
$$[e_{m+2}, e_n] = e_{m+1},\ [e_{m+3}, e_n] = e_{m+2}, \dots, [e_n, e_n] = e_{n-1}$$

are isomorphic.

For $m > n/2$, suppose that these algebras are isomorphic; i.e., there is an isomorphism φ from the first algebra onto the second. Then $\varphi(e_m) = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n$, where $\alpha_n \neq 0$. It is well known that every isomorphism takes generators into generators. Therefore,

$$[\varphi(e_n), \varphi(e_m)] = \varphi(e_{m-1}), \dots, [\varphi(e_2), \varphi(e_m)] = 0$$

(in view of $m > n - m$); a contradiction. This completes the proof of the lemma.

For convenience, in the case of $\dim Z(L) = n - 1$ we henceforth specify an algebra L by defining the operator of right multiplication by an element x, where $x \in Z(L)$.

Corollary 1. *Assume that $L \in N_n Z$ and $C(L) = (n_1, \dots, n_s)$. Then L is isomorphic to one of the algebras*

$$R_{e_{n_1}} = \begin{pmatrix} J_{n_1} & 0 & \cdots & 0 & 0 \\ 0 & J_{n_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & J_{n_s} \end{pmatrix}, \dots, R_{e_{n_1+\dots+n_s}} = \begin{pmatrix} J_{n_1} & 0 & \cdots & 0 & 0 \\ 0 & J_{n_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & J_{n_s} \end{pmatrix},$$

where $J_{n_1}, \dots, J_{n_s}$ are Jordan blocks of respective dimensions $n_1, \dots, n_s$. In particular, $R_{e_{n_1+\dots+n_{i-1}}} \cong R_{e_{n_1+\dots+n_{i-1}+n_i}}$ if and only if $n_{i-1} = n_i$.

Proof. Suppose that L satisfies the conditions of the lemma. Then the arguments similar to those in Lemma 2 show that L may be one of the algebras in the statement of the corollary. Assume that $n_{i-1} = n_i$, where $2 \leq i \leq s$. Changing the basis as follows

$$\bar{e}_{n_1+\dots+n_{i-2}+1} := e_{n_1+\dots+n_{i-2}+n_{i-1}+1},$$
$$\bar{e}_{n_1+\dots+n_{i-2}+2} := e_{n_1+\dots+n_{i-2}+n_{i-1}+2}, \dots, \bar{e}_{n_1+\dots+n_{i-1}} := e_{n_1+\dots+n_i},$$
$$\bar{e}_{n_1+\dots+n_{i-2}+n_{i-1}+1} := e_{n_1+\dots+n_{i-2}+1},$$
$$\bar{e}_{n_1+\dots+n_{i-2}+n_{i-1}+2} := e_{n_1+\dots+n_{i-2}+2}, \dots, \bar{e}_{n_1+\dots+n_i} := e_{n_1+\dots+n_{i-1}},$$
$$\bar{e}_i = e_i \text{ for the other indices,}$$

we obtain an isomorphism between the algebras $R_{e_{n_1+\dots+n_{i-1}}}$ and $R_{e_{n_1+\dots+n_i}}$. By analogy to Lemma 2, we can demonstrate that the algebra $R_{e_{n_1+\dots+n_{i-1}}}$ is not isomorphic to $R_{e_{n_1+\dots+n_i}}$ if $n_{i-1} \neq n_i$ for some i. This completes the proof of the corollary.

Under the assumptions of Corollary 1, we also have

Corollary 2. *The number of nonisomorphic algebras in $N_n Z$ equals the cardinality of the set $\{n_1, \dots, n_s\}$.*

Lemma 3. *Let L be an algebra in $N_n Z$ with a basis $\{e_1, \dots, e_n\}$. Then $L \in \overline{\mathrm{Orb}_n(L_0)}$ if and only if $C(L) = C(e_n)$.*

Proof. Putting $\bar{e}_i := e_{n+1-i}$ for $1 \leq i \leq n$, we obtain $C(L_0) = C(e_n)$; i.e., $L_0 \cong R_{\bar{e}_n} = J_n$. Suppose that L satisfies the conditions of the lemma; i.e.,

$$L \cong R_{e_n} = \begin{pmatrix} J_{n_1} & 0 & \cdots & 0 & 0 \\ 0 & J_{n_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & J_{n_s} \end{pmatrix}.$$

Consider the family of the matrices $(g_{\lambda_1})_{\lambda_1 \in R \setminus \{0\}}$ defined as follows:

$$g_{\lambda_1}(e_i) = \lambda_1^{-1} e_i \text{ for } 1 \leq i \leq n_1, \quad g_{\lambda_1}(e_i) = e_i \text{ for } n_1 + 1 \leq i \leq n.$$

Passing to the limit of this family as $\lambda_1 \to 0$, i.e., $\lim_{\lambda_1\to 0} g_{\lambda_1}^{-1}[g_{\lambda_1}(e_i), g_{\lambda_1}(e_j)]$, we obtain

$$L_0 \underset{\lambda_1\to 0}{\longrightarrow} R_{e_n} = \begin{pmatrix} J_{n_1} & 0 \\ 0 & J_{n-n_1} \end{pmatrix}.$$

Now, take the family of the matrices $(g_{\lambda_2})_{\lambda_2\in R\setminus\{0\}}$ defined by

$$g_{\lambda_2}(e_i) = \lambda_2^{-1} e_i \text{ for } n_1+1 \le i \le n_1+n_2,$$
$$g_{\lambda_2}(e_i) = e_i \text{ for } 1 \le i \le n_1 \text{ and } n_1+n_2+1 \le i \le n.$$

Taking the limit of this family as $\lambda_2 \to 0$, i.e., $\lim_{\lambda_2\to 0} g_{\lambda_2}^{-1}[g_{\lambda_2}(e_i), g_{\lambda_2}(e_j)]$, we obtain

$$L_0 \underset{\lambda_2\to 0}{\longrightarrow} R_{e_n} = \begin{pmatrix} J_{n_1} & 0 & 0 \\ 0 & J_{n_2} & 0 \\ 0 & 0 & J_{n-n_1-n_2} \end{pmatrix}.$$

Continuing the procedure s times, we conclude that the algebra defined by the operator

$$R_{e_n} = \begin{pmatrix} J_{n_1} & 0 & \cdots & 0 & 0 \\ 0 & J_{n_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & J_{n_s} \end{pmatrix}$$

belongs to $\overline{\mathrm{Orb}_n(L_0)}$. Assume that $L \in \overline{\mathrm{Orb}_n(L_0)}$. The multiplication in L is determined by that in L_0 as follows: $[e_i, e_j] = \lim_{\lambda\to 0} g_\lambda^{-1}[g_\lambda e_i, g_\lambda e_j]$. For every $\lambda \ne 0$ we have

$$g_\lambda(\mathrm{lin}(e_1, \dots, e_{n-1})) \subseteq \mathrm{lin}(e_1, \dots, e_{n-1}).$$

Therefore, $[e_i, e_j] = 0$ for $1 \le j \le n-1$. Thus, L is determined by the operator R_{e_n}. Let $Q^{-1}R_{e_n}Q = J$ (J is the Jordan form of the operator R_{e_n}). Taking the family $(g_\lambda Q)_{\lambda\in R\setminus\{0\}}$, we may assume that the operator R_{e_n} is in Jordan form; i.e., $C(L) = C(e_n)$, which completes the proof of the lemma.

Since the orbit of a null-filiform algebra is an open set in the affine variety N_n, from [5] we conclude that its closure is an irreducible component of N_n and the following theorem holds.

Theorem 1. *An irreducible component of the variety N_n, containing a null-filiform algebra, up to isomorphism consists of the following algebras:*

$$R_{e_n} = \begin{pmatrix} J_{n_1} & 0 & \cdots & 0 & 0 \\ 0 & J_{n_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & J_{n_s} \end{pmatrix},$$

where $n_1 + \dots + n_s = n$.

Proof ensues from Lemma 3 and Corollary 1.

Remark 1. Theorem 1 implies that the number of nonisomorphic algebras in the irreducible component of N_n containing the algebra L_0 equals $p(n)$, where $p(n)$ is the number of integer solutions of the equation $x_1 + x_2 + \dots + x_n = n$, $x_1 \ge x_2 \ge \dots \ge x_n \ge 0$. The asymptotic value of $p(n)$, given in [6] by the expression $p(n) \approx \frac{1}{4n\sqrt{3}} e^{A\sqrt{n}}$ with $A = \pi\sqrt{\frac{2}{3}}$ ($p(n) \approx g(n)$ means that $\lim_{n\to\infty} \frac{p(n)}{g(n)} = 1$), shows how small is the set of nonisomorphic Leibniz algebras in the irreducible component of N_n containing the algebra L_0; i.e., the number of orbits in this component is finite for every value of n.

§ 2. Classification of Naturally Graded Complex Filiform Leibniz Algebras

DEFINITION 6. A Leibniz algebra is a *filiform algebra* if $\dim L^i = n - i$, where $2 \le i \le n$.

Lemma 4. *Let L be an n-dimensional Leibniz algebra. Then the following are equivalent:*
(a) $C(L) = (n-1, 1)$;
(b) *L is a filiform Leibniz algebra;*
(c) $L^{n-1} \neq 0$ *and* $L^n = 0$.

PROOF. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are obvious.

(b) $\Rightarrow$ (a): Let $\{e_1, \dots, e_n\}$ be a basis for a filiform algebra L such that $\{e_3, \dots, e_n\} \subseteq L^2$, $\{e_4, \dots, e_n\} \subseteq L^3, \dots, \{e_n\} \subseteq L^{n-1}$.

Consider the products

$$[x, e_1 + \alpha e_2] = \gamma_1 e_3 + \alpha\beta_1 e_3, \quad [e_3, e_1 + \alpha e_2] = \gamma_2 e_4 + \alpha\beta_2 e_4,$$
$$[e_4, e_1 + \alpha e_2] = \gamma_3 e_5 + \alpha\beta_3 e_5, \dots, [e_n, e_1 + \alpha e_2] = 0,$$

where x is an arbitrary element of L and $|\gamma_i| + |\beta_i| \neq 0$ for any i. Choose α so that $\gamma_i + \alpha\beta_i \neq 0$ for any i. Then $z = e_1 + \alpha e_2 \in L \setminus [L, L]$ and $C(z) = (n-1, 1)$.

(c) $\Rightarrow$ (b): Assume that $L^n = 0$. Then we obtain a decreasing chain of subalgebras $L \supset L^2 \supset L^3 \supset \dots \supset L^{n-1} \supset L^n = 0$ of length n. Obviously, $\dim L^2 = n - 1$ or $\dim L^2 = n - 2$ (otherwise $L^{n-1} = 0$). Suppose that $\dim L^2 = n - 1$. Choose a basis $\{e_1, \dots, e_n\}$ for L that corresponds to the filtration $L \supset L^2 \supset L^3 \supset \dots \supset L^{n-1} \supset L^n = 0$. Suppose that $\dim L^s/L^{s+1} = 2$ ($s \neq 1$), i.e., $\{e_s, e_{s+1}\} \in L^s \setminus L^{s+1}$. Arguing as in the proof of Lemma 1 and appropriately changing variables, we may assume that $e_s = [[[e_1, e_1], e_1], \dots, e_1] + (^*)$ (the product is taken s times and $(^*) \in L^{s+1}$) and $e_{s+1} = [[[e_1, e_1], e_1], \dots, e_1] + (^{**})$ (the product is taken s times and $(^{**}) \in L^{s+1}$). Then $e_s - e_{s+1} \in L^{s+1}$. We arrive at a contradiction with the assumption that $\dim L^s/L^{s+1} = 2$. Therefore, $\dim L^i/L^{i+1} = 1$ ($1 \le i \le n-1$). Then the basis of the n-dimensional algebra L consists of $n - 1$ elements; a contradiction to the assumption $\dim L^2 = n - 1$. Thus, $\dim L^i = n - i$, where $n = \dim L$ and $2 \le i \le n$; i.e., L is a filiform algebra. The proof of the lemma is over.

Henceforth we represent an algebra L as a pair (V, μ), with V a vector space and μ the multiplication on V defining L.

Let (V, μ) be an $(n+1)$-dimensional complex filiform Leibniz algebra. Define a natural grading of (V, μ) by putting $V_1(\mu) = V$, $V_{i+1}(\mu) := \mu(V_i(\mu), V)$, and $W_i := V_i(\mu)/V_{i+1}(\mu)$. Then $V = W_1 + W_2 + \dots + W_n$, where $\dim W_1 = 2$, $\dim W_i = 1$, $2 \le i \le n$. By [7, Lemma 1] we have the embedding $\mu(W_i, W_j) \subseteq W_{i+j}$. We thus obtain a grading which is said to be *natural*.

By arguments similar to those in [8], over a field with infinitely many elements we can find a basis $e_0, e_1 \in W_1$, $e_i \in W_i$ ($i \ge 2$) for V and a bilinear mapping μ such that $\mu(e_i, e_0) = e_{i+1}$ and $\mu(e_n, e_0) = 0$, $1 \le i \le n$.

For convenience, we henceforth denote $\mu(x, y)$ by $[x, y]$.

CASE 1. Assume that $[e_0, e_0] = \alpha e_2$ ($\alpha \neq 0$). Then $e_2 \in Z(\mu)$ (where $Z(\mu)$ is the right annihilator of L). Hence, $e_3, \dots, e_n \in Z(\mu)$. Changing the basis to

$$\bar{e}_1 = \alpha e_1, \ \bar{e}_2 = \alpha e_2, \ \bar{e}_3 = \alpha e_3, \dots, \bar{e}_n = \alpha e_n,$$

we may assume that α equals to one. Thus, $[e_0, e_0] = e_2$, $[e_i, e_0] = e_{i+1}$, and $[e_n, e_0] = 0$. Suppose that $[e_0, e_1] = \beta e_2$ and $[e_1, e_1] = \gamma e_2$. Then

$$[e_0, [e_1, e_0]] = [[e_0, e_1], e_0] - [[e_0, e_0], e_1] \Rightarrow \beta e_3 = [e_2, e_1]$$

and

$$[e_1, [e_0, e_1]] = [[e_1, e_0], e_1] - [[e_1, e_1], e_0] \Rightarrow \gamma e_3 = [e_2, e_1].$$

It follows that $\beta = \gamma$. Inducting on the number of basis elements and using the equality $[e_i, [e_0, e_1]] = [[e_i, e_0], e_1] - [[e_i, e_1], e_0]$, we can easily prove that $[e_i, e_1] = \beta e_{i+1}$; i.e., in Case 1 we obtain the algebra

$$[e_0, e_0] = e_2, \quad [e_i, e_0] = e_{i+1}, \quad [e_1, e_1] = \beta e_2, \quad [e_i, e_1] = \beta e_{i+1}, \quad [e_0, e_1] = \beta e_2.$$

Case 2. $[e_0, e_0] = 0 \,\&\, [e_1, e_1] = \alpha e_2$ $(\alpha \neq 0)$. In this case $e_2 \in Z(\mu)$. Hence, $e_3, \dots, e_n \in Z(\mu)$. Putting

$$\bar{e}_0 = \alpha e_0,\ \bar{e}_2 = \alpha e_2,\ \bar{e}_3 = \alpha^2 e_3, \dots, \bar{e}_n = \alpha^{n-1} e_n,$$

we may assume that $\alpha = 1$; i.e., $[e_1, e_1] = e_2$, $[e_i, e_0] = e_{i+1}$. Put $[e_0, e_1] = \beta e_2$. Then

$$[e_0, [e_1, e_0]] = [[e_0, e_1], e_0] - [[e_0, e_0], e_1] \Rightarrow [[e_0, e_1], e_0] = 0;$$

i.e., $\beta[e_2, e_0] = \beta e_3 = 0 \Rightarrow \beta = 0$. Inducting on the number of basis elements and using the equality $[e_i, [e_0, e_1]] = [[e_{i+1}, e_0], e_1] - [[e_i, e_1], e_0]$, we can easily show that $[e_i, e_1] = e_{i+1}$; i.e., in Case 2 we obtain the algebra $[e_i, e_0] = e_{i+1}$, $[e_i, e_1] = e_{i+1}$ $(i \geq 1)$. Changing the variables by $\bar{e}_0 := e_0 - e_1$, $\bar{e}_1 := e_1$, we obtain the algebra $[\bar{e}_i, \bar{e}_1] = \bar{e}_{i+1}$. It is easy to see that this algebra is isomorphic to the algebra of Case 1 for $\beta = 1$ $(e_0' := e_0 - e_1,\ e_1' := e_1)$.

Case 3. $[e_0, e_0] = 0 \,\&\, [e_1, e_1] = 0$. Put $[e_0, e_1] = \alpha e_2$.

Subcase 1. Assume that $[e_0, e_1] = \alpha e_2$ $(\alpha \neq -1)$. Then $e_2 \in Z(\mu)$. Hence, $e_3, \dots, e_n \in Z(\mu)$. Since $\alpha \neq -1$, on putting $\bar{e}_1 = e_1 + e_0$ we obtain $\bar{e}_1^2 = (\alpha + 1)e_2$ and $[\bar{e}_1, e_0] = e_2$; i.e., we arrive at Case 2.

Subcase 2. $[e_0, e_1] = -e_2$. Before settling this subcase, we prove the following

Lemma 5. *Let (V, μ) be an $(n+1)$-dimensional naturally graded filiform Leibniz algebra with a basis $\{e_0, e_1, \dots, e_n\}$ satisfying the following equalities:* $[e_1, e_1] = [e_0, e_0] = 0$, $[e_0, e_1] = -e_2$, *and* $[e_i, e_0] = e_{i+1}$. *Then (V, μ) is a Lie algebra.*

Proof. Inducting on the number of basis elements and using the equality $[e_0, [e_i, e_0]] = [[e_0, e_i], e_0] - [[e_0, e_0], e_i]$, we can easily show that $[e_0, e_i] = -[e_i, e_0]$ $(1 \leq i \leq n)$. From the equality $[e_1, [e_1, e_0]] = [[e_1, e_1], e_0] - [[e_1, e_0], e_1]$ we have $[e_1, e_2] = -[e_2, e_1]$. From the chain of the equalities

$$\begin{aligned}
[e_1, e_{i+1}] &= [e_1, [e_i, e_0]] = [[e_1, e_i], e_0] - [[e_1, e_0], e_i] = -[[e_i, e_1], e_0] - [e_2, e_i] \\
&= [e_0, [e_i, e_1]] - [e_2, e_i] = [[e_0, e_i], e_1] - [[e_0, e_1], e_i] - [e_2, e_i] \\
&= [[e_0, e_i], e_1] + [e_2, e_i] - [e_2, e_i] = -[[e_i, e_0], e_1] = -[e_{i+1}, e_1]
\end{aligned}$$

and the induction base we obtain $[e_1, e_i] = -[e_i, e_1]$ $(1 \leq i \leq n)$. Thus, $[e_1, e_i] = -[e_i, e_1]$ and $[e_0, e_i] = -[e_i, e_0]$ $(0 \leq i \leq n)$. Let us prove the equality $[e_i, e_j] = -[e_j, e_i]$ for all i, j. We proceed by induction on i for a fixed j. Observe that j may be assumed to be greater than 1. Using the chain of the equalities

$$\begin{aligned}
[e_{i+1}, e_j] &= [[e_i, e_0], [e_{j-1}, e_0]] = [[[e_i, e_0], e_{j-1}], e_0] - [[[e_i, e_0], e_0], e_{j-1}] \\
&= -[e_0, [[e_i, e_0], e_{j-1}]] + [[e_0, [e_i, e_0]], e_{j-1}] = [e_0, [[e_0, e_i], e_{j-1}] - [[e_0, [e_0, e_i]], e_{j-1}] \\
&= [[e_0, [e_0, e_i]], e_{j-1}] - [[e_0, e_{j-1}, [e_0, e_i]] - [[[e_0, e_0], e_i], e_{j-1}] + [[e_0, e_i], e_0], e_{j-1}] \\
&= [[[e_0, e_0], e_i], e_{j-1}] - [[[e_0, e_i], e_0], e_{j-1}] - [[[e_0, e_0], e_i], e_{j-1}] - [[e_{j-1}, e_0], [e_i, e_0]] \\
&\quad + [[[e_0, e_i], e_0], e_{j-1}] = -[e_j, e_{i+1}],
\end{aligned}$$

we obtain anticommutativity of the basis elements of the algebra (V, μ). The proof of the lemma is over.

Thus, the naturally graded filiform Leibniz algebras that are not Lie algebras are as follows:

$$[e_0, e_0] = e_2, \quad [e_i, e_0] = e_{i+1}, \quad [e_i, e_1] = \beta e_{i+1}, \quad [e_0, e_1] = \beta e_2.$$

Assume that $\beta \neq 1$. Performing the change

$$\bar{e}_0 = (1 - \beta)e_0,\ \bar{e}_1 = -\beta e_0 + e_1,\ \bar{e}_2 = (1 - \beta)^2 e_2, \dots, \bar{e}_n = (1 - \beta)^n e_n,$$

we may assume that $\beta = 0$.

Now, consider the case in which $\beta = 1$, i.e., $[e_0, e_0] = e_2$, $[e_i, e_1] = e_{i+1}$, $[e_0, e_1] = e_2$ $(1 \le i \le n)$. Making the change $\bar{e}_1 = e_1 - e_0$, we have $[e_0, e_0] = e_2$, $[e_i, e_0] = e_{i+1}$ $(1 \le i \le n)$.

We demonstrate that the algebras $[e_0, e_0] = e_2$, $[e_i, e_0] = e_{i+1}$ $(1 \le i \le n-1)$, and $[e_0, e_0] = e_2$, $[e_i, e_0] = e_{i+1}$ $(2 \le i \le n-1)$ are nonisomorphic to one another.

Assume the contrary and let φ be an isomorphism from the first algebra into the second, i.e., $\varphi : L_1 \to L_2$ and $\varphi(e_i) = \sum_{j=0}^{n} \alpha_{ij} e_j$.

We have

$$[\varphi(e_0), \varphi(e_0)] = \left[\sum_{j=0}^{n} \alpha_{0j} e_j, \alpha_{00} e_0\right] = \alpha_{00}(\alpha_{00} e_2 + \alpha_{02} e_3 + \dots + \alpha_{0,n-1} e_n).$$

On the other hand,

$$\varphi([e_0, e_0]) = \varphi(e_2) = \sum_{j=0}^{n} \alpha_{2j} e_j.$$

Comparing the two equalities, we conclude that

$$\alpha_{20} = \alpha_{21} = 0, \quad \alpha_{22} = \alpha_{00}^2, \quad \alpha_{2,k} = \alpha_{00}\alpha_{0,k-1} \text{ for } 3 \le k \le n. \tag{2}$$

Consider the product

$$[\varphi(e_i), \varphi(e_0)] = \left[\sum_{j=0}^{n} \alpha_{ij} e_j, \alpha_{00} e_0\right] = \alpha_{00} \sum_{j=0}^{n} \alpha_{ij} [e_j, e_0]$$
$$= \alpha_{00}(\alpha_{i,0} e_2 + \alpha_{i,2} e_3 + \dots + \alpha_{i,n-1} e_n).$$

Also,

$$\varphi([e_i, e_0]) = \varphi(e_{i+1}) = \sum_{j=0}^{n} \alpha_{i+1,j} x_j$$

for $1 \le i \le n-1$. Comparing the two equalities, we deduce that

$$\begin{gathered} \alpha_{i+1,0} = \alpha_{i+1,1} = 0, \quad \alpha_{i+1,2} = \alpha_{00}\alpha_{i,0}, \\ \alpha_{i+1,k} = \alpha_{00}\alpha_{i,k-1} \text{ for } 3 \le k \le n,\ 1 \le i \le n-1. \end{gathered} \tag{3}$$

It follows from (3) that $\alpha_{22} = \alpha_{00}\alpha_{10}$. Since $\alpha_{00} \ne 0$ (otherwise φ is degenerate), (2) implies that $\alpha_{00} = \alpha_{10}$.

We have $\varphi([e_0, e_1]) = \varphi(0) = 0$. On the other hand,

$$[\varphi(e_0), \varphi(e_1)] = \left[\sum_{j=0}^{n} \alpha_{0j} e_j, \alpha_{10} e_0\right] = \alpha_{10} \sum_{j=0}^{n} \alpha_{0j} [e_j, e_0]$$
$$= \alpha_{10}(\alpha_{00} e_0 + \alpha_{02} e_3 + \dots + \alpha_{0,n-1} e_n) = 0.$$

Hence, $\alpha_{10}\alpha_{00} = 0$ and so $\alpha_{10} = 0$; i.e., the first column of the matrix of the isomorphism $[\varphi]$ is zero. Therefore, φ is degenerate.

We have thus proved the following

Theorem 2. *There are exactly two nonisomorhic naturally graded complex non-Lie filiform Leibniz algebras μ_0^n and μ_1^n of dimension $n+1$, where*

$$\mu_0^n : \mu_0^n(e_0, e_0) = e_2, \quad \mu_0^n(e_i, e_0) = e_{i+1} \text{ for } 1 \le i \le n-1,$$
$$\mu_1^n : \mu_1^n(e_0, e_0) = e_2, \quad \mu_1^n(e_i, e_0) = e_{i+1} \text{ for } 2 \le i \le n-1,$$

the other products vanish.

Remark 1. The naturally graded complex filiform Lie algebras were described in [8]. Thus, there is a classification for naturally graded complex Leibniz algebras.

Corollary 3. *Every $(n+1)$-dimensional complex non-Lie filiform Leibniz algebra is isomorphic to one of the algebras*

$$\mu(e_0,e_0)=e_2,\ \mu(e_i,e_0)=e_{i+1},\ \mu(e_0,e_1)=\alpha_3e_3+\alpha_4e_4+\cdots+\alpha_{n-1}e_{n-1}+\theta_ne_n,$$
$$\mu(e_i,e_1)=\alpha_3e_{i+2}+\alpha_4e_{i+3}+\cdots+\alpha_{n+1-i}e_n \text{ for } 1\le i\le n,$$

$$\mu(e_i,e_1)=\beta_3e_{i+2}+\beta_4e_{i+3}+\cdots+\beta_{n+1-i}e_n \text{ for } 2\le i\le n-1.$$
$$\mu(e_0,e_0)=e_2,\ \mu(e_i,e_0)=e_{i+1},\ \mu(e_0,e_1)=\beta_3e_3+\beta_4e_4+\cdots+\beta_ne_n,\ \mu(e_1,e_1)=\gamma e_n,$$

the other products vanish.

Proof. By immediate verification we can convince ourselves that the above-written algebras are Leibniz algebras. By Theorem 2, every $(n+1)$-dimensional complex non-Lie filiform Leibniz algebra μ is isomorphic to the algebra $\mu_0^n+\beta$, where $\beta(e_0,e_0)=0$, $\beta(e_i,e_0)=0$ for $1\le i\le n-1$, $\beta(e_i,e_j)\in\operatorname{lin}(e_{i+j+1},\dots,e_n)$ for $i\ne 0$, and $\beta(e_0,e_j)\in\operatorname{lin}(e_{j+2},\dots,e_n)$ for $1\le j\le n-2$, or to the algebra $\mu_1^n+\beta$, where $\beta(e_0,e_0)=0$, $\beta(e_i,e_0)=0$ for $2\le i\le n-1$, $\beta(e_i,e_j)\in\operatorname{lin}(e_{i+j+1},\dots,e_n)$ for $i,j\ne 0$, and $\beta(e_0,e_j)\in\operatorname{lin}(e_{j+2},\dots,e_n)$ for $1\le j\le n-2$.

Case 1. Assume that $\mu\cong\mu_0^n+\beta$. Then $\mu(e_0,e_0)=\mu_0^n(e_0,e_0)=e_2$ and $\mu(e_i,e_0)=\mu_0^n(e_i,e_0)=e_{i+1}$ for $1\le i\le n-1$; whence $e_2,e_3,\dots,e_n\in Z(\mu)$, so that $\mu(e_i,e_j)=0$ for $2\le j\le n$, $0\le i\le n$.

Put $\mu(e_1,e_1)=\alpha_3e_3+\alpha_4e_4+\cdots+\alpha_ne_n$. Consider

$$\mu(e_i,\mu(e_0,e_1))=\mu(\mu(e_i,e_0),e_1)-\mu(\mu(e_i,e_1),e_0).$$

Since $\mu(e_0,e_1)\in Z(\mu)$, we have $\mu(e_i,\mu(e_0,e_1))=0$ and so $\mu(\mu(e_i,e_0),e_1)=\mu(\mu(e_i,e_1),e_0)$ for all $i\ge 1$. Thus, $\mu(e_i,e_1)=\alpha_3e_{i+2}+\alpha_4e_{i+3}+\cdots+\alpha_{n+1-i}e_n$ for $1\le i\le n$.

Let $\mu(e_0,e_1)=\theta_3e_3+\theta_4e_4+\cdots+\theta_ne_n$. Consider

$$\mu(e_0,\mu(e_1,e_0))=\mu(\mu(e_0,e_1),e_0)-\mu(\mu(e_0,e_0),e_1).$$

We have

$$\mu(\mu(e_0,e_1),e_0)=\mu(\mu(e_0,e_0),e_1).$$

However, $\mu(e_0,e_0)=e_2$ and $\mu(e_i,e_0)=e_{i+1}$. Therefore,

$$\theta_3e_4+\theta_4e_5+\cdots+\theta_{n-1}e_n=\alpha_3e_4+\alpha_4e_5+\cdots+\alpha_{n-1}e_n;$$

whence

$$\mu(e_0,e_1)=\alpha_3e_3+\alpha_4e_4+\cdots+\alpha_{n-1}e_{n-1}+\theta_ne_n.$$

Thus, in Case 1 we obtain the following class:

$$\mu(e_0,e_0)=e_2,\mu(e_i,e_0)=e_{i+1},\mu(e_0,e_1)=\alpha_3e_3+\alpha_4e_4+\cdots+\alpha_{n-1}e_{n-1}+\theta_ne_n,$$

$$\mu(e_i,e_1)=\alpha_3e_{i+2}+\alpha_4e_{i+3}+\cdots+\alpha_{n+1-i}e_n \text{ for } 1\le i\le n.$$

Case 2. $\mu\cong\mu_1^n+\beta$. In this case $\mu(e_0,e_0)=\mu_1^n(e_0,e_0)=e_2$ and $\mu(e_i,e_0)=\mu_1^n(e_i,e_0)=e_{i+1}$ for $2\le i\le n-1$; whence $e_2,e_3,\dots,e_n\in Z(\mu)$ and so $\mu(e_i,e_j)=0$ for $2\le j\le n$, $0\le i\le n$.

Let $\beta(e_1,e_0)=\alpha_3e_3+\alpha_4e_4+\cdots+\alpha_ne_n$. Making the change $\bar e_1:=e_1-\alpha_3e_2-\alpha_4e_3-\cdots-\alpha_ne_{n-1}$, we obtain

$$\mu(\bar e_1,e_0)=\mu_1^n(\bar e_1,e_0)+\beta(\bar e_1,e_0)=\mu_1^n(-\alpha_3e_2-\alpha_4e_3-\cdots-\alpha_ne_{n-1},e_0)+\beta(e_1,e_0)=0.$$

We may thus assume that $\mu(e_1,e_0)=0$.

Let $\mu(e_0,e_1)=\beta_3e_3+\beta_4e_4+\cdots+\beta_ne_n$. Consider the product

$$\mu(e_0,\mu(e_1,e_0))=\mu(\mu(e_0,e_1),e_0)-\mu(\mu(e_0,e_0),e_1).$$

Since $\mu(e_1, e_0) \in Z(\mu)$, we have $\mu(\mu(e_0, e_1), e_0) = \mu(\mu(e_0, e_0), e_1)$. Therefore, $\mu(\mu(e_0, e_1), e_0) = \mu(e_2, e_1)$; i.e., $\mu(e_2, e_1) = \beta_3 e_4 + \beta_4 e_5 + \cdots + \beta_{n-1} e_n$.

Consider the product

$$\mu(e_1, \mu(e_0, e_1)) = \mu(\mu(e_1, e_0), e_1) - \mu(\mu(e_1, e_1), e_0).$$

In view of $\mu(e_0, e_1) \in Z(\mu)$ and $\mu(e_1, e_0) = 0$, we have $\mu(\mu(e_1, e_1), e_0) = 0$. However, e_0 left annihilates only e_n. Therefore, $\mu(e_1, e_1) = \gamma e_n$.

Look at the product

$$\mu(e_i, \mu(e_0, e_1)) = \mu(\mu(e_i, e_0), e_1) - \mu(\mu(e_i, e_1), e_0)$$

for $2 \leq i \leq n-1$. Since $\mu(e_0, e_1) \in Z(\mu)$, we have $\mu(\mu(e_i, e_0), e_1 = \mu(\mu(e_i, e_1), e_0)$. Thereby $\mu(e_{i+1}, e_1) = \mu(\mu(e_i, e_1), e_0)$; i.e., $\mu(e_i, e_1) = \beta_3 e_{i+2} + \beta_4 e_{i+3} + \cdots + \beta_{n+1-i} e_n$ for $2 \leq i \leq n-1$.

Thus, in Case 2 we obtain the following class:

$$\mu(e_0, e_0) = e_2, \mu(e_i, e_0) = e_{i+1}, \mu(e_0, e_1) = \beta_3 e_3 + \beta_4 e_4 + \cdots + \beta_n e_n = \gamma e_n.$$

$$\mu(e_i, e_1) = \beta_3 e_{i+2} + \beta_4 e_{i+3} + \cdots + \beta_{n+1-i} e_n \text{ for } 2 \leq i \leq n.$$

This completes the proof of the corollary.

Remark 2. The classes of algebras in Corollary 3 are disjoint, but the question of isomorphisms between these classes is open.

References

1. *Loday J.-L. and Pirashvili T.*, "Universal enveloping algebras of Leibniz algebras and (co)homology," Math. Ann., **296**, No. 1, 139–158 (1993).
2. *Loday J.-L.*, "Une version non commutative des algebres de Lie: Les algebres de Leibniz," Enseign. Math., **39**, No. 3–4, 269–293 (1993).
3. *Cabezas J. M., Gomez J. R., and Jimenez-Merchan A.*, "Family of p-filiform Lie algebras," in: Algebra and Operator Theory, Proceedings of the Colloquium in Tashkent, 1997, Kluwer Acad. Publ., Dordrecht, Boston, and London, 1998, pp. 93–102.
4. *Omirov B. A.*, Degeneracy of Jordan Algebras of Small Dimension [in Russian], Diplomnaya Rabota, Novosibirsk Univ., Novosibirsk (1993).
5. *Shafarevich I. P.*, Fundamentals of Algebraic Geometry. Vol. 1 [Russian], Nauka, Moscow (1988).
6. *Hall M.*, Combinatorial Theory [Russian], Mir, Moscow (1970).
7. *Ayupov Sh. A. and Omirov B. A.*, "On Leibniz algebras," in: Algebra and Operator Theory, Proceedings of the Colloquium in Tashkent, 1997, Kluwer Acad. Publ., Dordrecht, Boston, and London, 1998, pp. 1–13.
8. *Vergne M.*, "Cohomologie des algebres de Lie nilpotentes al'etude de la variete des algebras de Lie nilpotentes," Bull. Soc. Math. France, **98**, 81–116 (1970).