Leibniz algebras with associated Lie algebras sl_2\dot{+} R (dim R=2)

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LEIBNIZ ALGEBRAS WITH ASSOCIATED LIE ALGEBRA $sl_2\dot{+}R$ $(dimR = 2)$.

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Abstract. From the theory of Lie algebras it is known that every finite dimensional Lie algebra is decomposed into a semidirect sum of a semisimple subalgebra and solvable radical. Moreover, according to Mal'cev, the study of solvable Lie algebras is reduced to the study of nilpotent algebras.

For the finite dimensional Leibniz algebras the analogues of the mentioned results are not proved yet. In order to get some idea how to establish such results, we examine Leibniz algebras whose associated Lie algebra is a semidirect sum of a semisimple Lie algebra and the maximal solvable ideal. In this paper the class of complex Leibniz algebras for which the quotient algebra by the ideal I is isomorphic to the semidirect sum of the algebra sl_2 and a two-dimensional solvable ideal R is described.

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1. Introduction

The notion of Leibniz algebra was introduced in 1993 by J.-L. Loday [7] as a generalization of Lie algebras. In the last 20 years the theory of Leibniz algebras has been actively studied and many results of the theory of Lie algebras have been extended to Leibniz algebras.

According to Mal'cev's work [8] the study of finite dimensional Lie algebras is reduced to nilpotent ones. For later works on the description of finite-dimensional nilpotent Lie algebras see [5], [6], [11].

The nilpotency of a finite-dimensional Lie algebra is characterized by Engel's Theorem. In [6] the local nilpotency of a Lie algebra over a field of zero characteristic is proved satisfying the Engel's n-condition. Further, E.I. Zelmanov [11] generalized this result to global nilpotency of a Lie algebra with Engel's n-condition. In [9] the global nilpotency for the case of a Leibniz algebra with Engel's n-condition was extended.

An algebra L over a field F is called *Leibniz algebra* if for any elements $x, y, z \in L$ *the Leibniz identity* holds:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

where $[-, -]$ is multiplication of L.

Let L be a Leibniz algebra and $I = ideal < [x, x] \mid x \in L >$ be the ideal of L generated by all squares. Then I is the minimal ideal with respect to the property

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that L/I is a Lie algebra. The natural epimorphism $\varphi : L \to L/I$ determines the corresponding Lie algebra L/I of the Leibniz algebra L.

According to [5], a 3-dimensional simple Lie algebra L is said to be split if L contains an element h such that $ad(h)$ has a non-zero characteristic root ρ belonging to the base field. Such algebra has a basis $\{e, f, h\}$ with the multiplication table

$$\begin{array}{lll} [e,h] = 2e, & [f,h] = -2f, & [e,f] = h, \\ [h,e] = -2e, & [h,f] = 2f, & [f,e] = -h. \end{array}$$

This simple 3-dimensional Lie algebra is denoted by sl_2 and the basis $\{e, f, h\}$ is called *canonical basis*. Note that any 3-dimensional simple Lie algebra is isomorphic to sl_2.

The analogue of Levi-Mal'cev's Theorem for Leibniz algebras is not proved yet. We only know the result of [10], where the Leibniz algebras whose quotient algebra by an ideal I are isomorphic to the simple Lie algebra sl_2 are classified.

In fact, Dzhumadil'daev proposed the following construction of Leibniz algebras:

Let G be a simple Lie algebra and M be an irreducible skew-symmetric G-module (i.e. $[x, m] = 0$ for all $x \in G, m \in M$). Then the vector space $Q = G + M$ equipped with the multiplication

$$[x + m, y + n] = [x, y] + [m, y],$$

where $m, n \in M, x, y \in G$ is a Leibniz algebra. Moreover, the corresponding Lie algebra for this Leibniz algebra is a simple algebra.

The notion of simple Leibniz algebras was introduced in [1], [2].

A Leibniz algebra L is said to be *simple* if the only ideals of L are $\{0\}$, I, L and $[L, L] \neq I$. Obviously, when a Leibniz algebra is Lie, the ideal I is equal to zero. Therefore, this definition agrees with the definition of a simple Lie algebra.

Note that the above mentioned Leibniz algebra is a simple algebra. It is also easy to see that the corresponding Lie algebra is simple for the simple Leibniz algebra.

In this paper, we study the class of complex Leibniz algebras, for which its Lie algebra is isomorphic to the semidirect sum of the algebra sl_2 and a two-dimensional solvable ideal R.

The representation of sl_2 is determined by the images E, F, H of the base elements e, f, h and we have

$$[E, H] = 2E, \ [F, H] = -2F, \ [E, F] = H,$$

$$[H, E] = -2E, \ [H, F] = 2F, \ [F, E] = -H.$$

Conversely, any three linear transformations E, F, H satisfying these relations determine a representation of sl_2 and hence a sl_2-module.

We suppose that a base field is the field of the complex numbers. Then one has the following

Theorem 1.1. [5] *For each integer $m = 0, 1, 2, \ldots$ there exists one and, in the sense of isomorphism, only one irreducible sl_2-module M of dimension $m+1$. The module M has a basis $\{x_0, x_1, ..., x_m\}$ such that the representing transformations E, F and H corresponding to the canonical basis $\{e, f, h\}$ are given by:*

$$\begin{array}{ll} H(x_k) = (m - 2k)x_k, & 0 \leq k \leq m, \\ F(x_m) = 0, \ F(x_k) = x_{k+1}, & 0 \leq k \leq m-1, \\ E(x_0) = 0, \ E(x_k) = -k(m+1-k)x_{k-1}, & 1 \leq k \leq m. \end{array}$$

In [10], the authors described the complex finite dimensional Leibniz algebras whose L/I is isomorphic to sl_2 using Theorem 1.1.

In this work, we consider the Leibniz algebra L for which its corresponding Lie algebra is a semidirect sum of sl_2 and a two-dimensional solvable ideal R. In addition, we assume that I is a right irreducible module over sl_2.

By verifying antisymmetric and Jacobi identities, we derive that a semidirect sum of sl_2 and a two-dimensional solvable Lie algebra is the direct sum of the algebras.

Let $\{x_0, x_1, \dots, x_m\}$ be a basis of I and $\{e, f, h\}$ a basis of sl_2. Thus, if I is a right irreducible module over sl_2, then according to Theorem 1.1, the products $[I, sl_2]$ are defined as follows:

$$\begin{array}{ll} [x_k, h] = (m-2k)x_k & 0 \le k \le m, \\ [x_k, f] = x_{k+1}, & 0 \le k \le m-1, \\ [x_k, e] = -k(m+1-k)x_{k-1}, & 1 \le k \le m, \end{array}$$

where the omitted products are equal to zero.

2. On complex Leibniz algebras whose quotient Lie algebras are isomorphic to $sl_2\dot{+}R$.

Let L be a Leibniz algebra such that $L/I \simeq sl_2 \oplus R$, where R is a solvable Lie algebra and $\{\overline{e}, \overline{h}, \overline{f}\}$, $\{x_0, x_1, \dots, x_m\}$, $\{\overline{y_1}, \overline{y_2}, \dots, \overline{y_n}\}$ are the bases of sl_2, I, R respectively.

Let $\{e, h, f, x_0, x_1, \dots, x_m, y_1, y_2, \dots, y_n\}$ be a basis of the algebra L such that

$$\varphi(e) = \overline{e},\ \varphi(h) = \overline{h},\ \varphi(f) = \overline{f},\ \varphi(y_i) = \overline{y_i},\ 1 \le i \le n.$$

Then we have:

$$\begin{array}{lll} [e,h] = 2e + \sum\limits_{j=0}^{m} a^j_{eh} x_j, & [h,f] = 2f + \sum\limits_{j=0}^{m} a^j_{hf} x_j, & [e,f] = h + \sum\limits_{j=0}^{m} a^j_{ef} x_j, \\ [h,e] = -2e + \sum\limits_{j=0}^{m} a^j_{he} x_j & [f,h] = -2f + \sum\limits_{j=0}^{m} a^j_{fh} x_j, & [f,e] = -h + \sum\limits_{j=0}^{m} a^j_{fe} x_j, \\ [e,y_i] = \sum\limits_{j=0}^{m} \alpha_{ij} x_j & [f,y_i] = \sum\limits_{j=0}^{m} \beta_{ij} x_j, & [h,y_i] = \sum\limits_{j=0}^{m} \gamma_{ij} x_j, \\ [x_k,h] = (m-2k)x_k, & 0 \le k \le m, & \\ [x_k,f] = x_{k+1}, & 0 \le k \le m-1, & \\ [x_k,e] = -k(m+1-k)x_{k-1}, & 1 \le k \le m. & \end{array}$$

where $1 \le i \le n$.

It is easy to check that similarly as in paper [10] one can get

$$\begin{array}{lll} [e,h] = 2e, & [h,f] = 2f, & [e,f] = h, \\ [h,e] = -2e & [f,h] = -2f, & [f,e] = -h, \\ [e,e] = 0 & [f,f] = 0, & [h,h] = 0. \end{array}$$

Let us denote the following vector spaces:

$$sl_2^{-1} =< e, h, f >, \quad R^{-1} =< y_1, y_2, \dots y_n > .$$

The following result holds.

Lemma 2.1. *Let L be a Leibniz algebra whose quotient $L/I \cong sl_2 \oplus R$, where R is a solvable ideal and I is a right irreducible module over sl_2 with $dim(R) \neq 3$. Then $[sl_2^{-1}, R^{-1}] = 0$.*

Proof. It is known that it is sufficient to prove the equality for the basic elements of sl_2^{-1} and R^{-1}. Consider the Leibniz identity:

$$\begin{aligned}[e,[e,y_i]] &= [[e,e],y_i]-[[e,y_i],e] = -[[e,y_i],e] = \\ &= -\sum_{j=0}^{m}\alpha_{ij}[x_j,e] = \sum_{j=1}^{m}(-mj+j(j-1))\alpha_{ij}x_{j-1}, \qquad 1\le i\le n.\end{aligned}$$

On the other hand, we have that $[e,[e,y_i]] = [e,\sum\limits_{j=0}^{m}\alpha_{ij}x_j] = 0$ for $1\le i\le n$.

Comparing the coefficients at the basic elements we obtain $\alpha_{ij}=0$ for $1\le j\le m$, thus $[e,y_i]=\alpha_{i,0}x_0$ with $1\le i\le n$.

Consider the chain of equalities

$$\begin{aligned}0 &= [e,\sum_{j=0}^{m}\beta_{ij}x_j] = [e,[f,y_i]] = [[e,f],y_i]-[[e,y_i],f] = \\ &= [h,y_i]-\alpha_{i,0}[x_0,f] = [h,y_i]-\alpha_{i,0}x_1.\end{aligned}$$

Then we have that $[h,y_i]=\alpha_{i,0}x_1$ with $1\le i\le n$.

From the equalities

$$\begin{aligned}0 &= [e,[h,y_i]] = [[e,h],y_i]-[[e,y_i],h] = 2[e,y_i]-\alpha_{i,0}[x_0,h] = \\ &= 2\alpha_{i,0}x_0 - m\alpha_{i,0}x_0 = \alpha_{i,0}(2-m)x_0,\end{aligned}$$

it follows that $\alpha_{i,0}=0$ for $1\le i\le n$. Taking into account that $m\neq 2$ we get $[e,y_i]=[h,y_i]=0$ with $1\le i\le n$.

From the equalities

$$\begin{aligned}0 &= [f,[e,y_i]] = [[f,e],y_i]-[[f,y_i],e] = [h,y_i]-[[f,y_i],e] = -[[f,y_i],e] = \\ &= -\sum_{j=0}^{m}\beta_{ij}[x_j,e] = -\sum_{j=0}^{m}(-mj+j(j-1))\beta_{ij}x_{j-1},\end{aligned}$$

we derive $\beta_{i,j}=0$ for $1\le j\le m$. Consequence, $[f,y_i]=\beta_{i,0}x_0$, for all $1\le i\le n$.

Similarly, from

$$\begin{aligned}0 &= [f,[f,y_i]] = [[f,f],y_i]-[[f,y_i],f] = [[f,y_i],f] = \\ &= \beta_{i,0}[x_0,f] = \beta_{i,0}x_1,\end{aligned}$$

we obtain $[f,y_i]=0$ for all $1\le i\le n$.

Thus, we obtain $[e,y_i]=[f,y_i]=[h,y_i]=0$ with $1\le i\le n$, i.e $[sl_2^{-1},R^{-1}]=0$. □

3. On complex Leibniz algebras whose quotient Lie algebra is isomorphic to $sl_2\dot{+}R$, $dimR=2$.

Let R be a two-dimensional solvable Lie algebra, then from the classification of two-dimensional Lie algebras (see [5]) we know that in R there exists a basis $\{\overline{y_1},\overline{y_2}\}$ with the following table of multiplication

$$[\overline{y_1},\overline{y_2}]=\overline{y_1}, \qquad [\overline{y_2},\overline{y_1}]=-\overline{y_1}.$$

In the case when $dimR \neq 3$ and I a right irreducible module over sl_2, summarizing the results of Lemma 2.1 we get the following table of multiplication:

$$\begin{array}{lll}
[e,h]=2e, & [h,f]=2f, & [e,f]=h,\\
[h,e]=-2e & [f,h]=-2f, & [f,e]=-h,\\
[x_k,h]=(m-2k)x_k & 0\leq k\leq m, & \\
[x_k,f]=x_{k+1}, & 0\leq k\leq m-1, & \\
[x_k,e]=-k(m+1-k)x_{k-1}, & 1\leq k\leq m, & \\
[y_i,e]=\sum\limits_{j=0}^{m} a_{ie}^{j}x_j, & 1\leq i\leq 2, & \\
[y_i,f]=\sum\limits_{j=0}^{m} a_{if}^{j}x_j, & 1\leq i\leq 2, & \\
[y_i,h]=\sum\limits_{j=0}^{m} a_{ih}^{j}x_j, & 1\leq i\leq 2, & \\
[x_k,y_i]=\sum\limits_{j=0}^{m} a_{ij}^{k}x_j, & 0\leq k\leq m, & 1\leq i\leq 2,\\
[y_1,y_2]=y_1+\sum\limits_{j=0}^{m} a_{12}^{j}x_j, & [y_2,y_1]=-y_1, & \\
[y_1,y_1]=\sum\limits_{j=0}^{m} a_{1}^{j}x_j, & [y_2,y_2]=\sum\limits_{j=0}^{m} a_{2}^{j}x_j, &
\end{array} \tag{1}$$

where $\{e,h,f,x_0,x_1,\dots,x_m,y_1,y_2\}$ is a basis of L.

Let us present the following theorem which describes the Leibniz algebras with condition $L/I\cong sl_2\oplus R$, where $dimR\neq 3,\ n=2$ and I a right irreducible module over sl_2.

Theorem 3.1. *Let L be a Leibniz algebra whose quotient $L/I\cong sl_2\oplus R$, where R is a two-dimensional solvable ideal and I a right irreducible module over sl_2 ($dimR\neq 3$). Then there exists a basis $\{e,h,f,x_0,x_1,\dots,x_m,y_1,y_2\}$ of the algebra L such that the table of multiplication in L has the following form:*

$$\begin{array}{lll}
[e,h]=2e, & [h,f]=2f, & [e,f]=h,\\
[h,e]=-2e & [f,h]=-2f, & [f,e]=-h,\\
[x_k,h]=(m-2k)x_k & 0\leq k\leq m, & \\
[x_k,f]=x_{k+1}, & 0\leq k\leq m-1, & \\
[x_k,e]=-k(m+1-k)x_{k-1}, & 1\leq k\leq m, & \\
[y_1,y_2]=y_1 & [y_2,y_1]=-y_1, & \\
[x_k,y_2]=ax_k, & 0\leq k\leq m, \quad a\in F &
\end{array}$$

where the omitted products are equal to zero.

Proof. Let L be an algebra satisfying the conditions of the theorem, then we get the table of multiplication (1). Further we shall study the product $[I,R^{-1}]$.

We consider the chain of equalities

$$\begin{aligned}
0 &= [x_i,[h,y_1]] = [[x_i,h],y_1]-[[x_i,y_1],h] = (m-2i)[x_i,y_1]-\sum_{k=0}^{m} a_{1k}^{i}[x_k,h] =\\
&= (m-2i)\sum_{k=0}^{m} a_{1k}^{i}x_k - \sum_{k=0}^{m} a_{1k}^{i}(m-2k)x_k = \sum_{k=0}^{m} a_{1k}^{i}(m-2i-(m-2k))x_k =\\
&= \sum_{k=0}^{m} 2a_{1k}^{i}(k-i)x_k,
\end{aligned}$$

from which we have $a_{1k}^{i}=0$, with $0\leq i\leq m$ and $i\neq k$. Thus, $[x_i,y_1]=a_{1i}^{i}x_i=a_{1i}x_i$ with $0\leq i\leq m$.

Similarly,

$$\begin{aligned} 0 &= [x_i,[h,y_2]] = [[x_i,h],y_2] - [[x_i,y_2],h] = (m-2i)[x_i,y_2] - \sum_{k=0}^{m} a_{2k}^i [x_i,h] = \\ &= (m-2i)\sum_{k=0}^{m} a_{2k}^i x_i - \sum_{k=0}^{m} a_{2k}^i (m-2k)x_k = \sum_{k=0}^{m} a_{2k}^i (m-2i-(m-2k))x_k = \\ &= \sum_{k=0}^{m} 2a_{2k}^i (k-i)x_k, \end{aligned}$$

we get $[x_i,y_2] = a_{2i}^i x_i = a_{2i}x_i$ with $0 \le i \le m$.

From the identity $[x_i,[y_1,y_2]] = [[x_i,y_1],y_2] - [[x_i,y_2],y_1]$, we deduce

$$\begin{aligned} &[x_i, y_1 + \sum_{k=0}^{m} a_{12}^k x_k] = a_{1i}[x_i,y_2] - a_{2i}[x_i,y_1] \quad \Rightarrow \\ &\Rightarrow \quad [x_i,y_1] = a_{1i}a_{2i}x_i - a_{2i}a_{1i}x_i = 0, \end{aligned}$$

from which we have $[x_i,y_1] = 0$ with $0 \le i \le m$, i.e. $[I,y_1] = 0$.

We consider the identity $[x_i,[y_2,e]] = [[x_i,y_2],e] - [[x_i,e],y_2]$ for $0 \le i \le m$.

Then

$$\begin{aligned} 0 &= a_{2i}[x_i,e] - (-mi + i(i-1))[x_{i-1},y_2] = \\ &= a_{2i}(-mi+i(i-1))x_{i-1} - a_{2,i-1}(-mi+i(i-1))x_{i-1} = \\ &= -(-mi+i(i-1))(a_{2i} - a_{2,i-1})x_{i-1} = 0, \end{aligned}$$

which leads to $a_{2i} = a_{2,i-1} = a$, i.e. $[x_i,y_2] = ax_i$ with $0 \le i \le m$.

Now we shall study the products $[R^{-1},R^{-1}]$ and $[R^{-1},sl_2^{-1}]$.

Verifying the following

$$\begin{aligned} 0 &= [y_1, \sum_{j=0}^{m} a_{1f}^j x_j] = [y_1,[y_1,f]] = [[y_1,y_1],f] - [[y_1,f],y_1] = [[y_1,y_1],f] = \\ &= \sum_{j=0}^{m} a_1^j [x_j,f] = \sum_{j=0}^{m-1} a_1^j x_{j+1}, \end{aligned}$$

we obtain $a_1^j = 0$ for $0 \le j \le m-1$, i.e. $[y_1,y_1] = a_1{}^m x_m$.

Consider the equalities

$$0 = [y_1,[y_1,h]] = [[y_1,y_1],h] - [[y_1,h],y_1] = [[y_1,y_1],h] = a_1^m [x_m,h] = -ma_1^m x_m,$$

which deduce $a_1^m = 0$, hence $[y_1,y_1] = 0$.

From the following identities

$$\begin{aligned} 0 &= [y_2,[y_1,h]] = [[y_2,y_1],h] - [[y_2,h],y_1] = -[y_1,h], \\ 0 &= [y_2,[y_1,f]] = [[y_2,y_1],f] - [[y_2,f],y_1] = -[y_1,f], \\ 0 &= [y_2,[y_1,e]] = [[y_2,y_1],e] - [[y_2,e],y_1] = -[y_1,e], \end{aligned}$$

we obtain $[y_1,h] = [y_1,f] = [y_1,e] = 0$.

Using the above obtained equalities and the following

$$\begin{aligned} 0 &= [y_1,[y_2,f]] = [[y_1,y_2],f] - [[y_1,f],y_2] = [[y_1,y_2],f] = \\ &= [y_1 + \sum_{k=0}^{m} a_{12}^k x_k, f] = \sum_{k=0}^{m} a_{12}^k [x_k,f] = \sum_{k=0}^{m-1} a_{12}^k x_{k+1}, \end{aligned}$$

we get $a_{12}^i = 0$ with $0 \le i \le m-1$.

Now from

$$\begin{aligned} 0 &= [y_1,[y_2,h]] = [[y_1,y_2],h] - [[y_1,h],y_2] = [[y_1,y_2],h] = \\ &= [y_1 + a_{12}^m x_m, h] = -m a_{12}^m x_m, \end{aligned}$$

we get $a_{12}^m = 0$, consequently $a_{12}^m = 0$ for all $0 \le i \le m$, i.e. $[y_1,y_2] = y_1$.

Thus, we obtain the following table of multiplication:

$$\begin{array}{lll}
[e,h] = 2e, & [h,f] = 2f, & [e,f] = h, \\
[h,e] = -2e & [f,h] = -2f, & [f,e] = -h, \\
[x_k,h] = (m-2k)x_k & 0 \le k \le m, & \\
[x_k,f] = x_{k+1}, & 0 \le k \le m-1, & \\
[x_k,e] = -k(m+1-k)x_{k-1}, & 1 \le k \le m, & \\
[y_2,e] = \sum\limits_{j=0}^{m} a_{2e}^j x_j, & [y_2,f] = \sum\limits_{j=0}^{m} a_{2f}^j x_j, & [y_2,h] = \sum\limits_{j=0}^{m} a_{2h}^j x_j, \\
[y_1,y_2] = y_1, & [y_2,y_1] = -y_1, & [y_2,y_2] = \sum\limits_{j=0}^{m} a_2^j x_j, \\
[x_k,y_2] = a x_k, & 0 \le k \le m. &
\end{array}$$

In order to complete the proof of the theorem we need to prove that $[y_2,y_2] = 0$, and $[R^{-1}, sl_2^{-1}] = 0$.

Consider two cases:

Case 1.

Let $a \neq 0$, then taking the change of the basic element as follows

$$y_2{}' = y_2 - \sum_{j=0}^{m} \frac{a_2^j}{a} x_j,$$

we get

$$\begin{aligned} [y_2{}', y_2{}'] &= \left[y_2 - \sum_{j=0}^{m} \frac{a_2^j}{a} x_j, y_2 - \sum_{j=0}^{m} \frac{a_2^j}{a} x_j \right] = \\ &= [y_2,y_2] - \left[\sum_{j=0}^{m} \frac{a_2^j}{a} x_j, y_2 \right] = \sum_{j=0}^{m} a_2^j x_j - \sum_{j=0}^{m} a_2^j x_j = 0, \end{aligned}$$

which leads to $[y_2,y_2] = 0$.

Consider

$$\begin{aligned} 0 &= [y_2,[y_2,h]] = [[y_2,y_2],h] - [[y_2,h],y_2] = -[[y_2,h],y_2] = \\ &= -\sum_{j=0}^{m} a_{2h}^j [x_j,y_2] = -\sum_{j=0}^{m} a_{2h}^j a x_j, \end{aligned}$$

which gives $a_{2h}^j = 0$ for $0 \le j \le m$.

Similarly from the equalities

$$\begin{aligned} 0 &= [y_2,[y_2,f]] = [[y_2,y_2],f] - [[y_2,f],y_2] = -\sum_{j=0}^{m} a_{2f}^j [x_j,y_2] = -\sum_{j=0}^{m} a_{2f}^j a x_j, \\ 0 &= [y_2,[y_2,e]] = [[y_2,y_2],e] - [[y_2,e],y_2] = -\sum_{j=0}^{m} a_{2e}^j [x_j,y_2] = -\sum_{j=0}^{m} a_{2e}^j a x_j, \end{aligned}$$

we get $a_{2f}^j = a_{2e}^j = 0$ for $0 \le j \le m$. Hence, $[R^{-1}, sl_2^{-1}] = 0$.

Thus, we proved the theorem for $a \neq 0$.

Case 2.

Let $a=0$, then we consider the identity

$$[y_2,[y_2,f]]=[[y_2,y_2],f]-[[y_2,f],y_2]$$

and we derive

$$0=\sum_{j=0}^{m}a_2^i[x_i,f]=\sum_{j=0}^{m-1}a_2^ix_{i+1}\ \Rightarrow\ a_2^i=0,\ 0\leq i\leq m-1,\quad i.e.\ \ [y_2,y_2]=a_2^mx_m.$$

From the chain of the equalities

$$0=[y_2,[y_2,h]]=[[y_2,y_2],h]-[[y_2,h],y_2]=a_2^m[x_m,h]=-ma_2^mx_m,$$

we obtain $a_2^m=0$, that is $[y_2,y_2]=0$.

Let us take the change of the basic element in the form:

$$y_2{}'=y_2-\sum_{j=1}^{m}\frac{a_{2e}^{j-1}}{-mj+j(j-1)}x_j$$

Then

$$\begin{aligned}
[y_2{}',e] &=[y_2,e]-\sum_{j=1}^{m}\tfrac{a_{2e}^{j-1}}{-mj+j(j-1)}[x_j,e]=\\
&=[y_2,e]-\sum_{j=1}^{m}\tfrac{a_{2e}^{j-1}}{-mj+j(j-1)}(-mj+j(j-1))x_{j-1}=\\
&=\sum_{j=0}^{m}a_{2e}^jx_j-\sum_{j=1}^{m}a_{2e}^{j-1}x_{j-1}=\\
&=\sum_{j=0}^{m}a_{2e}^jx_j-\sum_{j=0}^{m-1}a_{2e}^jx_j=a_{2e}^mx_m.
\end{aligned}$$

Thus, we can assume that

$$[y_2,e]=a_{2e}^mx_m,\quad [y_2,h]=\sum_{j=0}^{m}a_{2h}^jx_j,\quad [y_2,f]=\sum_{j=0}^{m}a_{2f}^jx_j.$$

We have

$$\begin{aligned}
[y_2,[e,h]] &=[[y_2,e],h]-[[y_2,h],e]=a_{2e}^m[x_m,h]-\sum_{j=0}^{m}a_{2h}^j[x_j,e]=\\
&=-ma_{2e}^mx_m-\sum_{j=0}^{m}a_{2h}^j(-mj+j(j-1))x_{j-1}.
\end{aligned}$$

On the other hand $[y_2,[e,h]]=2[y_2,e]=2a_{2e}^mx_m$.

Comparing the coefficients at the basic elements, we get $a_{2e}^m=0$ and $a_{2h}^j=0$ where $1\leq j\leq m$. Hence, $[y_2,e]=0$, $[y_2,f]=\sum\limits_{j=0}^{m}a_{2f}^jx_j$, $[y_2,h]=a_{2h}^0x_0$.

Consider

$$\begin{aligned}
[y_2,[e,f]] &=[[y_2,e],f]-[[y_2,f],e]=-\sum_{j=0}^{m}a_{2f}^j[x_j,e]=\\
&=-\sum_{j=0}^{m}a_{2f}^j(mj+j(j-1))x_{j-1}=\\
&=ma_{2f}^1x_0-\sum_{j=2}^{m}a_{2f}^j(mj+j(j-1))x_{j-1}.
\end{aligned}$$

On the other hand

$$[y_2,[e,f]]=[y_2,h]=a_{2h}^0x_0.$$

Comparing the coefficients, we obtain $a^0_{2h} = ma^1_{2f}$ and $a^j_{2f} = 0$ for $2 \leq j \leq m$. Then we have the product $[y_2, f] = a^0_{2f}x_0 + a^1_{2f}x_1$.

Now we consider the equalities

$$-2[y_2, f] = [y_2, [f, h]] = [[y_2, f], h] - [[y_2, h], f] = [a^0_{2f}x_0 + a^1_{2f}x_1, h] - ma^1_{2f}[x_0, f],$$

and we have

$$-2a^0_{2f}x_0 - 2a^1_{2f}x_1 = ma^0_{2f}x_0 + a^1_{2f}(m-2)x_1 - ma^1_{2f}x_1 \Rightarrow a^0_{2f} = 0.$$

Therefore, $[y_2, f] = a^1_{2f}x_1$ and $[y_2, h] = ma^1_{2f}x_0$.

Taking the change $y_2{}' = y_2 - a^1_{2f}x_0$, we obtain

$$\begin{aligned} [y_2{}', f] &= [y_2, f] - a^1_{2f}[x_0, f] = a^1_{2f}x_1 - a^1_{2f}x_1 = 0, \\ [y_2{}', h] &= [y_2, h] - a^1_{2f}[x_0, h] = ma^1_{2f}x_0 - ma^1_{2f}x_0 = 0. \end{aligned}$$

Thus, we have $[R^{-1}, sl_2^{-1}] = 0$ which completes the proof of the theorem. $\square$

In the case when the dimension of the ideal I is equal to three, we have the family of Leibniz algebras with the following table of multiplication:

$$\begin{array}{lll}
[e,h] = 2e, & [h,f] = 2f, & [e,f] = h, \\
[h,e] = -2e & [f,h] = -2f, & [f,e] = -h, \\
[x_1,e] = -2x_0 & [x_2,e] = -2x_1, & [x_0,f] = x_1, \\
[x_1,f] = x_2 & [x_0,h] = 2x_0, & [x_2,h] = -2x_2, \\
[e,y_1] = \lambda x_0 & [f,y_1] = \frac{1}{2}\lambda x_2, & [h,y_1] = \lambda x_1, \\
[e,y_2] = \mu x_0 & [f,y_2] = \frac{1}{2}\mu x_2, & [h,y_2] = \mu x_1, \\
[y_1,y_2] = y_1 & [y_2,y_1] = -y_1, & [y_2,y_2] = -\frac{ab}{2}x_2, \\
[x_0,y_2] = ax_0 & [x_1,y_2] = ax_1, & [x_2,y_2] = ax_2, \\
[y_2,e] = bx_1 & [y_2,h] = bx_2, &
\end{array}$$

Verifying the Leibniz identity of the above family of algebras, using the software *Mathematica* [3], we get the condition $\lambda(1-a) = 0$.

Taking the change in the form $y_2{}' = y_2 + \frac{b}{2}x_2$ we obtain

$$\begin{aligned} [y_2{}', e] &= [y_2 + \tfrac{b}{2}x_2, e] = [y_2, e] + \tfrac{b}{2}[x_2, e] = bx_1 - bx_1 = 0, \\ [y_2{}', h] &= [y_2 + \tfrac{b}{2}x_2, h] = [y_2, h] + \tfrac{b}{2}[x_2, h] = bx_2 - bx_2 = 0, \\ [y_2{}', y_2{}'] &= [y_2 + \tfrac{b}{2}x_2, y_2 + \tfrac{b}{2}x_2] = [y_2, y_2] + \tfrac{b}{2}[x_2, y_2] = -\tfrac{ab}{2}bx_2 + \tfrac{ab}{2}bx_2 = 0. \end{aligned}$$

Thus, we can assume that $[y_2{}'e] = [y_2{}', h] = [y_2{}', y_2{}'] = 0$ and we have the family of algebras $L(\lambda, \mu, a)$.

$$\begin{array}{lll}
[e,h] = 2e, & [h,f] = 2f, & [e,f] = h, \\
[h,e] = -2e & [f,h] = -2f, & [f,e] = -h, \\
[x_1,e] = -2x_0 & [x_2,e] = -2x_1, & [x_0,f] = x_1, \\
[x_1,f] = x_2 & [x_0,h] = 2x_0, & [x_2,h] = -2x_2, \\
[e,y_1] = \lambda x_0 & [f,y_1] = \frac{1}{2}\lambda x_2, & [h,y_1] = \lambda x_1, \\
[e,y_2] = \mu x_0 & [f,y_2] = \frac{1}{2}\mu x_2, & [h,y_2] = \mu x_1, \\
[x_0,y_2] = ax_0 & [x_1,y_2] = ax_1, & [x_2,y_2] = ax_2, \\
[y_1,y_2] = y_1 & [y_2,y_1] = -y_1, &
\end{array}$$

with the condition $\lambda(1-a)=0$.

Theorem 3.2. *Let L be a Leibniz algebra such that $L/I \cong sl_2 \oplus R$, where R is a two-dimensional solvable ideal and I is a three-dimensional right irreducible module over sl_2. Then L is isomorphic to one of the following pairwise non isomorphic algebras :*

$$L(1,0,1);\ L(0,1,a);\ L(0,0,a),\ \text{with } a \in F.$$

Proof. Similarly we derive $\lambda(1-a)=0$.

Let $\lambda \neq 0$, then $a=1$. By change ${y_1}' = \frac{1}{\lambda}y_1$, ${y_2}' = -\frac{\mu}{\lambda}y_1 + y_2$ we deduce

$$\begin{aligned}
&[e,{y_1}'] = [e,\tfrac{1}{\lambda}y_1] = \tfrac{1}{\lambda}\lambda x_0 = x_0,\\
&[f,{y_1}'] = [e,\tfrac{1}{\lambda}y_1] = \tfrac{1}{2\lambda}\lambda x_2 = \tfrac{1}{2}x_2,\\
&[h,{y_1}'] = [h,\tfrac{1}{\lambda}y_1] = \tfrac{1}{\lambda}\lambda x_1 = x_1,\\
&[e,{y_2}'] = [e,-\tfrac{\mu}{\lambda}y_1+y_2] = -\tfrac{\mu}{\lambda}[e,y_1]+[e,y_2] = -\tfrac{\mu}{\lambda}\lambda x_0 + \mu x_0 = 0,\\
&[f,{y_2}'] = [f,-\tfrac{\mu}{\lambda}y_1+y_2] = -\tfrac{\mu}{\lambda}[f,y_1]+[f,y_2] = -\tfrac{\mu}{2\lambda}\lambda x_2 + \tfrac{1}{2}\mu x_2 = 0,\\
&[h,{y_2}'] = [h,-\tfrac{\mu}{\lambda}y_1+y_2] = -\tfrac{\mu}{\lambda}[h,y_1]+[h,y_2] = -\tfrac{\mu}{\lambda}\lambda x_1 + \mu x_1 = 0.
\end{aligned}$$

Thus, we can assume that $\lambda = 1$ and $\mu = 0$. Hence, we get the algebra

$$L(1,0,1)$$

If $\lambda = 0$, then when $\mu \neq 0$ by scale of basis of I, we can suppose that $\mu = 1$, i.e. we obtain the algebra $L(0,1,a)$.

If $\lambda = 0$, then when $\mu = 0$, we get the algebra $L(0,0,a)$.

By using the software *Mathematica* [4], we obtain that these algebras are non isomorphic. The theorem is proved. □

Analyzing the above obtained results we can formulate

Conjecture: Any Leibniz algebra is decomposed into a semidirect sum of its corresponding Lie algebra and the ideal I.

References

[1] Abdykassymova S., *Simple Leibnniz algebras of rank 1 in the characteristic p*, Ph.D. thesis, Almaty State University, Kazakhstan, (2001).

[2] Abdykassymova S., Dzhumadil'daev A.S. *Leibniz algebras in characteristic p*, C.R. Acad. Sci. Paris Sr. I Math. 332(12), (2001), 1047–1052.

[3] Camacho L.M., Gómez J.R., González A.J., Omirov B.A., *Naturally graded quasifiliform Leibniz algebras,* Journal Symbolic Computation, 44(5), (2009), 527–539.

[4] Camacho L.M., Cañete E.M., Gómez J.R., Omirov B.A., *Computational Applications for Maximum Length in Laibniz algebras*, submitted to Advances in Computational Mathematics, (2011).

[5] Jacobson N., Lie algebras, *Interscience Publishers*, Wiley, New York, 1962.

[6] Kostrikin A.I., *Around Bernside* [in Russian], Nauka, Moskow, 1986.

[7] Loday J.-L., Une version non commutative des algebres de Lie: les algebres de Leibniz, *Enseign. Math.*, **39**, (1993), 269-293.

[8] Mal'cev A., On solvable Lie algebras, (in Russian), *Bull.Acad.Sci.URSS, Ser.Math. (Izvestia Akad. Nauk SSSR)*, **9**, (1945),329-356.

[9] Omirov B.A., On finite dimensional nilpotent Leibniz algebras, Ph. D. Thesis, [in Russian], Tashkent, Uzbekistan, 2001.

[10] Rakhimov I.S., Omirov B.A. and Turdibaev R.M., On description of Leibniz algebras corresponding to sl_2, to appear 2012.

[11] Zelmanov E.I., On the global nilpotency of Engel Lie algebras in characteristic zero, [in Russian], *DAN SSSR*, **4** (1986), 12-26.

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