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SOME REMARKS ON LEIBNIZ ALGEBRAS WHOSE SEMISIMPLE PART RELATED WITH sl_2.

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Abstract. In this paper we identify the structure of complex finite-dimensional Leibniz algebras with associated Lie algebras $sl_2^1 \oplus sl_2^2 \oplus \cdots \oplus sl_2^s \oplus R$, where R is a solvable radical. The classifications of such Leibniz algebras in the cases $dimR = 2, 3$ and $dimI \neq 3$ are obtained. Moreover, we classify Leibniz algebras with $L/I \cong sl_2^1 \oplus sl_2^2$ and some conditions on ideal I.

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1. Introduction.

The notion of Leibniz algebras has been first introduced by Loday in [8], [9] as a non-antisymmetric generalization of Lie algebras. During the last 20 years the theory of Leibniz algebras has been actively studied and many results of the theory of Lie algebras have been extended to Leibniz algebras. A lot of works have so far been devoted to the description of finite-dimensional nilpotent Leibniz algebras [2], [3]. However, just a few works are related to the semisimple part of Leibniz algebras [6], [5], [11].

We know from the classical theory of finite-dimensional Lie algebras, that an arbitrary Lie algebra is decomposed into a semidirect sum of the solvable radical and its semisimple subalgebra (Levi's Theorem [7]). According to the Cartan-Killing theory, a semisimple Lie algebra can be represented as a direct sum of simple ideals, which are completely classified [7].

Recently, Barnes has proved an analogue of Levi's Theorem for the case of Leibniz algebras [5]. Namely, a Leibniz algebra is decomposed into a semidirect sum of its solvable radical and a semisimple Lie algebra.

The inherent properties of non-Lie Leibniz algebras imply that the subspace spanned by squares of elements of the algebra is a non-trivial ideal (further denoted by I). Moreover, the ideal I is abelian and hence, it belongs to the solvable radical. Although Barnes's result reduces the semisimple part of a Leibniz algebra to the Lie algebras case, we still need to study the relationship between the products of a semisimple Lie algebra and the ideal I (see [10] and [11]). In order to analyze the general case, we study the case when semisimple Leibniz part is a direct sum of sl_2 algebras since the exact description of the irreducible modules is established only for the algebra sl_2.

The present work aims at describing the structure of Leibniz algebras with the associated Lie algebras $sl_2^1 \oplus sl_2^2 \oplus \cdots \oplus sl_2^s \oplus R$ and with I a right irreducible sl_2^k-module for some k in order to classify the Leibniz algebras with semisimple part $sl_2^1 \oplus sl_2^2$ and some conditions on the ideal I.

Content is organized into different sections as follows. In Section 2, we give some necessary notions and preliminary results about Leibniz algebras with associated Lie algebra $sl_2 \dot{+} R$. Section 3, is devoted to the study of the structure of the Leibniz algebras whose semisimple part is a direct sum of sl_2 algebras and it is under some conditions to the ideal I. In Section 4, we classify Leibniz algebras whose semisimple part is a direct sum of sl_2^1, sl_2^2 and I is decomposed into a direct sum of two irreducible modules $I_{1,1}, I_{1,2}$ over sl_2^1 such that $dimI_{1,1} = dimI_{1,2}$.

Throughout the work, the vector spaces and the algebras are finite-dimensional over the field of complex numbers. Moreover, in the table of multiplication of an algebra the omitted products are assumed to be zero. We shall use the following symbols: $+$, $\oplus$ and $\dot{+}$ for notations of the direct sum of the vector spaces, the direct and semidirect sums of algebras, respectively.

2. Preliminaries

In this section we give some necessary definitions and preliminary results.

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Definition 2.1. [8] *An algebra $(L, [\cdot,\cdot])$ over a field F is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds true.

Let L be a Leibniz algebra and let $I = ideal < [x, x] \mid x \in L >$ be the ideal of L generated by all squares. The natural epimorphism $\varphi : L \to L/I$ determines the associated Lie algebra L/I of the Leibniz algebra L. It is clear that ideal I is the minimal ideal with respect to the property that the quotient algebra by this ideal is a Lie algebra.

In [5] we note that the ideal I coincides with the space spanned by squares of elements of an algebra.

According to [7] there exists a unique (up to isomorphism) simple 3-dimensional Lie algebra with the following table of multiplication:

$$sl_2: \quad [e, h] = -[h, e] = 2e, \qquad [h, f] = -[f, h] = 2f, \qquad [e, f] = -[f, e] = h,$$

The basis $\{e, f, h\}$ is called the *canonical basis*.

[10] describes the Leibniz algebras for which the quotient Lie algebras are isomorphic to sl_2. Let us present a Leibniz algebra L with the table of multiplication in a basis $\{e, f, h, x_0^j, \dots, x_{t_j}^j,\ 1 \le j \le p\}$ and the quotient algebra L/I is sl_2:

$$\begin{array}{lll}
[e, h] = -[h, e] = 2e, & [h, f] = -[f, h] = 2f, & [e, f] = -[f, e] = h,\\
[x_k^j, h] = (t_j - 2k)x_k^j, & 0 \le k \le t_j, & \\
[x_k^j, f] = x_{k+1}^j, & 0 \le k \le t_j - 1, & \\
[x_k^j, e] = -k(t_j + 1 - k)x_{k-1}^j, & 1 \le k \le t_j. &
\end{array}$$

where $L = sl_2 + I_1 + I_2 + \dots + I_p$ and $I_j = \langle x_1^j, \dots, x_{t_j}^j \rangle$, $1 \le j \le p$.

The last three types of products of the above table of multiplication are characterized asan irreducible sl_2-module with the canonical basis of sl_2 [7].

Now we introduce the notion of semisimplicity for Leibniz algebras.

Definition 2.2. *A Leibniz algebra L is called semisimple if its maximal solvable ideal is equal to I.*

Since in the Lie algebras case the ideal I is equal to zero, this definition also agrees with the definition of semisimple Lie algebra.

Although Levi's Theorem is proved for the left Leibniz algebras [5], it is also true for right Leibniz algebras (here we consider the right Leibniz algebras).

Theorem 2.3. [5] *(Levi's Theorem). Let L be a finite dimensional Leibniz algebra over a field of characteristic zero and R be its solvable radical. Then there exists a semisimple subalgebra S of L, such that $L = S \dot{+} R$.*

An algebra L is called *simple* if it only has only ideals $\{0\}, \{I\}, \{L\}$ and $L^2 \neq I$, see [1]. From the proof of Theorem 2.3, it is not difficult to see that S is a semisimple Lie algebra. Therefore, we have that a simple Leibniz algebra is a semidirect sum of simple Lie algebra S and the irreducible right module I, i.e. $L = S \dot{+} I$. Hence, we get the description of the simple Leibniz algebras in terms of simple Lie algebras and ideals I.

Definition 2.4. [7] *A non-zero module M over a Lie algebra whose only submodules are the module itself and zero module is called irreducible module. A non-zero module M which admits decomposition into a direct sum of irreducible modules is said to be completely reducible.*

Further, we shall use the following result of the classical theory of Lie algebras.

Theorem 2.5. [7] *Let G be a semisimple Lie algebra over a field of characteristic zero. Then every finite dimensional module over G is completely reducible.*

Now we present the results of the classification of Leibniz algebras with the conditions $L/I \cong sl_2 \oplus R,\ dimR = 2, 3$ and I a right irreducible module over sl_2 $(dimI \neq 3)$.

Theorem 2.6. [4] *Let L be a Leibniz algebra whose quotient $L/I \cong sl_2 \oplus R$, where R is a two-dimensional solvable ideal and I is a right irreducible module over sl_2 $(dimI \neq 3)$. Then there exists a* *basis* $\{e,h,f,x_0,x_1,\dots,x_m,y_1,y_2\}$ *of the algebra* L *such that the table of multiplication in* L *has the following form:*

$$\begin{cases} [e,h]=-[h,e]=2e, & [h,f]=-[f,h]=2f, \quad [e,f]=-[f,e]=h,\\ [y_1,y_2]=-[y_2,y_1]=y_1, & [x_k,y_2]=ax_k, \quad 0\le k\le m,\ a\in\mathbb{C},\\ [x_k,h]=(m-2k)x_k & 0\le k\le m,\\ [x_k,f]=x_{k+1}, & 0\le k\le m-1,\\ [x_k,e]=-k(m+1-k)x_{k-1}, & 1\le k\le m.\end{cases}$$

The following theorem extends Theorem 2.6 for the case $dimR=3$.

Theorem 2.7. [11] *Let* L *be a Leibniz algebra whose quotient* $L/I\cong sl_2\oplus R$, *where* R *is a three-dimensional solvable ideal and* I *is a right irreducible module over* sl_2 $(dimI\neq 3)$. *Then there exists a basis* $\{e,h,f,x_0,x_1,\dots,x_m,y_1,y_2,y_3\}$ *of the algebra* L *such that the table of multiplication in* L *has one of the following two forms:*

$$L_1(\alpha,a):\begin{cases} [e,h]=-[h,e]=2e, & [h,f]=-[f,h]=2f, \quad [e,f]=-[f,e]=h,\\ [y_1,y_2]=-[y_2,y_1]=y_1, & [y_3,y_2]=-[y_2,y_3]=\alpha y_3,\\ [x_k,h]=(m-2k)x_k, & 0\le k\le m,\\ [x_k,f]=x_{k+1}, & 0\le k\le m-1,\\ [x_k,e]=-k(m+1-k)x_{k-1}, & 1\le k\le m,\\ [x_i,y_2]=ax_i, & 0\le i\le m.\end{cases}$$

$$L_2(a):\begin{cases} [e,h]=-[h,e]=2e, & [h,f]=-[f,h]=2f, \quad [e,f]=-[f,e]=h,\\ [y_1,y_2]=-[y_2,y_1]=y_1+y_3, & [y_3,y_2]=-[y_2,y_3]=y_3,\\ [x_k,h]=(m-2k)x_k, & 0\le k\le m,\\ [x_k,f]=x_{k+1}, & 0\le k\le m-1,\\ [x_k,e]=-k(m+1-k)x_{k-1}, & 1\le k\le m,\\ [x_i,y_2]=ax_i, & 0\le i\le m.\end{cases}$$

For a semisimple Lie algebra S we consider a semisimple Leibniz algebra L such that $L=(sl_2\oplus S)\dot{+}I$.. We put $I_1=[I,sl_2]$.

Let I_1 is a reducible over sl_2. Then by Theorem 2.5 we have the decomposition:

$$I_1=I_{1,1}\oplus I_{1,2}\oplus\dots\oplus I_{1,p},$$

where $I_{1,j}$ are the irreducible modules over sl_2 for every j, $1\le j\le p$.

Theorem 2.8. [6] *Let* $dimI_{1,j_1}=dimI_{1,j_2}=\dots=dimI_{1,j_s}=t+1$ *be with* $1\le s\le p$. *Then there exist* $(t+1)$*-pieces of* s*-dimensional submodules* $I_{2,1},I_{2,2},\dots I_{2,t+1}$ *of the module* $I_2=[I,S]$ *such that*

$$I_{2,1}+I_{2,2}+\dots+I_{2,t+1}=I_1\cap I_2.$$

3. The structure of Leibniz algebras with associated Lie algebras $sl_2^1\oplus sl_2^2\oplus\dots\oplus sl_2^s\oplus R$ and I is a right irreducible sl_2^k-module for some k.

In this section, we will consider a Leibniz algebra satisfying the following conditions:

(i) the quotient algebra L/I is isomorphic to the direct sum $sl_2^1\oplus sl_2^2\oplus\dots\oplus sl_2^s\oplus R$, where R is n-dimensional solvable Lie algebra;

(ii) the ideal I is a right irreducible sl_2^k-module for some $k\in\{1,\dots,s\}$.

We put $dimI=m+1$.

Let us introduce the following notations:

$$sl_2^i=<e_i,f_i,h_i>,\ \ 1\le i\le s,\quad I=<x_0,\dots,x_m>,\quad R=<y_1,\dots,y_n>.$$

Without loss of generality one can assume that $k=1$. Then due to [7] we have

$$\begin{array}{ll} [e_1,h_1]=-[h_1,e_1]=2e_1, & [h_1,f_1]=-[f_1,h_1]=2f_1, \quad [e_1,f_1]=-[f_1,e_1]=h_1,\\ [x_k,h_1]=(m-2k)x_k, & 0\le k\le m,\\ [x_k,f_1]=x_{k+1}, & 0\le k\le m-1,\\ [x_k,e_1]=-k(m+1-k)x_{k-1}, & 1\le k\le m.\end{array}$$

Lemma 3.1. *Let L be a Leibniz algebra satisfying the conditions (i)-(ii). Then $[I, sl_2^j] = 0$ for any $j \in \{2, \dots, s\}$.*

Proof. For a fixed j $(2 \leq j \leq s)$ we put

$$[x_0, e_j] = \sum_{i=0}^{m} \alpha_{j,i} x_i, \quad [x_0, f_j] = \sum_{i=0}^{m} \beta_{j,i} x_i, \quad [x_0, h_j] = \sum_{i=0}^{m} \gamma_{j,i} x_i.$$

Applying the Leibniz identity we have

$$[[x_0, e_j], f_1] = [x_0, [e_j, f_1]] + [[x_0, f_1], e_j] = [[x_0, f_1], e_j] = [x_1, e_j].$$

On the other hand,

$$[[x_0, e_j], f_1] = \sum_{i=0}^{m} \alpha_{j,i} [x_i, f_1] = \sum_{i=0}^{m-1} \alpha_{j,i} x_{i+1}.$$

Consequently, we obtain $[x_1, e_j] = \sum\limits_{i=0}^{m-1} \alpha_{j,i} x_{i+1}$.

Using the equality

$$[[x_i e_j], f_1] = [x_i, [e_j, f_1]] + [[x_i, f_1], e_j]$$

and the mathematical induction, we prove the following expression

$$[x_i, e_j] = \sum_{k=0}^{m-i} \alpha_{j,k} x_{k+i}, \ 2 \leq i \leq m.$$

From the chain of the equalities

$$-\sum_{i=1}^{m} i(m-i+1)\alpha_{j,i} e_{i-1} = \sum_{i=0}^{m} \alpha_{j,i} [x_i, e_1] = [[x_0, e_j], e_1] = [x_0, [e_j, e_1]] + [[x_0, e_1], e_j] = 0$$

we conclude that $\alpha_{j,i} = 0$ with $1 \leq i \leq m$, that is, $[x_k, e_j] = \alpha_{j,0} x_k, \ 0 \leq k \leq m$.

Similarly, we obtain

$$[x_k, f_j] = \beta_{j,0} x_k, \qquad [x_k, h_j] = \gamma_{j,0} x_k, \ 0 \leq k \leq m.$$

The equalities

$$2[x_i, e_j] = [x_i, [e_j, h_j]] = [[x_i, e_j], h_j] - [[x_i, h_j], e_j] = \alpha_{j,0}[x_i, h_j] - \gamma_{j,0}[x_i, e_j] = \alpha_{j,0}\gamma_{j,0} x_i - \gamma_{j,0}\alpha_{j,0} x_i = 0$$

imply that $[x_i, e_j] = 0$ with $0 \leq i \leq m$.

Similarly, from

$$[x_i, [f_j, h_j]] = [[x_i, f_j], h_j] - [[x_i, h_j], f_j],$$
$$[x_i, [e_j, f_j]] = [[x_i, e_j], f_j] - [[x_i, f_j], e_j],$$

we derive $[x_i, f_j] = 0, \quad [x_i, h_j] = 0, \ 0 \leq i \leq m$. Thus, $[I, sl_2^j] = 0$ with $2 \leq j \leq s$. ◻

We need the following lemma.

Lemma 3.2. *Let L be a Leibniz algebra satisfying the conditions (i)-(ii). Then $[sl_2^j, sl_2^j] = sl_2^j$ with $2 \leq j \leq s$.*

Proof. We set

$$[e_j, h_j] = 2e_j + \sum_{k=0}^{m} a_{j,k} x_k, \quad [f_j, h_j] = -2f_j + \sum_{k=0}^{m} b_{j,k} x_k, \quad [e_j, f_j] = h_j + \sum_{k=0}^{m} c_{j,k} x_k.$$

Take the basis transformation in the following form:

$$e_j' = e_j + \frac{1}{2} \sum_{k=0}^{m} a_{j,k} x_k, \quad f_j' = f_j - \frac{1}{2} \sum_{k=0}^{m} b_{j,k} x_k, \quad h_j' = h_j + \sum_{k=0}^{m} c_{j,k} x_k.$$

Then, thanks to Lemma 3.1, we can conclude

$$[e_j, h_j] = 2e_j, \quad [f_j, h_j] = -2f_j, \quad [e_j, f_j] = h_j. \tag{3.1}$$

Taking into account that $[I, sl_2^j] = 0$ we have

$$2[h_j, e_j] = [h_j, [e_j, h_j]] = [[h_j, e_j], h_j] - [[h_j, h_j], e_j] = -2[e_j, h_j] \Rightarrow [e_j, h_j] = -[h_j, e_j].$$

Analogously, we obtain

$$[f_j, h_j] = [f_j, [e_j, f_j]] = [[f_j, e_j], f_j] - [[f_j, f_j], e_j] = -[h_j, f_j] \Rightarrow [f_j, h_j] = -[h_j, f_j].$$

Now, we denote

$$[e_j, e_j] = \sum_{i=0}^{m} \lambda_{j,i} x_i, \quad [f_j, f_j] = \sum_{i=0}^{m} \mu_{j,i} x_i,$$
$$[h_j, h_j] = \sum_{i=0}^{m} \tau_{j,i} x_i, \quad [f_j, e_j] = -h_j + \sum_{i=0}^{m} \eta_{j,i} x_i.$$

From the chain of the equalities

$$\sum_{i=0}^{m} \lambda_{j,i}(m-2i)x_i = \sum_{i=0}^{m} \lambda_{j,i}[x_i, h_1] = [[e_j, e_j], h_1] = [[e_j, h_1], e_j] + [e_j, [e_j, h_1]] = 0$$

we derive $\sum\limits_{i=0}^{m} \lambda_{j,i}(m-2i)x_i = 0$.

- If m is odd, then $\lambda_{j,i} = 0$ with $0 \le i \le m$, that is, we have $[e_j, e_j] = 0$ for $2 \le j \le s$.

- If m is even, then $[e_j, e_j] = \lambda_{j,\frac{m}{2}} x_{\frac{m}{2}}$.
  The equalities
  $$0 = [e_j, [f_1, e_j]] = [[e_j, f_1], e_j] - [[e_j, e_j], f_1] = -[[e_j, e_j], f_1] =$$
  $$= -\lambda_{j,\frac{m}{2}}[x_{\frac{m}{2}}, f_1] = -\lambda_{j,\frac{m}{2}} x_{\frac{m}{2}+1}$$
  imply that $[e_j, e_j] = 0$ for an even value of m and $2 \le j \le s$, as well.

Consider

$$\sum_{i=0}^{m} \mu_{j,i}(m-2i)x_i = \sum_{i=0}^{m} \mu_{j,i}[x_i, h_1] =$$
$$= [[f_j, f_j], h_1] = [[f_j, h_1], f_j] + [f_j, [f_j, h_1]] = 0.$$

Then, $\sum\limits_{i=0}^{m} \mu_{j,i}(m-2i)x_i = 0$.

Evidently, for an odd value of m the products $[f_j, f_j]$ are equal to zero and for an even value of m we have $[f_j, f_j] = \mu_{j,\frac{m}{2}} x_{\frac{m}{2}}$.

The equalities

$$0 = [f_j, [f_1, f_j]] = [[f_j, f_1], f_j] - [[f_j, f_j], f_1] = -[[f_j, f_j], f_1] =$$
$$= -\mu_{j,\frac{m}{2}}[x_{\frac{m}{2}}, f_1] = -\mu_{j,\frac{m}{2}} x_{\frac{m}{2}+1}$$

imply that $[f_j, f_j] = 0$ for any value of m and $2 \le j \le s$.

In a similar way from the equations

$$-\sum_{i=1}^{m} i(m+1-i)\tau_{j,i} x_{i-1} = \sum_{i=0}^{m} \tau_{j,i}[x_i, e_1] = \sum_{i=0}^{m} \tau_{j,i}[x_i, e_1] =$$
$$= [[h_j, h_j], e_1] = [[h_j, e_1], h_j] + [h_j, [h_j, e_1]] =$$
$$= [[h_j, h_j], f_1] = [[h_j, f_1], h_j] + [h_j, [h_j, f_1]] = 0,$$

we derive $[h_j, h_j] = 0$ for $2 \le j \le s$.

Finally, from

$$0 = [h_j, h_j] = [h_j, [e_j, f_j]] = [[h_j, e_j], f_j] - [[h_j, f_j], e_j] = -2[e_j, f_j] - 2[f_j, e_j]$$

we deduce $[e_j, f_j] = -[f_j, e_j]$ for $2 \le j \le s$.

Taking into account the obtained equalities:

$$[e_j, h_j] = -[h_j, e_j], \quad [e_j, f_j] = -[f_j, e_j], \quad [f_j, h_j] = -[h_j, f_j], \quad [e_j, e_j] = [f_j, f_j] = [h_j, h_j] = 0$$

and (3.1) complete the proof of lemma. □

The following result establishes the multiplication of sl_2^i and sl_2^j with $i \neq j$.

Lemma 3.3. *Let L be Leibniz algebra satisfying the conditions (i)-(ii). Then*

$$[sl_2^i, sl_2^j] = 0, \quad 1 \le i, j \le s, \ i \neq j.$$

Proof. Firstly we shall prove that $[sl_2^1, sl_2^j] = 0$ for some $j \in \{2, \dots s\}$.

For a fixed element b of sl_2^j, we put

$$[e_1, b] = \sum_{k=0}^{m} \theta_k x_k, \qquad [f_1, b] = \sum_{k=0}^{m} \rho_k x_k.$$

Consider

$$0 = [e_1, [h_1, b]] = [[e_1, h_1], b] - [[e_1, b], h_1] = 2[e_1, b] - \sum_{k=0}^{m} \theta_k [x_k, h_1] =$$
$$= 2\sum_{k=0}^{m} \theta_k x_k - \sum_{k=0}^{m} \theta_k (m - 2k) x_k = \sum_{k=0}^{m} \theta_k (-m + 2k + 2) x_k.$$

Consequently,

$$[e_1, b] = \begin{cases} 0, & m \quad \text{odd}, \\ \theta_{\frac{m}{2}-1} x_{\frac{m}{2}-1}, & m \quad \text{even}. \end{cases}$$

If m is even, $m \neq 2$, the equalities

$$0 = [e_1, [e_1, b]] = [[e_1, e_1], b] - [[e_1, b], e_1] = -[[e_1, b], e_1] =$$
$$= -\theta_{\frac{m}{2}-1}[x_{\frac{m}{2}-1}, e_1] = -\theta_{\frac{m}{2}-1}(\tfrac{m}{2} - 1)(\tfrac{m}{2} + 2) x_{\frac{m}{2}-2}$$

imply $[e_1, b] = 0$.

Similarly as above, from

$$0 = [f_1, [h_1, b]] = [[f_1, h_1], b] - [[f_1, b], h_1] = -2[f_1, b] - [[f_1, b], h_1] =$$
$$= -2\sum_{k=0}^{m} \rho_k x_k - \sum_{k=0}^{m} \rho_k [x_k, h_1] = -2\sum_{k=0}^{m} \rho_k x_k - \sum_{k=0}^{m} \rho_k (m - 2k) x_k =$$
$$= \sum_{k=0}^{m} \rho_k (-m + 2k - 2) x_k,$$
$$0 = [f_1, [f_1, b]] = [[f_1, f_1], b] - [[f_1, b], f_1] = -[[f_1, b], f_1],$$

we get $[f_1, b] = 0$.

The equality $[h_1, b] = 0$ follows from

$$0 = [e_1, [f_1, b]] = [[e_1, f_1], b] - [[e_1, b], f_1] = [[e_1, f_1], b] = [h_1, b].$$

Thus, we have proved that $[sl_2^1, sl_2^j] = 0$ with $j \in \{2, \dots, s\}$ and $m \neq 2$.

If $m = 2$, we have

$$\begin{array}{lll} [e_1, e_j] = a_j x_0, & [e_1, f_j] = b_j x_0, & [e_1, h_j] = c_j x_0, \\ [f_1, e_j] = 0, & [f_1, f_j] = 0, & [f_1, h_j] = 0, \\ [h_1, e_j] = a_j x_1, & [h_1, f_j] = b_j x_1, & [h_1, h_j] = c_j x_1. \end{array}$$

Considering the Leibniz identity for the following triples of elements:

$$\{e_1, e_j, h_j\}, \ \{e_1, h_j, f_j\}, \ \{e_1, e_j, f_j\}$$

we lead to $a_j = b_j = c_j = 0, \ 2 \leq j \leq s$. Hence, $[sl_2^1, sl_2^j] = 0$ with $2 \leq j \leq s$ and $m = 2$.

For an arbitrary element c of sl_2^1, we apply the Leibniz identity for the following triples of elements:

$$\{e_j, h_j, c\}, \ \{h_j, f_j, c\}, \ \{e_j, f_j, c\}.$$

Then we deduce $[e_j, c] = [f_j, c] = [h_j, c]$, that is, $[sl_2^j, sl_2^1] = 0$.

Let $a \in sl_2^i$ with $2 \leq i \leq s, \ i \neq j$. From the equalities

$$0 = [a, [b, e_1]] = [[a, b], e_1] - [[a, e_1], b] = [[a, b], e_1]$$

$$0 = [a, [b, f_1]] = [[a, b], f_1] - [[a, f_1], b] = [[a, b], f_1],$$

we conclude that $[a, b] = 0$ □

Below we show that the solvable ideal R annihilate to both sides of each $sl_2^i, \ 2 \leq i \leq s$.

Lemma 3.4. *Let L be a Leibniz algebra satisfying the conditions (i)-(ii). Then*

$$[R, sl_2^i] = [sl_2^i, R] = 0, \quad 2 \leq i \leq s.$$

Proof. Applying the Leibniz identity for the following triples

$$\{y_s, e_1, a\},\ \{y_s, f_1, a\},\ \{a, y_s, e_1\},\ \{a, y_s, f_1\}$$

we lead to

$$[y_s, a] = 0, \quad [a, y_s] = 0,\ 1 \le s \le n$$

for an arbitrary element $a \in sl_2^i,\ 2 \le i \le s$. □

Summarizing the results of Lemmas 3.1-3.4, we obtain the following theorem.

Theorem 3.5. *Let L be a finite-dimensional Lebniz algebra satisfying the conditions:*

(i) $L/I \cong sl_2^1 \oplus sl_2^2 \oplus \cdots \oplus sl_2^s \oplus R$, where R is an n-dimensional solvable Lie algebra;

(ii) the ideal I is a right irreducible sl_2^k-module for some $k \in \{1, \dots, s\}$.

Then, $L = ((sl_2^1 \oplus R) \dotplus I) \oplus sl_2^2 \oplus \cdots \oplus sl_2^s$.

As a result of the Theorems 2.6–2.7 and 3.5, we have the following corollaries.

Corollary 3.6. *Let $L/I \cong sl_2^1 \oplus sl_2^2 \oplus \cdots \oplus sl_2^s \oplus R$ with $dimR = 2$ and $dimI \neq 3$. Then L is isomorphic to the following algebra:*

$$\begin{cases} [e_j, h_j] = -[h_j, e_j] = 2e_j, & [h_j, f_j] = -[f_j, h_j] = 2f_j, \\ [e_j, f_j] = -[f_j, e_j] = h_j, & 1 \le j \le s, \\ [y_1, y_2] = -[y_2, y_1] = y_1, & \\ [x_k, h_1] = (m - 2k)x_k, & 0 \le k \le m, \\ [x_k, f_1] = x_{k+1}, & 0 \le k \le m-1, \\ [x_k, e_1] = -k(m+1-k)x_{k-1}, & 1 \le k \le m, \\ [x_k, y_2] = ax_k, & 0 \le k \le m,\ a \in \mathbb{C}. \end{cases}$$

Corollary 3.7. *Let $L/I \cong sl_2^1 \oplus sl_2^2 \oplus \cdots \oplus sl_2^s \oplus R$, with $dimR = 3$ and $dimI \neq 3$. Then L is isomorphic to the following non-isomorphic algebras:*

$$L_1(\alpha, a): \begin{cases} [e_j, h_j] = -[h_j, e_j] = 2e_j, & [h_j, f_j] = -[f_j, h_j] = 2f_j, \\ [e_j, f_j] = -[f_j, e_j] = h_j, & 1 \le j \le s, \\ [y_1, y_2] = -[y_2, y_1] = y_1, & [y_3, y_2] = -[y_2, y_3] = \alpha y_3, \\ [x_k, h_1] = (m - 2k)x_k, & 0 \le k \le m, \\ [x_k, f_1] = x_{k+1}, & 0 \le k \le m-1, \\ [x_k, e_1] = -k(m+1-k)x_{k-1}, & 1 \le k \le m, \\ [x_i, y_2] = ax_i, & 0 \le i \le m, \end{cases}$$

$$L_2(a): \begin{cases} [e_j, h_j] = -[h_j, e_j] = 2e_j, & [h_j, f_j] = -[f_j, h_j] = 2f_j, \\ [e_j, f_j] = -[f_j, e_j] = h_j, & 1 \le j \le s, \\ [y_1, y_2] = -[y_2, y_1] = y_1 + y_3, & [y_3, y_2] = -[y_2, y_3] = y_3, \\ [x_k, h_1] = (m - 2k)x_k, & 0 \le k \le m, \\ [x_k, f_1] = x_{k+1}, & 0 \le k \le m-1, \\ [x_k, e_1] = -k(m+1-k)x_{k-1}, & 1 \le k \le m, \\ [x_i, y_2] = ax_i, & 0 \le i \le m. \end{cases}$$

4. The description of Leibniz algebras with semisimple part $sl_2^1 \oplus sl_2^2$ and some conditions on ideal I.

Let L be a Leibniz algebra and the quotient Lie algebra L/I isomorphic to a direct sum of two copies of the sl_2 ideals. In this section, we shall investigate the case when the ideal I is reducible over only one copy of sl_2. Thus, we have $L/I \cong sl_2^1 \oplus sl_2^2$. One can assume that I is reducible over sl_2^1. Due to Theorem 2.5 we have the following decomposition:

$$I = I_{1,1} \oplus I_{1,2} \oplus ... \oplus I_{1,s+1},$$

where $I_{1,j},\ 1 \leq j \leq s+1$ are the irreducible sl_2^1-modules.

We shall focus our study on the case when $dim I_{1,1} = dim I_{1,2} = \cdots = dim I_{1,s+1} = m+1$.

Let us introduce the notations as follows:

$$I_{1,j} = < x_0^j, x_1^j, \dots, x_m^j >,\ 1 \leq j \leq s+1$$

and

$$[x_i^j, e_2] = \sum_{k=1}^{s+1} \sum_{p=0}^{m} a_{i,j,p}^k x_p^k, \quad [x_i^j, f_2] = \sum_{k=1}^{s+1} \sum_{p=0}^{m} b_{i,j,p}^k x_p^k, \quad [x_i^j, h_2] = \sum_{k=1}^{s+1} \sum_{p=0}^{m} c_{i,j,p}^k x_p^k,$$

where $0 \leq i \leq m,\ 1 \leq j \leq s+1$.

Without loss of generality, one can assume that the products $[I_{1,j}, sl_2^1],\ 1 \leq j \leq s+1$ are expressed as follows:

$$\begin{array}{ll} [x_i^j, e_1] = -i(m+1-i)x_{i-1}^j, & 1 \leq i \leq m, \\ [x_i^j, f_1] = x_{i+1}^j, & 0 \leq i \leq m-1, \\ [x_i^j, h_1] = (m-2i)x_i^j, & 0 \leq i \leq m. \end{array}$$

Proposition 4.1. *Let $L/I \cong sl_2^1 \oplus sl_2^2$ and $I = I_{1,1} \oplus I_{1,2} \oplus \dots \oplus I_{1,s+1}$, with $dim I_{1,j} = m+1$ and $I_{1,j}$ are the irreducible sl_2^1-modules for $1 \leq j \leq s+1$. Then,*

$$[x_i^j, e_2] = \sum_{k=1}^{s+1} a_j^k x_i^k, \quad [x_i^j, f_2] = \sum_{k=1}^{s+1} b_j^k x_i^k, \quad [x_i^j, h_2] = \sum_{k=1}^{s+1} c_j^k x_i^k,$$

where $0 \leq i \leq m,\ 1 \leq j \leq s+1$.

Proof. Applying the Leibniz identity for the following triples of elements:

$$\{x_0^j, e_1, e_2\}, \quad \{x_1^j, e_1, e_2\}, \quad \{x_1^j, h_1, e_2\}, \qquad 1 \leq j \leq s+1$$

we derive the restrictions:

$$a_{0,j,p}^k = 0,\ 1 \leq p \leq m, \quad a_{1,j,1}^k = a_{0,j,0}^k, \quad a_{1,j,p}^k = 0,\ 2 \leq p \leq m, \quad a_{1,j,0}^k = 0, \quad 1 \leq k \leq s+1.$$

Consequently, we obtain

$$[x_0^j, e_2] = \sum_{k=1}^{s+1} a_{0,j,0}^k x_0^k, \quad [x_1^j, e_2] = \sum_{k=1}^{s+1} a_{0,j,0}^k x_1^k, \quad 1 \leq j \leq s+1.$$

By induction, we shall prove the equality

$$[x_i^j, e_2] = \sum_{k=1}^{s+1} a_{0,j,0}^k x_i^k, \quad 0 \leq i \leq m. \tag{4.1}$$

Using this assumption in the following chain of the equalities:

$$0 = [x_{i+1}^j, [e_1, e_2]] = [[x_{i+1}^j, e_1], e_2] - [[x_{i+1}^j, e_2], e_1] = -[(i+1)(m-i)x_i^j, e_2] -$$
$$- \sum_{k=1}^{s+1} \sum_{p=0}^{m} a_{i+1,j,p}^k [x_p^k, e_1] == -(i+1)(m-i) \sum_{k=1}^{s+1} a_{0,j,0}^k x_i^k + \sum_{k=1}^{s+1} \sum_{p=1}^{m} a_{i+1,j,p}^k p(m+1-p) x_{p-1}^k,$$

we conclude that

$$a_{i+1,j,i+1}^k = a_{0,j,0}^k, \quad a_{i+1,j,p}^k = 0, \quad p \neq i+1, \quad 1 \leq p \leq m, \quad 1 \leq k \leq s+1.$$

Hence,

$$[x_{i+1}^j, e_2] = \sum_{k=1}^{s+1} a_{i+1,j,0}^k x_0^k + \sum_{k=1}^{s+1} a_{0,j,0}^k x_{i+1}^k.$$

The following equalities

$$0 = [x_{i+1}^j, [h_1, e_2]] = [[x_{i+1}^j, h_1], e_2] - [[x_{i+1}^j, e_2], h_1] = (m-2i-2)[x_{i+1}^j, e_2] -$$
$$-[\sum_{k=1}^{s+1} a_{i+1,j,0}^k x_0^k + \sum_{k=1}^{s+1} a_{0,j,0}^k x_{i+1}^k, h_1] = (m-2i-2)(\sum_{k=1}^{s+1} a_{i+1,j,0}^k x_0^k + \sum_{k=1}^{s+1} a_{0,j,0}^k x_{i+1}^k) -$$
$$-m \sum_{k=1}^{s+1} a_{i+1,j,0}^k x_0^k - (m-2i-2) \sum_{k=1}^{s+1} a_{0,j,0}^k x_{i+1}^k = -2(i+1) \sum_{k=1}^{s+1} a_{i+1,j,0}^k x_0^k,$$

complete the proof of Equality 4.1.

Putting $a_j^k = a_{0,j,0}^k$, we have $[x_i^j, e_2] = \sum\limits_{k=1}^{s+1} a_j^k x_i^k,\ 1 \leq j \leq s+1,\ 0 \leq i \leq m$.

Applying the Leibniz identity for the triples of elements:

$$\{x_0^j, e_1, f_2\},\ \{x_1^j, e_1, f_2\},\ \{x_1^j, h_1, f_2\} \quad 1 \leq j \leq s+1$$

we get

$$b_{0,j,p}^k = 0,\ 1 \leq p \leq m, \quad b_{1,j,1}^k = b_{0,j,0}^k, \quad b_{1,j,p}^k = 0,\ 2 \leq p \leq m, \quad b_{1,j,0}^k = 0, \quad 1 \leq k \leq s+1.$$

Therefore, we obtain

$$[x_0^j, f_2] = \sum_{k=1}^{s+1} b_{0,j,0}^k x_0^k, \quad [x_1^j, f_2] = \sum_{k=1}^{s+1} b_{0,j,0}^k x_1^k,\ 1 \leq j \leq s+1.$$

Applying the induction and the following chain of equalities

$$0 = [x_{i+1}^j, [e_1, f_2]] = [[x_{i+1}^j, e_1], f_2] - [[x_{i+1}^j, f_2], e_1] = -[(i+1)(m-i)x_i^j, f_2] -$$
$$-\sum_{k=1}^{s+1}\sum_{p=0}^{m} b_{i+1,j,p}^k [x_p^k, e_1] = -(i+1)(m-i)\sum_{k=1}^{s+1} b_{0,j,0}^k x_i^k + \sum_{k=1}^{s+1}\sum_{p=1}^{m} b_{i+1,j,p}^k p(m+1-p)x_{p-1}^k,$$
$$0 = [x_{i+1}^j, [h_1, f_2]] = [[x_{i+1}^j, h_1], f_2] - [[x_{i+1}^j, f_2], h_1] = (m-2i-2)[x_{i+1}^j, f_2] -$$
$$-[\sum_{k=1}^{s+1} b_{i+1,j,0}^k x_0^k + \sum_{k=1}^{s+1} b_{0,j,0}^k x_{i+1}^k, h_1] = (m-2i-2)(\sum_{k=1}^{s+1} b_{i+1,j,0}^k x_0^k + \sum_{k=1}^{s+1} b_{0,j,0}^k x_{i+1}^k) -$$
$$-m\sum_{k=1}^{s+1} b_{i+1,j,0}^k x_0^k - (m-2i-2)\sum_{k=1}^{s+1} b_{0,j,0}^k x_{i+1}^k = -2(i+1)\sum_{k=1}^{s+1} b_{i+1,j,0}^k x_0^k.$$

we derive the equality

$$[x_i^j, f_2] = \sum_{k=1}^{s+1} b_{0,j,0}^k x_i^k,\ 0 \leq i \leq m,\ 1 \leq j \leq s+1.$$

Setting $b_j^k = b_{0,j,0}^k$, we obtain $[x_i^j, f_2] = \sum_{k=1}^{s+1} b_j^k x_i^k,\ 0 \leq i \leq m,\ 1 \leq j \leq s+1.$

Analogously, one can prove the equality $[x_i^j, h_2] = \sum_{k=1}^{s+1} c_j^k x_i^k$ with $1 \leq j \leq s+1$. □

Now we shall describe the Leibniz algebras such that $L/I \cong sl_2^1 \oplus sl_2^2$ and $I = I_{1,1} \oplus I_{1,2}$, where $I_{1,1}, I_{1,1}$ are the irreducible sl_2^1-modules. Without loss of generality we can suppose

$$\begin{array}{ll} [x_k^j, h_1] = (m-2k)x_k^j, & 0 \leq k \leq m, \\ [x_k^j, f_1] = x_{k+1}^j, & 0 \leq k \leq m-1, \\ [x_k^j, e_1] = -k(m+1-k)x_{k-1}^j, & 1 \leq k \leq m. \end{array}$$

for $j = 1, 2$.

Thanks to the Proposition 4.1, one can assume

$$\begin{array}{ll} [x_i^1, e_2] = a_1 x_i^1 + a_2 x_i^2, & [x_i^2, e_2] = a_3 x_i^1 + a_4 x_i^2, \\ [x_i^1, f_2] = b_1 x_i^1 + b_2 x_i^2, & [x_i^2, f_2] = b_3 x_i^1 + b_4 x_i^2, \\ [x_i^1, h_2] = c_1 x_i^1 + c_2 x_i^2, & [x_i^2, h_2] = c_3 x_i^1 + c_4 x_i^2, \end{array}$$

where $0 \leq i \leq m$.

From the following chains of the equalities obtained applying the Leibniz identity

$$2(a_1 x_0^1 + a_2 x_0^2) = 2[x_0^1, e_2] = [x_0^1, [e_2, h_2]] = (a_2 c_3 - c_2 a_3)x_0^1 + (a_1 c_2 + a_2 c_4 - c_1 a_2 - c_2 a_4)x_0^2,$$
$$2(a_3 x_0^1 + a_4 x_0^2) = 2[x_0^2, e_2] = [x_0^2, [e_2, h_2]] = (a_3 c_1 + a_4 c_3 - c_3 a_1 - c_4 a_3)x_0^1 + (a_3 c_2 - a_2 c_3)x_0^2,$$
$$-2(b_1 x_0^1 + b_2 x_0^2) = -2[x_0^1, f_2] = [x_0^1, [f_2, h_2]] = (b_2 c_3 - c_2 b_3)x_0^1 + (b_1 c_2 + b_2 c_4 - c_1 b_2 - c_2 b_4)x_0^2,$$
$$-2(b_3 x_0^1 + b_4 x_0^2) = 2[x_0^2, e_2] = [x_0^2, [f_2, h_2]] = (b_3 c_1 + b_4 c_3 - c_3 b_1 - c_4 b_3)x_0^1 + (b_3 c_2 - b_2 c_3)x_0^2,$$
$$-c_1 x_1^1 - c_2 x_1^2 = -[x_1^1, h_2] = [x_1^1, [f_2, e_2]] = (a_3 b_2 - a_2 b_3)x_1^1 + 2(a_2 b_1 - a_1 b_2)x_1^2,$$
$$-c_3 x_1^1 - c_4 x_1^2 = -[x_1^2, h_2] = [x_1^2, [f_2, e_2]] = 2(a_1 b_3 - a_3 b_1)x_1^1 + (a_2 b_3 - a_3 b_2)x_1^2.$$

we derive:

$$
(4.2)\qquad \begin{cases}
2a_1 = a_2c_3 - a_3c_2,\\
2a_2 = a_1c_2 + a_2c_4 - c_1a_2 - c_2a_4,\\
2a_3 = a_3c_1 + a_4c_3 - c_3a_1 - c_4a_3,\\
2a_4 = a_3c_2 - a_2c_3,\\
-2b_1 = b_2c_3 - c_2b_3,\\
-2b_2 = b_1c_2 + b_2c_4 - c_1b_2 - c_2b_4,\\
-2b_3 = b_3c_1 + b_4c_3 - c_3b_1 - c_4b_3,\\
-2b_4 = b_3c_2 - b_2c_3,\\
c_1 = a_2b_3 - a_3b_2,\\
c_2 = 2(a_1b_2 - a_2b_1),\\
c_3 = 2(a_3b_1 - a_1b_3),\\
c_4 = a_3b_2 - a_2b_3.
\end{cases}
$$

It is easy to see that $a_4 = -a_1, b_4 = -b_1$ and $c_4 = -c_1$. By substituting the above relations in the restrictions (4.2), we will have:

$$
(4.3)\qquad \begin{cases}
a_1 = 2a_2a_3b_1 - a_1a_2b_3 - a_1a_3b_2,\\
a_2 = 2a_1^2b_2 - 2a_1a_2b_1 - a_2^2b_3 + a_2a_3b_2,\\
a_3 = a_2a_3b_3 - a_3^2b_2 - 2a_1a_3b_1 + 2a_1^2b_3,\\
b_1 = 2a_1b_2b_3 - a_2b_1b_3 - a_3b_1b_2,\\
b_2 = 2a_2b_1^2 - 2a_1b_1b_2 + a_2b_2b_3 - a_3b_2^2,\\
b_3 = a_3b_2b_3 - a_2b_3^2 + 2b_1^2a_3 - 2a_1b_1b_3.
\end{cases}
$$

Thus, we obtain the following products:

$$
(4.4)\qquad \begin{array}{ll}
[x_i^1, e_2] = a_1x_i^1 + a_2x_i^2, & [x_i^1, f_2] = b_1x_i^1 + b_2x_i^2,\\
[x_i^2, e_2] = a_3x_i^1 - a_1x_i^2, & [x_i^2, f_2] = b_3x_i^1 - b_1x_i^2,\\
[x_i^1, h_2] = (a_2b_3 - a_3b_2)x_i^1 + 2(a_1b_2 - a_2b_1)x_i^2, & \\
[x_i^2, h_2] = 2(a_3b_1 - a_1b_3)x_i^1 - (a_2b_3 - a_3b_2)x_i^2, &
\end{array}
$$

where the structure constants a_1, a_2, a_3 and b_1, b_2, b_3 satisfy the relations (4.3).

We present the classification of Leibniz algebras satisfying the following conditions below:

(i) $L/I \cong sl_2^1 \oplus sl_2^2$;
(ii) $I = I_{1,1} \oplus I_{1,2}$ such that $I_{1,1}, I_{1,2}$ are the irreducible sl_2^1-modules and $dim I_{1,1} = dim I_{1,2}$;
(iii) $I = I_{2,1} \oplus I_{2,2} \oplus ... \oplus I_{2,m+1}$ such that $I_{2,k}$ are the irreducible sl_2^2-modules with $1 \le k \le m+1$.

Theorem 4.2. *An arbitrary Leibniz algebra satisfying the conditions (i)-(iii) is isomorphic to the following algebra:*

$$
\begin{cases}
[e_i, h_i] = -[h_i, e_i] = 2e_i, & \\
[e_i, f_i] = -[f_i, e_i] = h_i, & \\
[h_i, f_i] = -[f_i, h_i] = 2f_i, & \\
[x_k^i, h_1] = (m-2k)x_k^i, & 0 \le k \le m,\\
[x_k^i, f_1] = x_{k+1}^i, & 0 \le k \le m-1,\\
[x_k^i, e_1] = -k(m+1-k)x_{k-1}^i, & 1 \le k \le m,\\
[x_j^1, e_2] = [x_j^2, h_2] = x_j^2, & \\
[x_j^1, h_2] = [x_j^2, f_2] = -x_j^1, &
\end{cases}
$$

with $1 \le i \le 2$ *and* $0 \le j \le m$.

Proof. We set $dim I_{1,1} = dim I_{1,2} = m+1$. Then, according to Theorem 2.8, we obtain $dim I_{2,k} = 2$ for $1 \le k \le m+1$.

Let $\{x_0^1, x_1^1, ..., x_m^1\}$, $\{x_0^2, x_1^2, ..., x_m^2\}$ and $\{y_0^k, y_1^k\}$ be the bases of I_{11}, I_{12} and $I_{2,k}$, $1 \le k \le m+1$, respectively. We set

$$
y_0^i = \sum_{k=1}^{2}\sum_{s=0}^{m} \alpha_{is}^k x_s^k, \quad y_1^i = \sum_{k=1}^{2}\sum_{s=0}^{m} \beta_{is}^k x_s^k,
$$

with $1 \leq i \leq m+1$.

Taking into account the products (4.4) for $1 \leq i \leq m+1$ we consider the equalities

$$0 = [y_1^i, f_2] = [\sum_{s=0}^{m} \beta_{i,s}^1 x_s^1 + \sum_{s=0}^{m} \beta_{i,s}^2 x_s^2, f_2] = \sum_{s=0}^{m} \beta_{i,s}^1 (b_1 x_s^1 + b_2 x_s^2) +$$
$$+ \sum_{s=0}^{m} \beta_{i,s}^2 (b_3 x_s^1 - b_1 x_s^2) = \sum_{s=0}^{m} (\beta_{i,s}^1 b_1 + \beta_{i,s}^2 b_3) x_s^1 + \sum_{s=0}^{m} (\beta_{i,s}^1 b_2 - \beta_{i,s}^2 b_1) x_s^2.$$

Therefore,

$$\left\{ \begin{array}{l} \beta_{i,s}^1 b_1 + \beta_{i,s}^2 b_3 = 0, \\ \beta_{i,s}^1 b_2 - \beta_{i,s}^2 b_1 = 0, \end{array} \right. \tag{4.5}$$

with $1 \leq i \leq m+1$ and $0 \leq s \leq m$.

If $b_1^2 + b_2 b_3 \neq 0$, then the system of equations (4.5) has only the trivial solution, which is a contradiction. Hence, $b_1^2 + b_2 b_3 = 0$.

Similarly, from

$$0 = [y_0^i, e_2] = \sum_{s=0}^{m} (\alpha_{i,s}^1 a_1 + \alpha_{i,s}^2 a_3) x_s^1 + \sum_{s=0}^{m} (\alpha_{i,s}^1 a_2 - \alpha_{i,s}^2 a_1) x_s^2$$

we derive $a_1^2 + a_2 a_3 = 0$.

Thus, we have $a_1 = i\sqrt{a_2 a_3}$ and $b_1 = i\sqrt{b_2 b_3}$.

Substituting the relations $a_1 = i\sqrt{a_2 a_3},\ b_1 = i\sqrt{b_2 b_3}$ by the restrictions (4.3) we get

$$\left\{ \begin{array}{l} \sqrt{a_2 a_3}(1 + a_3 b_2 + a_2 b_3 - 2\sqrt{a_2 a_3 b_2 b_3}) = 0, \\ a_2(1 + a_3 b_2 + a_2 b_3 - 2\sqrt{a_2 a_3 b_2 b_3}) = 0, \\ a_3(1 + a_3 b_2 + a_2 b_3 - 2\sqrt{a_2 a_3 b_2 b_3}) = 0, \\ \sqrt{b_2 b_3}(1 + a_3 b_2 + a_2 b_3 - 2\sqrt{a_2 a_3 b_2 b_3}) = 0, \\ b_2(1 + a_3 b_2 + a_2 b_3 - 2\sqrt{a_2 a_3 b_2 b_3}) = 0, \\ b_3(1 + a_3 b_2 + a_2 b_3 - 2\sqrt{a_2 a_3 b_2 b_3}) = 0. \end{array} \right.$$

Consequently, $1 + a_3 b_2 + a_2 b_3 - 2\sqrt{a_2 a_3 b_2 b_3} = 0$ (otherwise $[I_{1,1}, sl_2^2] = [I_{1,2}, sl_2^2] = 0$ which is a contradiction with the assumption of the theorem).

Let us summarize the obtained products:

$$\begin{array}{ll} [x_i^1, e_2] = a_1 x_i^1 + a_2 x_i^2, & [x_i^1, f_2] = b_1 x_i^1 + b_2 x_i^2, \\ [x_i^2, e_2] = a_3 x_i^1 - a_1 x_i^2, & [x_i^2, f_2] = b_3 x_i^1 - b_1 x_i^2, \\ [x_i^1, h_2] = (a_2 b_3 - a_3 b_2) x_i^1 + 2(a_1 b_2 - a_2 b_1) x_i^2, & \\ [x_i^2, h_2] = 2(a_3 b_1 - a_1 b_3) x_i^1 - (a_2 b_3 - a_3 b_2) x_i^2, & \end{array} \tag{4.6}$$

with $0 \leq i \leq m$ and the relations $a_1^2 + a_2 a_3 = b_1^2 + b_2 b_3 = 1 + a_3 b_2 + a_2 b_3 - 2\sqrt{a_2 a_3 b_2 b_3} = 0$.

Taking the following basis transformation:

$$x_i^{1'} = A x_i^1 + B x_i^2,\ x_i^{2'} = (A a_1 + B a_3) x_i^1 + (A a_2 - B a_1) x_i^2,\ 0 \leq i \leq m$$

we can assume that the products (4.6) have the following form:

$$\begin{array}{ll} [x_i^1, e_2] = x_i^2, & [x_i^2, e_2] = 0, \\ [x_i^1, f_2] = b_1 x_i^1 + b_1^2 x_i^2, & [x_i^2, f_2] = -x_i^1 - b_1 x_i^2, \\ [x_i^1, h_2] = -x_i^1 - 2b_1 x_i^2, & [x_i^2, h_2] = x_i^2. \end{array}$$

Applying the change of basis as follows

$$x_i^{1'} = x_i^1 + b_1 x_i^2,\ x_i^{2'} = x_i^2,\quad 0 \leq i \leq m,$$

we complete the proof of theorem. □

The following theorem establishes that condition (iii) can be omitted because of if conditions (i)-(ii) are true, then condition (iii) is always executable.

Theorem 4.3. *There is no Leibniz algebra satisfying the conditions (i)-(ii), which does not satisfy condition (iii).*

Proof. Let a Leibniz algebra satisfying conditions $(i)-(ii)-(iii)$. There exists r with $1 \leq r \leq m+1$ such that $I_{2,i}$ for $1 \leq i \leq r$ are the reducible sl_2^2-modules. Then from Theorem 2.5, we conclude that $I_{2,i}$ are the fully reducible modules over sl_2^2 with $1 \leq i \leq r$. Therefore, $I_{2,i} =< y_0^i > \oplus < y_1^i >$ where $< y_0^i >$, $< y_1^i >$ are the one-dimensional trivial sl_2^2-modules, that is,

$$[y_0^j, e_2] = [y_1^j, e_2] = [y_0^j, f_2] = [y_1^j, f_2] = [y_0^j, h_2] = [y_1^j, h_2] = 0.$$

Similar to the proof of Theorem 4.2 we obtain

$$a_1 = i\sqrt{a_2a_3}, \quad b_1 = i\sqrt{b_2b_3}, \quad 1 + a_3b_2 + a_2b_3 - 2\sqrt{a_2a_3b_2b_3} = 0. \tag{4.7}$$

Consider the chain of the equalities

$$0 = [y_1^1, h_2] = \sum_{s=0}^{m} \beta_{i,s}^1 [x_s^1, h_2] + \sum_{s=0}^{m} \beta_{i,s}^2 [x_s^2, h_2] = \sum_{s=0}^{m} ((a_2b_3 - a_3b_2)\beta_{i,s}^1 +$$
$$+2(a_3b_1 - a_1b_3)\beta_{i,s}^2)x_s^1 + \sum_{s=0}^{m} (2(a_1b_2 - a_2b_1)\beta_{i,s}^1 + (a_3b_2 - a_2b_3)\beta_{i,s}^2)x_s^2.$$

Then we have

$$\begin{cases} (a_2b_3 - a_3b_2)\beta_{i,s}^1 + 2(a_3b_1 - a_1b_3)\beta_{i,s}^2 = 0, \\ 2(a_1b_2 - a_2b_1)\beta_{i,s}^1 + (a_3b_2 - a_2b_3)\beta_{i,s}^2 = 0, \end{cases}$$

with $1 \leq i \leq r$ and $0 \leq s \leq m$.

Taking into account the relations (4.7), we conclude that the determinant of the above system of the equations is equal to 1. Indeed,

$$\begin{aligned}
&(a_2b_3 - a_3b_2)(a_3b_2 - a_2b_3) - 4(a_3b_1 - a_1b_3)(a_1b_2 - a_2b_1) = \\
&= (a_2b_3 - a_3b_2)(a_3b_2 - a_2b_3) - 4(ia_3\sqrt{b_2b_3} - ib_3\sqrt{a_2a_3})(ib_2\sqrt{a_2a_3} - ia_2\sqrt{b_2b_3}) = \\
&= (a_2b_3 - a_3b_2)(a_3b_2 - a_2b_3) + 4(a_3\sqrt{b_2b_3} - b_3\sqrt{a_2a_3})(b_2\sqrt{a_2a_3} - a_2\sqrt{b_2b_3}) = \\
&= a_2a_3b_2b_3 - a_2^2b_3^2 - a_3^2b_2^2 + a_2a_3b_2b_3 + 4a_3b_2\sqrt{a_2a_3b_2b_3} - 4a_2a_3b_2b_3 - 4a_2a_3b_2b_3 + 4a_2b_3\sqrt{a_2a_3b_2b_3} = \\
&= -(a_2^2b_3^2 + a_3^2b_2^2 + 6a_2a_3b_2b_3 - 4a_3b_2\sqrt{a_2a_3b_2b_3} - 4a_2b_3\sqrt{a_2a_3b_2b_3}) = \\
&= -(a_3b_2 + a_2b_3 - 2\sqrt{a_2a_3b_2b_3})^2 = -1.
\end{aligned}$$

Consequently, $\beta_{i,s}^1 = \beta_{i,s}^2 = 0$ for $1 \leq i \leq r$ and $0 \leq s \leq m$ and we obtain $y_1^i = 0$. Thus, we get a contradiction. We complete the proof of the theorem. □

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