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An algorithm for the classification of 3-dimensional complex Leibniz algebras

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ABSTRACT

We propose an algorithm using Gröbner bases that decides in terms of the existence of a non-singular matrix P if two Leibniz algebra structures over a finite dimensional $\mathbb{C}$-vector space are representative of the same isomorphism class.

We apply this algorithm in order to obtain a reviewed classification of the 3-dimensional Leibniz algebras given by Ayupov and Omirov. The algorithm has been implemented in a Mathematica notebook.

1. Introduction

A classical problem in Lie algebras theory is to know how many different (up to isomorphisms) finite-dimensional Lie algebras exist for each dimension [12,13].

The classical methods to obtain the classifications essentially solve the system of equations given by the bracket laws, that is, for a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ with basis $\{a_1, \ldots, a_n\}$, the bracket is completely determined by the scalars $c_{ij}^k \in \mathbb{K}$ such that

$$[a_i, a_j] = \sum_{k=1}^{n} c_{ij}^k a_k. \tag{1.1}$$

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Therefore, the Lie algebra structure is determined by means of the computation of the structure constants c_{ij}^k, which satisfy the following equations:

$$c_{ii}^k = 0 = c_{ij}^k + c_{ji}^k \tag{1.2}$$

$$\sum_{l=1}^{n} \left(c_{il}^m c_{jk}^l + c_{jl}^m c_{ki}^l + c_{kl}^m c_{ij}^l \right) = 0 \tag{1.3}$$

for all $1 \leqslant i, j, k, m \leqslant n$. The solutions of the system derived from (1.2) and (1.3) can be computed by different methods, including Gröbner bases techniques [9–11], nevertheless the classification must be presented by means of isomorphism classes.

Leibniz algebras, introduced by Loday [14] when he studied periodicity phenomenons in algebraic K-theory, are $\mathbb{K}$-vector spaces $\mathfrak{g}$ endowed with a bilinear operation $[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \quad \text{for all } x, y, z \in \mathfrak{g}.$$

They are a non-skew-symmetric version of Lie algebras. The relationship between Lie algebras and associative algebras can be translated into an analogous relationship between Leibniz algebras and associative dialgebras (see [15]), which are a generalization of associative algebras with two bilinear operations and provide a natural setting for Leibniz algebras. The main motivation to study Leibniz algebras is the existence of a (co)homology theory for Leibniz algebras which restricted to Lie algebras provides new invariants.

The classification of Leibniz algebras in low dimensions is obtained for specific classes of Leibniz algebras (solvable, nilpotent, filiform, etc.) [1–5,8]. The classification problem is very difficult to handle because the space of solutions of the system of equations given by the structure constants (1.1) and the equations provided by the relations

$$\sum_{l=1}^{n} \left(c_{jk}^l c_{il}^r - c_{ij}^l c_{lk}^r + c_{ik}^l c_{lj}^r \right) = 0, \quad 1 \leqslant i, j, k, r \leqslant n, \tag{1.4}$$

becomes very hard to compute, especially for dimensions $n \geqslant 3$ since it is necessary to solve a system of n^4 equations in n^3 unknowns, causing frequent errors in the literature.

In [6] we have developed an algorithm for testing the Leibniz algebra structure using techniques of Gröbner bases. We have applied this test to the classification of 3-dimensional complex Leibniz algebras showed in [4] and we have detected that the isomorphism class whose representative element is the algebra with basis $\{x, y, z\}$ and bracket given by $[x, y] = \alpha x$; $[x, z] = \alpha x$, $[z, y] = x$ and 0 otherwise, does not correspond with a Leibniz algebra structure.

Our goal in the present paper is to obtain a complete classification of the 3-dimensional Leibniz algebras over the field $\mathbb{C}$. To do this, first of all we compute all the solutions of the system of equations obtained from (1.4) considering the decomposition $\mathfrak{g} = \mathfrak{g}^{\mathrm{ann}} \oplus \mathfrak{g}_{\mathrm{Lie}}$ (see Definition 2.2 below), the dimension of $\mathfrak{g}^{\mathrm{ann}}$, the Leibniz identity and Gröbner bases computations.

To reach our goal, in Section 2 we present an algorithm using Gröbner bases that compares two solutions and decides whether there exists an isomorphism between them or not in terms of the existence of a non-singular matrix P, that is, given two different structures $(\mathfrak{g}, [-,-]_1)$ and $(\mathfrak{g}, [-,-]_2)$ which are solutions of the system (1.4), we must verify if they are isomorphic. For that, it is necessary to check the existence of a non-singular matrix P satisfying the equation

$$P \cdot [a_i, a_j]_2 = [P \cdot a_i, P \cdot a_j]_1 \tag{1.5}$$

for all $i, j \in \{1, 2, 3\}$. In this step we use computational methods based on Gröbner bases techniques to check the existence (and also to perform the construction) of a matrix P. With the computations carried out with this algorithm, we present in Section 3 a reviewed classification of 3-dimensional

Leibniz algebras given in [4], obtaining the following conclusions: the class AO5 does not correspond to a Leibniz algebra, the classes AO9 and AO10 of Table 2 are isomorphic and the existence of a new isomorphism class 2(f) that is not contained in that table. In Section 4 we show some examples of these computations using a Mathematica notebook in which is implemented the algorithm.

This technique can be extended to large enough dimensions.

2. On Leibniz algebras

Definition 2.1. A Leibniz algebra $\mathfrak{g}$ is a $\mathbb{K}$-vector space equipped with a bilinear map $[-,-]\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \text{ for all } x,y,z \in \mathfrak{g}. \tag{2.1}$$

When the bracket satisfies $[x,x] = 0$ for all $x \in \mathfrak{g}$, then the Leibniz identity (2.1) becomes the Jacobi identity, so a Leibniz algebra is a Lie algebra. Hence, there is a canonical inclusion functor from the category **Lie** of Lie algebras to the category **Leib** of Leibniz algebras.

Definition 2.2. The inclusion functor inc: **Lie** $\to$ **Leib** has a left adjoint, the Liezation functor $(\ )_{\mathrm{Lie}}\colon$ **Leib** $\to$ **Lie** which assigns to a Leibniz algebra $\mathfrak{g}$ the Lie algebra $\mathfrak{g}_{\mathrm{Lie}} = \mathfrak{g}/\mathfrak{g}^{\mathrm{ann}}$, where $\mathfrak{g}^{\mathrm{ann}} = \mathrm{ideal}\langle\{[x,x], x \in \mathfrak{g}\}\rangle$.

Example 2.3.

1. Lie algebras.
2. Let A be a K-associative algebra equipped with a K-linear map $D\colon A \to A$ satisfying

$$D(a(Db)) = DaDb = D((Da)b), \text{ for all } a,b \in A. \tag{2.2}$$

   Then A with the bracket $[a,b] = a(Db) - (Db)a$ is a Leibniz algebra.
   If $D = \mathrm{Id}$, we obtain the Lie algebra structure associated to an associative algebra. If D is an idempotent algebra endomorphism ($D^2 = D$) or D is a derivation of square zero ($D^2 = 0$), then D satisfies equation (2.2) and the bracket gives rise to a structure of non-Lie Leibniz algebra.
3. Let $\mathfrak{g}$ be a differential Lie algebra, then $(\mathfrak{g},[-,-]_d)$ with $[x,y]_d :=[x,dy]$ is a non-Lie Leibniz algebra.

A homomorphism of Leibniz algebras is a $\mathbb{K}$-linear map $\Phi\colon \mathfrak{g} \to \mathfrak{h}$ such that $\Phi([x,y]_{\mathfrak{g}}) = [\Phi(x),\Phi(y)]_{\mathfrak{h}}$, for all $x,y \in \mathfrak{g}$. In case of finite dimensional Leibniz algebras $\mathfrak{g}$ and $\mathfrak{h}$, the homomorphism Φ can be represented by means of a matrix P.

Proposition 2.4. *Consider two Leibniz algebras $(\mathfrak{g},[-,-]_1)$ and $(\mathfrak{g},[-,-]_2)$ with the same underlying $\mathbb{K}$-vector space, the same basis $\{a_1,\dots,a_n\}$ and different structures given by the brackets $[-,-]_1$ and $[-,-]_2$. There exists a non-singular matrix P such that the change of variables given by P provides the following commutative diagram:*

$$\begin{tikzcd}
\mathfrak{g}\times\mathfrak{g} \arrow[r, "{[-,-]_1}"] & \mathfrak{g} \arrow[d, "P^{-1}"] \\
\mathfrak{g}\times\mathfrak{g} \arrow[u, "P\times P"] \arrow[r, "{[-,-]_2}"] & \mathfrak{g};
\end{tikzcd}$$

that is, the following identity holds

$$P\cdot[a_i,a_j]_2 = [P\cdot a_i, P\cdot a_j]_1,$$

if and only if the Leibniz algebras are isomorphic.

Proposition 2.5 (Consistency algorithm [7]). *If we have polynomials $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$ then $f_1, \ldots, f_s$ have no common zero in $\mathbb{C}^n$ if and only if the Gröbner basis of the ideal generated by $f_1, \ldots, f_s$ is* $\{1\}$.

So, if we put together this two facts we have the following algorithm:

Algorithm 2.6 (*Leibniz algebra isomorphism test)*.

Input: Two Leibniz algebras $(\mathfrak{g}, [-,-]_1)$ and $(\mathfrak{g}, [-,-]_2)$ with $\dim_{\mathbb{C}}(\mathfrak{g}) = n$ and basis $\{a_1, \ldots, a_n\}$.
Output: True if $(\mathfrak{g}, [-,-]_1)$ is isomorphic to $(\mathfrak{g}, [-,-]_2)$ and False in other case.

1. Compute the following system of equations

$$P \cdot [a_i, a_j]_2 - [P \cdot a_i, P \cdot a_j]_1 = 0; \quad i, j \in \{1, \ldots, n\}.$$

2. To ensure that P is going to be non-singular, add the following relation with a new variable Y:

$$\det[P] \cdot Y - 1 = 0.$$

3. Compute a Gröbner basis G of the ideal $\langle \{P \cdot [a_i, a_j]_2 - [P \cdot a_i, P \cdot a_j]_1\}_{i,j \in \{1,\ldots,n\}} \cup \{\det[P] \cdot Y - 1\} \rangle$ in the polynomial ring $\mathbb{C}[p_{ij}, Y]$, where $P = (p_{ij})$.
4. $G = \{1\}$?

4.1. Yes.
 Return False.

4.2. No.
 Output: True; Return G.

Remark 2.7. If the output of the algorithm is True, then it provides the equations of an algebraic variety whose points are all the possible values for P; hence we can obtain a matrix P satisfying (1.5).

3. Application

We devote the present section, where we apply our technique together with the test developed in [6], to obtaining the classification of the 3-dimensional complex Leibniz algebras and to compare it with the classification given in [4].

Lemma 3.1. *If $\mathfrak{g}$ is a non-trivial Leibniz algebra then $\mathfrak{g} \neq \mathfrak{g}^{\mathrm{ann}}$.*

Proof. Suppose that $\mathfrak{g} = \mathfrak{g}^{\mathrm{ann}}$. If $y, z \in \mathfrak{g} = \mathfrak{g}^{\mathrm{ann}} = \mathrm{ideal}\langle \{[x,x], x \in \mathfrak{g}\} \rangle$, then $[y,z] = 0$ for all $y, z \in \mathfrak{g}$. Since any element of $\mathfrak{g}^{\mathrm{ann}}$ is a linear combination of elements of the form $[x,x]$, we have $\mathfrak{g} = 0$, which contradicts the hypothesis. $\square$

We will use $\mathfrak{g}^{\mathrm{ann}}$ as an invariant of classification and taking into account that

$$0 \longrightarrow \mathfrak{g}^{\mathrm{ann}} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}_{\mathrm{Lie}} \longrightarrow 0 \tag{3.1}$$

is a split (as $\mathbb{K}$-vector spaces) short exact sequence, we obtain that $\mathfrak{g} = \mathfrak{g}^{\mathrm{ann}} \oplus \mathfrak{g}_{\mathrm{Lie}}$ as $\mathbb{K}$-vector spaces.

In the sequel, let $\mathfrak{g}$ be a non-trivial 3-dimensional Leibniz algebra over $\mathbb{C}$, therefore $\dim \mathfrak{g}^{\mathrm{ann}} < 3$ by Lemma 3.1, and $\{a_1, a_2, a_3\}$ a $\mathbb{C}$-basis of $\mathfrak{g}$ such that $a_i \in \mathfrak{g}^{\mathrm{ann}}$ or $a_i \in \mathfrak{g}_{\mathrm{Lie}}$, $i \in \{1, 2, 3\}$.

$\mathfrak{g}_{\mathrm{Lie}}$ is a Lie algebra, so we can suppose that we have chosen a $\mathbb{C}$-basis of $\mathfrak{g}$, which verifies that the restriction of the bracket to $\mathfrak{g}_{\mathrm{Lie}}$ is in a canonical form.

So, if $\mathfrak{g}^{\mathrm{ann}} = \langle\{a_1\}\rangle$ the bracket will be of the form

$$\begin{pmatrix} \theta & \alpha_1 \cdot a_1 & \alpha_2 \cdot a_1 \\ \theta & \alpha_3 \cdot a_1 & \alpha_4 \cdot a_1 + x \cdot a_2 \\ \theta & \alpha_5 \cdot a_1 - x \cdot a_2 & \alpha_6 \cdot a_1 \end{pmatrix}, \quad x \in \{0, 1\}, \tag{3.2}$$

where $\theta = (0, 0, 0)$, $a_1 = (1, 0, 0)$, $a_2 = (0, 1, 0)$, $a_3 = (0, 0, 1)$ and $[a_i, a_j]$ is the entry of the matrix (3.2) placed at row i column j.

If $\mathfrak{g}^{\mathrm{ann}} = \langle\{a_1, a_2\}\rangle$ we will have

$$\begin{pmatrix} \theta & \theta & \alpha_1 \cdot a_1 + \alpha_2 \cdot a_2 \\ \theta & \theta & \alpha_3 \cdot a_1 + \alpha_4 \cdot a_2 \\ \theta & \theta & \alpha_5 \cdot a_1 + \alpha_6 \cdot a_2 \end{pmatrix}. \tag{3.3}$$

The case $\dim \mathfrak{g}^{\mathrm{ann}} = 0$ is not considered because it implies that $\mathfrak{g}$ is a Lie algebra and its classification is well known [13].

The following step is to apply Leibniz identity to each case, and thus we will obtain a system of equations.

If $\mathfrak{g}^{\mathrm{ann}} = \langle\{a_1\}\rangle$ and $x = 0$ the system is:

$$\left.\begin{aligned} \alpha_2 \cdot (\alpha_3 \cdot \alpha_6 - \alpha_4 \cdot \alpha_5) = 0 \\ -\alpha_2 \cdot \alpha_5 + \alpha_1 \cdot \alpha_6 = 0 \\ -\alpha_2 \cdot \alpha_3 + \alpha_1 \cdot \alpha_4 = 0 \end{aligned}\right\}.$$

If $\mathfrak{g}^{\mathrm{ann}} = \langle\{a_1\}\rangle$ and $x = 1$ the system is:

$$\left.\begin{aligned} \alpha_1 = 0 \\ \alpha_3 \cdot (\alpha_4 - \alpha_5) = 0 \\ -\alpha_4 - \alpha_5 + \alpha_2 \cdot \alpha_5 = 0 \\ \alpha_3 \cdot (2 - \alpha_2) = 0 \end{aligned}\right\}.$$

And finally, if $\mathfrak{g}^{\mathrm{ann}} = \langle\{a_1, a_2\}\rangle$ then Leibniz identity does not generate any equation except the trivial equation $0 = 0$.

From the discussion of each system of equations we will obtain many Leibniz algebras but these algebras are sometimes isomorphic, then we will apply Algorithm 2.6 to obtain a classification of Leibniz algebras in isomorphism classes. If we work in this way we reach the following classification of 3-dimensional Leibniz algebras. All the non-written brackets are equal to zero.

1. Case 1: $\dim_{\mathbb{C}}(\mathfrak{g}^{\mathrm{ann}}) = 0$ (Lie algebras case).
   (a) $\mathfrak{g}$ abelian ($\mathbb{C}^3$).
   (b) $[a_2, a_3] = a_1$; $[a_3, a_2] = -a_1$ (Heisenberg algebra $\mathcal{H}$).
   (c) $[a_1, a_2] = a_1$; $[a_2, a_1] = -a_1$ ($\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$, where $\mathfrak{r}_2(\mathbb{C})$ is the unique two-dimensional non-abelian Lie algebra over $\mathbb{C}$).
   (d) $[a_1, a_3] = a_1$; $[a_2, a_3] = \alpha \cdot a_2$; $[a_3, a_1] = -a_1$; $[a_3, a_2] = -\alpha \cdot a_2$, $\alpha \in \mathbb{C} - \{0\}$.
   (e) $[a_1, a_3] = a_1 + a_2$; $[a_2, a_3] = a_2$; $[a_3, a_1] = -a_1 - a_2$; $[a_3, a_2] = -a_2$.
   (f) $[a_1, a_2] = a_3$; $[a_1, a_3] = -a_2$; $[a_2, a_1] = -a_3$;
   $[a_2, a_3] = a_1$; $[a_3, a_1] = a_2$; $[a_3, a_2] = -a_1$.

This algebra is $\mathfrak{sl}(2, \mathbb{C})$, all 2×2 matrices of trace 0 over $\mathbb{C}$, where $\Phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$ is given by $\Phi(e) = a_1 + ia_2$, $\Phi(f) = -a_1 + ia_2$, $\Phi(h) = 2ia_3$ and $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a basis of $\mathfrak{sl}(2, \mathbb{C})$.

2. Case 2: $\dim_{\mathbb{C}}(\mathfrak{g}^{\mathrm{ann}}) = 1$ (non-Lie Leibniz algebras).
   (a) $[a_2, a_2] = \gamma \cdot a_1, \gamma \in \mathbb{C}$; $[a_3, a_2] = a_1$, $[a_3, a_3] = a_1$.
   (b) $[a_3, a_3] = a_1$.
   (c) $[a_2, a_2] = a_1$; $[a_3, a_3] = a_1$.
   (d) $[a_1, a_3] = a_1$.
   (e) $[a_1, a_3] = \alpha \cdot a_1, \alpha \in \mathbb{C} - \{0\}$; $[a_2, a_3] = a_2$; $[a_3, a_2] = -a_2$.
   (f) $[a_2, a_3] = a_2$; $[a_3, a_2] = -a_2$; $[a_3, a_3] = a_1$.
   (g) $[a_1, a_3] = 2 \cdot a_1$; $[a_2, a_2] = a_1$; $[a_2, a_3] = a_2$; $[a_3, a_2] = -a_2$; $[a_3, a_3] = a_1$.
3. Case 3: $\dim_{\mathbb{C}}(\mathfrak{g}^{\mathrm{ann}}) = 2$ (non-Lie Leibniz algebras).
   (a) $[a_1, a_3] = \beta \cdot a_1, \beta \in \mathbb{C} - \{0\}$; $[a_2, a_3] = a_2$.
   (b) $[a_1, a_3] = a_1 + a_2$; $[a_2, a_3] = a_2$.
   (c) $[a_1, a_3] = a_2$; $[a_3, a_3] = a_1$.
   (d) $[a_1, a_3] = a_2$; $[a_2, a_3] = a_2$; $[a_3, a_3] = a_1$.

Remark 3.2.

1. If $\alpha_1, \alpha_2 \in \mathbb{C} - \{0\}$ such that $\alpha_1 \neq \alpha_2$, then the corresponding two Lie algebras of the family 1(d) are isomorphic if and only if $\alpha_1 = \frac{1}{\alpha_2}$.
2. If $\gamma_1, \gamma_2 \in \mathbb{C}$ such that $\gamma_1 \neq \gamma_2$, then the corresponding two Leibniz algebras of the family 2(a) are not isomorphic.
3. If $\alpha_1, \alpha_2 \in \mathbb{C} - \{0\}$ such that $\alpha_1 \neq \alpha_2$, then the corresponding two Leibniz algebras of the family 2(e) are not isomorphic.
4. If $\beta_1, \beta_2 \in \mathbb{C} - \{0\}$ such that $\beta_1 \neq \beta_2$, then the corresponding two Leibniz algebras of the family 3(a) are isomorphic if and only if $\beta_1 = \frac{1}{\beta_2}$.
5. If we choose two Leibniz algebras in different families, these algebras are not isomorphic.

We will recall some algebraic invariants of Leibniz algebras in order to check the obtained isomorphic classes.

Definition 3.3. Let $\mathfrak{g}$ be a Leibniz algebra. We call left and right center of $\mathfrak{g}$ to the respective $\mathbb{K}$-vector subspaces

$$Z^l(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [g, x] = 0, \ \forall g \in \mathfrak{g}\},$$
$$Z^r(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, g] = 0, \ \forall g \in \mathfrak{g}\}.$$

We call center of $\mathfrak{g}$ to the $\mathbb{K}$-vector subspace

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, g] = [g, x] = 0, \ \forall g \in \mathfrak{g}\}.$$

Definition 3.4. Let $\mathfrak{g}$ be a Leibniz algebra. We call the lower central series to the following sequence

$$\mathfrak{g}^{\langle 1 \rangle} = \mathfrak{g}, \quad \mathfrak{g}^{\langle n+1 \rangle} = [\mathfrak{g}^{\langle n \rangle}, \mathfrak{g}].$$

The algebra $\mathfrak{g}$ is said right nilpotent if $\mathfrak{g}^{\langle n \rangle} = 0$ for some $n \in \mathbb{N}$.

Table 1
Algebraic invariants.

	$\dim \mathfrak{g}^{\text{ann}}$	$\dim Z(\mathfrak{g})$	$\dim Z^r(\mathfrak{g})$	$\dim Z^l(\mathfrak{g})$	Nilpotent	I_2
2(a)	1	1	1	1	Yes	1
2(b)	1	2	2	2	Yes	
2(c)	1	1	1	1	Yes	2
2(d)	1	1	2	2	Non	
2(e)	1	0	0	1	Non	1
2(f)	1	1	1	1	Non	
2(g)	1	0	0	1	Non	0
	$\dim \mathfrak{g}^{\text{ann}}$	$\dim Z(\mathfrak{g})$	$\dim Z^r(\mathfrak{g})$	$\dim Z^l(\mathfrak{g})$	Nilpotent	I_3
3(a)	2	0	1	2	Non	3
3(b)	2	0	1	2	Non	2
3(c)	2	1	1	2	Yes	
3(d)	2	1	1	2	Non	

Table 2
Ayupov–Omirov's classification.

	$\dim Z(\mathfrak{g})$	Bracket
AO1	2	$[y,z]=\alpha_1 x;\ [x,z]=\alpha_2 x;\ [z,z]=x$
AO2	2	$[z,z]=x;\ [x,z]=y;\ [y,z]=\alpha_1 x+\alpha_2 y$
AO3	2	$[y,z]=\alpha_2 x+y;\ [z,z]=x;\ [x,z]=\alpha_1 x$
AO4	1	$[y,y]=x;\ [z,z]=\beta x;\ [x,y]=\alpha x;$ $[x,z]=\alpha x;\ [z,y]=\beta x;\ [y,z]=x,\ \alpha\neq 0$
AO5	1	$[x,y]=\alpha x;\ [x,z]=\alpha x;\ [z,y]=x$
AO6	1	$[x,z]=\alpha x;\ [z,y]=\beta x+y;\ [y,z]=-\beta(1+\alpha)x-y$
AO7	1	$[y,y]=x;\ [z,z]=\alpha x;\ [x,z]=-2x;$ $[z,y]=\beta x+y;\ [y,z]=\beta x-y$
AO8	1	$[y,y]=x;\ [z,z]=\alpha x;\ [z,y]=\beta x;\ [y,z]=x$
AO9	1	$[y,y]=x;\ [z,z]=x;\ [z,y]=\beta x$
AO10	1	$[z,z]=x;\ [z,y]=x;\ [y,z]=\alpha x,\ \alpha\neq 0$

Remark 3.5. It is proved in [4] that the concepts of right nilpotent and nilpotent are equivalent. It is also introduced in [4] the concept of solvable Leibniz algebra.

Definition 3.6. Let $\mathfrak{h}$ and $\mathfrak{k}$ be $\mathbb{K}$-vector subspaces of a Leibniz algebra $\mathfrak{g}$. We call centralizer of $\mathfrak{h}$ and $\mathfrak{k}$ over $\mathfrak{g}$ to the $\mathbb{K}$-vector subspace

$$C_{\mathfrak{g}}(\mathfrak{h},\mathfrak{k})=\{x\in\mathfrak{g}\mid [h,x]\in\mathfrak{k},\ [x,h]\in\mathfrak{k},\ \forall h\in\mathfrak{h}\}.$$

Remark 3.7. In the case that $\mathfrak{h}=\mathfrak{g}$ and $\mathfrak{k}=0$, we obtain the previous notion of center of $\mathfrak{g}$.

In Table 1 we show some invariants to distinguish isomorphism classes of non-Lie Leibniz algebras. All these algebras are solvable. We denote by I_2 and I_3 the following invariants:

$$I_2=\dim C_{\mathfrak{g}}\left(\langle a_3\rangle,0\right),$$
$$I_3=\dim C_{\mathfrak{g}}\left(\langle a_1\rangle,\langle a_1\rangle\right).$$

We reproduce in Table 2 the classification of 3-dimensional complex Leibniz algebras given by Ayupov and Omirov in [4] (here x, y, z are the basic elements), and in Table 3 we compare the isomorphism classes with our classification, establishing the correspondence between them.

All the computations are done with a computer program, whose source code is publicly available as detailed in Section 4, where some application examples are also shown. The main conclusions are that there are 11 isomorphism classes of non-Lie Leibniz algebras. The class AO5 does not correspond with a Leibniz algebra because the identity $[z,[y,z]]=[[z,y],z]-[[z,z],y]$ does not hold. Moreover, the classes AO9 and AO10 are isomorphic (this will be checked in Example 4.2).

Table 3
Comparison between classifications.

AO's class	Parameters	
AO1	$\alpha_1 \neq 0,\ \alpha_2 = 0$	2(a) with $\gamma = 0$
AO1	$\alpha_1 = \alpha_2 = 0$	2(b)
AO1	$\alpha_2 \neq 0$	2(d)
AO2	$\alpha_1 \neq 0,\ \alpha_2 = 0$	3(a)
AO2	$\alpha_1 \neq 0,\ \alpha_2 \neq 0,\ \alpha_1 \neq -\frac{\alpha_2^2}{4}$	3(a)
AO2	$\alpha_1 \neq 0,\ \alpha_2 \neq 0,\ \alpha_1 = -\frac{\alpha_2^2}{4}$	3(b)
AO2	$\alpha_1 = 0,\ \alpha_2 = 0$	3(c)
AO2	$\alpha_1 = 0,\ \alpha_2 \neq 0$	3(d)
AO3	$\alpha_1 = 1,\ \alpha_2 = 0$	3(a)
AO3	$\alpha_1 = 1,\ \alpha_2 \neq 0$	3(b)
AO3	$\alpha_1 \neq 0,\ \alpha_1 \neq 1$	3(a) with $\beta = \alpha_1$ or $\beta = \frac{1}{\alpha_1}$
AO3	$\alpha_1 = 0$	3(d)
AO4	$\alpha = 0,\ \beta \neq 1$	2(a) with $\gamma = 0$
AO4	$\alpha = 0,\ \beta = 1$	2(b)
AO4	$\alpha \neq 0$	2(d)
AO5		It is not Leibniz algebra
AO6		2(e)
AO7		2(g)
AO8	$\alpha \neq \beta,\ \beta \neq 1$	2(a) with $\gamma = \frac{\alpha-\beta}{(\beta-1)^2}$
AO8	$\alpha = \beta,\ \alpha \neq 0,\ \alpha \neq 1$	2(a) with $\gamma = 0$
AO8	$\alpha = \beta = 0$	2(a) with $\gamma = 0$
AO8	$\alpha = \beta = 1$	2(b)
AO8	$\alpha \neq 1,\ \beta = 1$	2(c)
AO9	$\beta \neq 0$	2(a) with $\gamma = \frac{1}{\beta^2}$
AO9	$\beta = 0$	2(c)
AO10	$\alpha \neq 0,\ \alpha \neq 1$	2(a) with $\gamma = -\frac{\alpha}{(\alpha-1)^2}$
AO10	$\alpha = 1$	2(c)
		2(f)

4. Some computations

This section is devoted to showing some examples of computations with a Mathematica program that implements Algorithm 2.6 discussed in Section 2. This program establishes the existence of a non-singular matrix P satisfying the equation (1.5). The Mathematica notebook `Iso_Leibniz.nb` together with some examples are available at http://www.usc.es/regaca/mladra/Iso_Leibniz.nb.

The following example shows the application of Algorithm 2.6 on two Leibniz algebras structures corresponding to the family 3(a). The example checks the case 4 of Remark 3.2.

Example 4.1. Let $(\mathfrak{g} = \langle\{a_1, a_2, a_3\}\rangle, [-,-]_1)$ and $(\mathfrak{g} = \langle\{a_1, a_2, a_3\}\rangle, [-,-]_2)$ be two Leibniz algebras such that $[a_1, a_3]_1 = \beta \cdot a_1, \quad [a_2, a_3]_1 = a_2$ (0 otherwise) and $[a_1, a_3]_2 = \frac{1}{\beta} \cdot a_1, \quad [a_2, a_3]_2 = a_2$ (0 otherwise).

We check if the two Leibniz algebras are isomorphic as follows:

```
BracketEqZero[3]

BracketOne[1, 3] := {β, 0, 0}
BracketOne[2, 3] := {0, 1, 0}

BracketTwo[1, 3] := {1/β, 0, 0}
BracketTwo[2, 3] := {0, 1, 0}

IsoLeibnizQ[3]
```

```
Algebras are isomorphic
To obtain the change of basis
matrix P you only need to get a point of the variety generated by
(take into account that Y=1/Det[P]):

{-1 - 1/β + p[3, 3] + β p[3, 3],
-1/β^2 - 1/β + p[3, 3] + p[3, 3]/β, -1/β^2 + p[3, 3]^2,
 p[3, 2], p[3, 1], p[2, 3], -p[2, 2] + β^2 p[2, 2],
 p[2, 2]/β - β p[2, 2], -p[2, 2] +  p[2, 2] p[3, 3],
-p[2, 1]/β +  p[2, 1] p[3, 3],
-p[2, 1] p[2, 2] + β p[2, 1] p[2, 2],
 p[1, 3], -p[1, 2]/β +  p[1, 2] p[3, 3],
-p[1, 2] p[2, 2] + β p[1, 2] p[2, 2],
-p[1, 1] + β^2 p[1, 1],  p[1, 1]/β - β p[1, 1],
-p[1, 1] + p[1, 1] p[3, 3], -p[1, 1] p[2, 1] + β p[1, 1] p[2, 1],
-p[1, 1] p[1, 2] + β p[1, 1] p[1, 2],
 1/β - β + β^2  - Y p[1, 2] p[2, 1] + Y β p[1, 2] p[2, 1] - p[3, 3],
 1 + 1/β - β - Y p[1, 2] p[2, 1] + Y p[1, 2] p[2, 1]/β - p[3, 3],
 1/β - β - Y p[1, 2] p[2, 1] + Y p[1, 1] p[2, 2] - p[3, 3]}
```

The output provides the equations of the algebraic variety that satisfies the matrix P.

For example, a point of this variety is $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta} \end{pmatrix}$.

The following example checks that the cases AO9 and AO10 are isomorphic.

Example 4.2. Let $(\mathfrak{g} =< \{a_1, a_2, a_3\} >, [-,-]_1)$ and $(\mathfrak{g} =< \{a_1, a_2, a_3\} >, [-,-]_2)$ be two Leibniz algebras such that $[a_2, a_2]_1 = a_1,\ [a_3, a_2]_1 = \beta \cdot a_1,\ \beta \neq 0,\ [a_3, a_3]_1 = a_1$ (0 otherwise) and $[a_2, a_3]_2 = \alpha \cdot a_1,\ 0 \neq \alpha \neq 1,\ [a_3, a_3]_2 = a_1,\ [a_3, a_2]_2 = a_1$ (0 otherwise).

```
BracketEqZero[3]

BracketOne[2,2] := {1, 0, 0}
BracketOne[3,2] := {β, 0, 0}
BracketOne[3,3] := {1, 0, 0}


BracketTwo[2,3] := {α, 0, 0}
BracketTwo[3,2] := {1, 0, 0}
BracketTwo[3,3] := {1, 0, 0}

IsoLeibnizQ[3]

Algebras are isomorphic
```

The first term of the output of the polynomials that generate the variety whose points are all the possible values of P is $1-2\alpha+\alpha^2+\alpha\beta^2 = 0$, which is equivalent to $\frac{1}{\beta^2} = -\frac{\alpha}{(\alpha-1)^2}$, and corresponds to the case 2(a) or AO9 and AO10 (see Table 3). Taking, for example, $\alpha = -1$ and $\beta = 2$, a matrix P satisfying (1.5) is $P = \begin{pmatrix} 1 & p_{12} & p_{13} \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$.

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