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Classification of solvable Leibniz algebras with naturally graded filiform nilradical

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ABSTRACT

In this paper we show that the method for describing solvable Lie algebras with given nilradical by means of non-nilpotent outer derivations of the nilradical is also applicable to the case of Leibniz algebras. Using this method we extend the classification of solvable Lie algebras with naturally graded filiform Lie algebra to the case of Leibniz algebras. Namely, the classification of solvable Leibniz algebras whose nilradical is a naturally graded filiform Leibniz algebra is obtained.

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1. Introduction

Solvable Lie algebras have played a significant role in recent decades, where they have been applied systematically to integrable systems, in the formulation of non-abelian gauge theories, in quantum gravity and string theories in the low-energy supergravity limit (see, e.g. [1,2]). The need for a classification of solvable Lie algebras of higher dimensions in physics arises in particular in the classification of higher dimensional Einstein spaces, or other pseudo-Riemannian spaces that can occur in string theories, brane cosmology and other elementary particle theories.

Leibniz algebras were introduced at the beginning of the 90s of the past century by J.-L. Loday in [3]. They are a "non-commutative" generalization of Lie algebras. Leibniz algebras inherit an important property of Lie algebras which is that the right multiplication operator on an element of a Leibniz algebra is a derivation. Active investigations on Leibniz algebra theory show that many results of the theory of Lie algebras can be extended to Leibniz algebras. Of course, distinctive properties of non-Lie Leibniz algebras have also been studied [4,5].

In fact, for a Leibniz algebra we have the corresponding Lie algebra, which is the quotient algebra by the two-sided ideal I generated by the square elements of a Leibniz algebra. Notice that this ideal is the minimal one such that the quotient algebra is a Lie algebra and in the case of non-Lie Leibniz algebras it is always non trivial (moreover, it is abelian).

From the theory of Lie algebras it is well known that the study of finite dimensional Lie algebras was reduced to the nilpotent ones [6,7]. In the Leibniz algebra case we have an analogue of Levi's theorem [5]. Namely, the decomposition of a Leibniz algebra into a semidirect sum of its solvable radical and a semisimple Lie algebra is obtained. The semisimple part can be described from simple Lie ideals and therefore, the main problem is to study the solvable radical, i.e. in a similar way as in the case of Lie algebras, the description of Leibniz algebras is reduced to the description of the solvable ones. The analysis of works devoted to the study of solvable Lie algebras (for example [8–12], where solvable Lie algebras with various types of nilradical were studied, such as naturally graded filiform and quasi-filiform algebras, abelian, triangular, etc.) shows that we can also apply similar methods to solvable Leibniz algebras with a given nilradical. Some results of Lie algebra theory generalized to Leibniz algebras [13] allow us to apply the technique of description of solvable extensions of nilpotent Lie algebras to the case of Leibniz algebras.

The aim of the present paper is to classify solvable Leibniz algebras with naturally graded filiform nilradical. Thanks to the works [4,14], we already have the classification of naturally graded filiform Leibniz algebras.

In order to achieve our goal we organize the paper as follows. In Section 2 we give some necessary notions and preliminary results about Leibniz algebras and solvable Lie algebras with naturally graded filiform radical. Section 3 is devoted to the classification of solvable Leibniz algebras whose nilradical is a naturally graded filiform Lie algebra and in Section 4 we describe, up to isomorphisms, solvable Leibniz algebras whose nilradical is a naturally graded filiform non-Lie Leibniz algebra.

Throughout the paper vector spaces and algebras are finite-dimensional over the field of the complex numbers. Moreover, in the table of multiplication of an algebra the omitted products are assumed to be zero and, if it is not noted, we shall consider non-nilpotent solvable algebras.

2. Preliminaries

In this section we give necessary definitions and preliminary results.

Definition 2.1. A vector space with bilinear bracket $(L, [-, -])$ over a field F is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds, or equivalently, $[[x, y], z] = [[x, z], y] + [x, [y, z]]$.

Here, we adopt the right Leibniz identity; since the bracket is not skew-symmetric, there exists the version corresponding to the left Leibniz identity,

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]].$$

From the Leibniz identity we conclude that the elements $[x, x]$, $[x, y] + [y, x]$, for any $x, y \in L$, lie in $\mathrm{Ann}_r(L) = \{x \in L \mid [y, x] = 0, \text{ for all } y \in L\}$, the *right annihilator* of the Leibniz algebra L. Moreover, we also get that $\mathrm{Ann}_r(L)$ is a two-sided ideal of L.

The two-sided ideal $\mathrm{Center}(L) = \{x \in L \mid [x, y] = 0 = [y, x], \text{ for all } y \in L\}$ is said to be the *center* of L.

Definition 2.2. A linear map $d \colon L \to L$ of a Leibniz algebra $(L, [-, -])$ is said to be a derivation if for all $x, y \in L$, the following condition holds:

$$d([x, y]) = [d(x), y] + [x, d(y)].$$

For a given element x of a Leibniz algebra L, the right multiplication operators $\mathcal{R}_x \colon L \to L$, $\mathcal{R}_x(y) = [y, x]$, $y \in L$, are derivations (for a left Leibniz algebra L, the left multiplication operators $\mathcal{L}_x \colon L \to L$, $\mathcal{L}_x(y) = [x, y]$, $y \in L$, are derivations). This kind of derivations are said to be *inner derivations*. Any Leibniz algebra L has associated the algebra of right multiplications $\mathcal{R}(L) = \{\mathcal{R}_x \mid x \in L\}$. $\mathcal{R}(L)$ is endowed with a structure of Lie algebra by means of the bracket $[\mathcal{R}_x, \mathcal{R}_y] = \mathcal{R}_x\mathcal{R}_y - \mathcal{R}_y\mathcal{R}_x = \mathcal{R}_{[y,x]}$. Moreover, there is an antisymmetric isomorphism between $\mathcal{R}(L)$ and the quotient algebra $L/\mathrm{Ann}_r(L)$.

Definition 2.3. For a given Leibniz algebra $(L, [-, -])$ the sequences of two-sided ideals defined recursively as follows:

$$L^1 = L,\ L^{k+1} = [L^k, L], \quad k \geqslant 1, \qquad L^{[1]} = L,\ L^{[s+1]} = [L^{[s]}, L^{[s]}], \quad s \geqslant 1,$$

are said to be the lower central and the derived series of L, respectively.

Definition 2.4. A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^{[m]} = 0$). The minimal number n (respectively, m) with such property is said to be the index of nilpotency (respectively, of solvability) of the algebra L.

Evidently, the index of nilpotency of an n-dimensional nilpotent algebra is not greater than $n + 1$.

Definition 2.5. An n-dimensional Leibniz algebra L is said to be null-filiform if $\dim L^i = n+1-i,\ 1 \leqslant i \leqslant n + 1$.

Evidently, null-filiform Leibniz algebras have maximal index of nilpotency.

Theorem 2.6 [4]**.** *An arbitrary n-dimensional null-filiform Leibniz algebra is isomorphic to the algebra*

$$NF_n : \quad [e_i, e_1] = e_{i+1}, \quad 1 \leqslant i \leqslant n - 1,$$

where $\{e_1, e_2, \ldots, e_n\}$ *is a basis of the algebra* NF_n.

Actually, a nilpotent Leibniz algebra is null-filiform if and only if it is one-generated algebra. Notice that this notion has no sense in Lie algebras case, because they are at least two-generated.

Definition 2.7. An n-dimensional Leibniz algebra L is said to be filiform if $\dim L^i = n-i$, for $2 \leqslant i \leqslant n$.

Now let us define a natural graduation for a filiform Leibniz algebra.

Definition 2.8. Given a filiform Leibniz algebra L, put $L_i = L^i/L^{i+1}$, $1 \leqslant i \leqslant n-1$, and $\mathrm{gr}(L) = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1}$. Then $[L_i, L_j] \subseteq L_{i+j}$ and we obtain the graded algebra $\mathrm{gr}(L)$. If $\mathrm{gr}(L)$ and L are isomorphic, then we say that an algebra L is naturally graded.

Thanks to [14] it is well known that there are two types of naturally graded filiform Lie algebras. In fact, the second type will appear only in the case when the dimension of the algebra is even.

Theorem 2.9 [14]**.** *Any complex naturally graded filiform Lie algebra is isomorphic to one of the following non isomorphic algebras:*

$$n_{n,1} : [e_i, e_1] = -[e_1, e_i] = e_{i+1}, \quad 2 \leqslant i \leqslant n-1.$$

$$Q_{2n} : \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leqslant i \leqslant 2n-2, \\ [e_i, e_{2n+1-i}] = -[e_{2n+1-i}, e_i] = (-1)^i e_{2n}, & 2 \leqslant i \leqslant n. \end{cases}$$

In the following theorem we recall the classification of the naturally graded filiform non-Lie Leibniz algebras given in [4].

Theorem 2.10 [4]**.** *Any complex n-dimensional naturally graded filiform non-Lie Leibniz algebra is isomorphic to one of the following non isomorphic algebras:*

$$F_n^1 = \begin{cases} [e_1, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, \ 2 \leqslant i \leqslant n-1, \end{cases}$$

$$F_n^2 = \begin{cases} [e_1, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, \ 3 \leqslant i \leqslant n-1. \end{cases}$$

Definition 2.11. The maximal nilpotent ideal of a Leibniz algebra is said to be the nilradical of the algebra.

Notice that the nilradical is not the radical in the sense of Kurosh, because the quotient Leibniz algebra by its nilradical may contain a nilpotent ideal (see [6]).

All solvable Lie algebras whose nilradical is the naturally graded filiform Lie algebra $n_{n,1}$ are classified in [15]. Further solvable Lie algebras whose nilradical is the naturally graded filiform Lie algebra Q_{2n} are classified in [16].

Using the above classifications, we shall give the classification of solvable non-Lie Leibniz algebras whose nilradical is a naturally graded filiform Lie algebra.

It is proved that the dimension of a solvable Lie algebra whose nilradical is isomorphic to an n-dimensional naturally graded filiform Lie algebra is not greater than $n+2$. Below, we present their classification.

In order to agree with the tables of multiplications of algebras in Theorems 2.9 and 2.10, we make the following change of basis in the classification of [15]:

$$e_i' = e_{n+1-i}, \quad 1 \leqslant i \leqslant n, \qquad x = -f.$$

We also use different notation to denote the algebras that appear in [15]. That way the results would be:

Theorem 2.12 [15]**.** *There are three types of solvable Lie algebras of dimension $n+1$ with nilradical isomorphic to $n_{n,1}$, for any $n \geqslant 4$. The isomorphism classes in the basis $\{e_1, \ldots, e_n, x\}$ are represented by the following algebras:*

$$S_{n+1}(\alpha,\beta):\begin{cases}[e_i,e_1]=-[e_1,e_i]=e_{i+1}, & 2\leqslant i\leqslant n-1,\\ [e_i,x]=-[x,e_i]=((i-2)\alpha+\beta)\,e_i, & 2\leqslant i\leqslant n,\\ [e_1,x]=-[x,e_1]=\alpha e_1.\end{cases}$$

The mutually non-isomorphic algebras of this type are $S_{n+1,1}(\beta)=S_{n+1}(1,\beta)$ (depending on the value of β, in this case there are three different classes, $\beta=0$, $\beta=n-2$ and $\beta\notin\{0,n-2\}$) and $S_{n+1,2}=S_{n+1}(0,1)$.

$$S_{n+1,3}:\begin{cases}[e_i,e_1]=-[e_1,e_i]=e_{i+1}, & 2\leqslant i\leqslant n-1,\\ [e_i,x]=-[x,e_i]=(i-1)\,e_i, & 2\leqslant i\leqslant n,\\ [e_1,x]=-[x,e_1]=e_1+e_2.\end{cases}$$

$$S_{n+1,4}(\alpha_3,\alpha_4,\dots,\alpha_{n-1}):\begin{cases}[e_i,e_1]=-[e_1,e_i]=e_{i+1}, \quad 2\leqslant i\leqslant n-1,\\ [e_i,x]=-[x,e_i]=e_i+\displaystyle\sum_{l=i+2}^{n}\alpha_{l+1-i}\,e_l,\ 2\leqslant i\leqslant n,\end{cases}$$

where at least one α_i satisfies $\alpha_i\neq 0$ and the first non-vanishing parameter $\{\alpha_3,\dots,\alpha_{n-1}\}$ can be assumed to be equal to 1.

Theorem 2.13 [15]. *There exists only one class of solvable Lie algebras of dimension $n+2$ with nilradical $n_{n,1}$. It is represented by a basis $\{e_1,e_2,\dots,e_n,x,y\}$ and the Lie brackets are*

$$S_{n+2}:\begin{cases}[e_i,e_1]=-[e_1,e_i]=e_{i+1}, & 2\leqslant i\leqslant n-1,\\ [e_i,x]=-[x,e_i]=(i-2)\,e_i, & 2\leqslant i\leqslant n,\\ [e_1,x]=-[x,e_1]=e_1, & \\ [e_i,y]=-[y,e_i]=e_i, & 2\leqslant i\leqslant n.\end{cases}$$

Now we recall the classification given in [16] after the following change of basis:

$$e_1'=-e_1,\qquad x'=-Y_1,\qquad y'=-Y_2.$$

Proposition 2.14 [16]. *Any solvable Lie algebra of dimension 2n+1 with nilradical isomorphic to Q_{2n} is isomorphic to one of the following algebras:*

$$Q_{2n+1,1}(\alpha):\begin{cases}[e_i,e_1]=-[e_1,e_i]=e_{i+1}, & 2\leqslant i\leqslant 2n-2,\\ [e_i,e_{2n+1-i}]=-[e_{2n+1-i},e_i]=(-1)^i\,e_{2n}, & 2\leqslant i\leqslant n,\\ [e_1,x]=-[x,e_1]=e_1, & \\ [e_i,x]=-[x,e_i]=(i-2+\alpha)\,e_i, & 2\leqslant i\leqslant 2n-1,\\ [e_{2n},x]=-[x,e_{2n}]=(2n-3-2\alpha)\,e_{2n}. & \end{cases}$$

$$Q_{2n+1,2}:\begin{cases}[e_i,e_1]=-[e_1,e_i]=e_{i+1}, & 2\leqslant i\leqslant 2n-2,\\ [e_i,e_{2n+1-i}]=-[e_{2n+1-i},e_i]=(-1)^i\,e_{2n}, & 2\leqslant i\leqslant n,\\ [e_1,x]=-[x,e_1]=e_1+\varepsilon\,e_{2n}, & \varepsilon=0,1,\\ [e_i,x]=-[x,e_i]=(i-n)\,e_i, & 2\leqslant i\leqslant 2n-1,\\ [e_{2n},x]=-[x,e_{2n}]=e_{2n}. & \end{cases}$$

$$Q_{2n+1,3}(\alpha):\begin{cases} [e_i,e_1]=-[e_1,e_i]=e_{i+1}, & 2\leqslant i\leqslant 2n-2,\\ [e_i,e_{2n+1-i}]=-[e_{2n+1-i},e_i]=(-1)^i\,e_{2n}, & 2\leqslant i\leqslant n,\\ [e_{2+i},x]=-[x,e_{2+i}]=e_{2+i}+\sum\limits_{k=2}^{\lfloor\frac{2n-3-i}{2}\rfloor}\alpha^{2k+1}e_{2k+1+i}, & \\ & 0\leqslant i\leqslant 2n-6,\\ [e_{2n-i},x]=-[x,e_{2n-i}]=e_{2n-i}, & i=1,2,3,\\ [e_{2n},x]=-[x,e_{2n}]=2e_{2n}. \end{cases}$$

Proposition 2.15 [16]. *For any $n\geqslant 3$ there is only one $(2n+2)$-dimensional solvable Lie algebra having a nilradical isomorphic to Q_{2n}:*

$$\begin{cases} [e_i,e_1]=-[e_1,e_i]=e_{i+1}, & 2\leqslant i\leqslant 2n-2,\\ [e_i,e_{2n+1-i}]=-[e_{2n+1-i},e_i]=(-1)^i\,e_{2n}, & 2\leqslant i\leqslant n,\\ [e_i,x]=-[x,e_i]=i\,e_i, & 1\leqslant i\leqslant 2n-1,\\ [e_{2n},x]=-[x,e_{2n}]=(2n+1)\,e_{2n}, & \\ [e_i,y]=-[y,e_i]=e_i, & 1\leqslant i\leqslant 2n-1,\\ [e_{2n},y]=-[y,e_{2n}]=2\,e_{2n}. \end{cases}$$

Let R be a solvable Leibniz algebra with nilradical N. We denote by Q the complementary vector space of the nilradical N to the algebra R. Let us consider the restrictions to N of the right multiplication operator on an element $x\in Q$ (denoted by $\mathcal{R}_{x|_N}$). If the operator $\mathcal{R}_{x|_N}$ is nilpotent, then we assert that the subspace $\langle x+N\rangle$ is a nilpotent ideal of the algebra R. Indeed, since for a solvable Leibniz algebra R we get the inclusion $R^2\subseteq N$ [13], and hence the subspace $\langle x+N\rangle$ is an ideal. The nilpotency of this ideal follows from the Engel's theorem for Leibniz algebras [13]. Therefore, we have a nilpotent ideal which strictly contains the nilradical, which is in contradiction with the maximality of N. Thus, we obtain that for any $x\in Q$, the operator $\mathcal{R}_{x|_N}$ is a non-nilpotent derivation of N.

Let $\{x_1,\dots,x_m\}$ be a basis of Q, then for any scalars $\{\alpha_1,\dots,\alpha_m\}\in\mathbb{C}\setminus\{0\}$, the matrix $\alpha_1\mathcal{R}_{x_1|_N}+\dots+\alpha_m\mathcal{R}_{x_m|_N}$ is not nilpotent, which means that the elements $\{x_1,\dots,x_m\}$ are nil-independent [17]. Therefore, we have that the dimension of Q is bounded by the maximal number of nil-independent derivations of the nilradical N. Moreover, similar to the case of Lie algebras, for a solvable Leibniz algebra R the inequality $\dim N\geqslant\frac{\dim R}{2}$ holds.

3. Solvable Leibniz algebras whose nilradical is a Lie algebra

It is not difficult to see that if R is a solvable non-Lie Leibniz algebra with nilradical isomorphic to the algebras $n_{n,1}$ or Q_{2n}, then the dimension of R is also not greater than $n+2$ and $2n+2$, respectively.

Let $n_{n,1}$ or Q_{2n} be the nilradical of a solvable Leibniz algebra R. Since the ideal $I=\langle\{[x,x]\mid x\in R\}\rangle$ is contained in $\mathrm{Ann}_r(R)$, then I is abelian, hence it is contained in the nilradical. Taking into account the multiplication in $n_{n,1}$ (respectively Q_{2n}) we conclude that $I=\langle\{e_n\}\rangle$.

Having in mind that an $(n+1)$-dimensional algebra R is solvable, then the quotient algebra R/I is also a solvable Lie algebra with nilradical $n_{n,1}$ (whose lists of tables of multiplication are given in Theorems 2.12 and 2.13).

Case $n_{n,1}$**.** Let us assume that R has dimension $n+1$, then the table of multiplication in R will be equal to the table of multiplication of $S_{n+1,i}$, $(i=1,2,3,4)$, except the following products:

$$\begin{array}{lll} [e_1,x]=\alpha_1e_1+\gamma_4e_n, & [e_2,x]=\beta_1e_2+\gamma_5e_n, & \\ [x,e_1]=-\alpha_1e_1+\gamma_1e_n, & [x,e_2]=-\beta_1e_2+\gamma_2e_n, & [x,x]=\gamma_3e_n, \end{array}$$

where $(\gamma_1+\gamma_4,\gamma_2+\gamma_5,\gamma_3)\neq(0,0,0)$.

Note that taking the change of basis

$$e_1' = \alpha_1 e_1 + \gamma_4 e_n, \qquad e_2' = \beta_1 e_2 + \gamma_5 e_n$$

we can assume that $\gamma_4 = \gamma_5 = 0$, i.e., $[e_1, x] = \alpha e_1$ and $[e_2, x] = \beta e_2$.

It is not difficult to see that, for the omitted products, the antisymmetric identity holds, i.e.

$$\begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [e_i, x] = -[x, e_i], & 3 \leqslant i \leqslant n. \end{cases}$$

We have $[e_n, x] = 0$ because $0 = [x, e_n] = -[e_n, x]$.

Consider

$$0 = [x, e_n] = \big[x, [e_{n-1}, e_1]\big] = \big[[x, e_{n-1}], e_1\big] - \big[[x, e_1], e_{n-1}\big] = -(n-2+\beta)\, e_n.$$

In the list of Theorem 2.12 only the algebra $S_{n+1,1}(\beta)$ is representative of the class for which the equality $[e_n, x] = 0$ holds. This class is defined by $\beta = 2 - n$.

Therefore, in the case of $\dim R = n+1$ whose nilradical is $n_{n,1}$, we have the following family:

$$R_{n+1,1}(\gamma_1, \gamma_2, \gamma_3): \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [e_1, x] = e_1, & \\ [x, e_1] = -e_1 + \gamma_1 e_n, & \\ [e_2, x] = (2-n)\, e_2, & \\ [x, e_2] = (n-2)\, e_2 + \gamma_2 e_n, & \\ [e_i, x] = -[x, e_i] = (i-n)\, e_i, & 3 \leqslant i \leqslant n-1, \\ [x, x] = \gamma_3 e_n, & \end{cases}$$

where $(\gamma_1, \gamma_2, \gamma_3) \neq (0, 0, 0)$.

Applying a similar argument and the table of multiplication of the algebra in Theorem 2.13, we conclude that solvable non-Lie Leibniz algebras of dimension $n+2$ with nilradical $n_{n,1}$ do not exist.

Theorem 3.1. *Any $(n+1)$-dimensional solvable Leibniz algebra with nilradical $n_{n,1}$ is isomorphic to one of the following pairwise non isomorphic algebras:*

$$R_{n+1,1}(0, 0, 1), \quad R_{n+1,1}(0, 1, 0), \quad R_{n+1,1}(1, 1, 0), \quad R_{n+1,1}(1, 0, 0).$$

Proof. We consider the general change of basis in the family $R_{n+1,1}(\gamma_1, \gamma_2, \gamma_3)$:

$$e_1' = \sum_{i=1}^{n} A_i e_i, \qquad e_2' = \sum_{i=1}^{n} B_i e_i, \qquad x' = D x + \sum_{i=1}^{n} C_i e_i,$$

where $(A_1 B_2 - B_1 A_2) D \neq 0$.

Using $[e_i', e_1'] = e_{i+1}'$, $2 \leqslant i \leqslant n-1$, the table of multiplication of $R_{n+1,1}(\gamma_1, \gamma_2, \gamma_3)$ and an induction, we obtain

$$e_i' = A_1^{i-3} \sum_{j=i}^{n} (A_1 B_{j+2-i} - B_1 A_{j+2-i})\, e_j, \quad 3 \leqslant i \leqslant n.$$

From the equalities

$$0 = [e_3', e_2'] = B_1 \sum_{j=4}^{n} (A_1 B_{j-2} - B_1 A_{j-2})\, e_j,$$

we have $B_1 = 0$.

Consider the multiplications

$$\begin{aligned}[e_1', x'] &= A_1 D\, e_1 - D\sum_{i=2}^{n-1} A_i(n-i)\, e_i + \sum_{i=3}^{n}(A_{i-1}C_1 - A_1C_{i-1})\, e_i \\ &= A_1De_1 - A_2D(n-2)\, e_2 + \sum_{i=3}^{n-1}\big(A_{i-1}C_1 - A_1C_{i-1} - (n-i)A_iD\big)\, e_i \\ &\quad + (A_{n-1}C_1 - A_1C_{n-1})\, e_n.\end{aligned}$$

On the other hand, we have

$$[e_1', x'] = e_1' = \sum_{i=1}^{n} A_i\, e_i.$$

Comparing the coefficients of the basis elements we derive:

$$\begin{aligned}&D = 1, \quad A_2 = 0, \quad A_{i+1} = \frac{A_1C_i - A_iC_1}{i-n-1}, \qquad 2 \leqslant i \leqslant n-2, \\ &A_n = A_1C_{n-1} - A_{n-1}C_n.\end{aligned}$$

From the equalities

$$\begin{aligned}-(n-2)\sum_{i=2}^{n} B_ie_i &= -(n-2)e_2' = [e_2', x'] = \left[\sum_{i=2}^{n} B_ie_i, x + \sum_{i=1}^{n} C_ie_i\right] \\ &= -\sum_{i=2}^{n-1} B_i(n-i)e_i + C_1\sum_{i=3}^{n} B_{i-1}e_i \\ &= -B_2(n-2)e_2 + \sum_{i=3}^{n-1}\big(B_{i-1}C_1 - B_i(n-i)\big)e_i + B_{n-1}C_1e_n,\end{aligned}$$

we deduce the following restrictions:

$$B_i = (-1)^i \frac{B_2C_1^{i-2}}{(i-2)!}, \quad 3 \leqslant i \leqslant n.$$

In an analogous way, comparing coefficients at the basis element e_n in the equalities, we obtain:

$$\gamma_3' A_1^{n-2} B_2\, e_n = \gamma_3'\, e_n' = [x', x'] = (\gamma_3 + C_1\gamma_1 + C_2\gamma_2)\, e_n,$$

and thus

$$\gamma_3' = \frac{\gamma_3 + C_1\gamma_1 + C_2\gamma_2}{A_1^{n-2}B_2}.$$

With a similar argument, we obtain

$$-e_1' + A_1^{n-2}B_2\gamma_1'\, e_n = -e_1' + \gamma_1'\, e_n' = [x', e_1'] = -e_1' + A_1\gamma_1\, e_n,$$

and

$$-(n-2)\, e_2' + A_1^{n-2}B_2\gamma_2'\, e_n = (n-2)\, e_2' + \gamma_2'\, e_n' = [x', e_2'] = (n-2)\, e_2' + B_2\gamma_2\, e_n,$$

and hence

$$\gamma_1' = \frac{\gamma_1}{A_1^{n-3}B_2} \quad \text{and} \quad \gamma_2' = \frac{\gamma_2}{A_1^{n-2}}.$$

Now we shall consider the possible cases of the parameters $\{\gamma_1, \gamma_2, \gamma_3\}$.

Case 1. Let $\gamma_1 = 0$. Then $\gamma_1' = 0$.

If $\gamma_2 = 0$, then $\gamma_2' = 0$ and $\gamma_3' = \frac{\gamma_3}{A_1^{n-2}B_2} \neq 0$. Putting $B_2 = \frac{\gamma_3}{A_1^{n-2}}$, then we have that $\gamma_3' = 1$, and thus the algebra is $R_{n+1,1}(0, 0, 1)$.

If $\gamma_2 \neq 0$, then putting $A_1 = \sqrt[n-2]{\gamma_2}$ and $C_2 = -\frac{\gamma_3}{\gamma_2}$, we get $\gamma_2' = 1$ and $\gamma_3' = 0$, i.e. we obtain the algebra $R_{n+1,1}(0, 1, 0)$.

Case 2. Let $\gamma_1 \neq 0$. Then putting $B_2 = \frac{\gamma_1}{A_1^{n-3}}$ and $C_1 = -\frac{\gamma_3+C_2\gamma_2}{\gamma_1}$, we have:

$$\gamma_1' = 1, \qquad \gamma_2' = \frac{\gamma_2}{A_1^{n-2}}, \qquad \gamma_3' = 0.$$

If $\gamma_2 \neq 0$, then putting $A_1 = \sqrt[n-2]{\gamma_2}$ we have that $\gamma_2' = 1$, and thus we obtain the algebra $R_{n+1,1}(1, 1, 0)$.

If $\gamma_2 = 0$, then we get the algebra $R_{n+1,1}(1, 0, 0)$. $\square$

Case Q_{2n}**.** Similarly as above, from Propositions 2.14 and 2.15, we conclude that solvable non-Lie Leibniz algebras with nilradical Q_{2n} exist only in the case of $\dim R = 2n + 1$ and they are isomorphic to $Q_{2n+1,1}(\alpha)$ for $\alpha = \frac{2n-3}{2}$. Thus, we have

$$R_{2n+1,1}: \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leqslant i \leqslant 2n-2, \\ [e_i, e_{2n+1-i}] = -[e_{2n+1-i}, e_i] = (-1)^i e_{2n}, & 2 \leqslant i \leqslant n, \\ [e_1, x] = e_1, \\ [x, e_1] = -e_1 + \gamma_1 e_n, \\ [e_2, x] = \dfrac{2n-3}{2} e_2, \\ [x, e_2] = -\dfrac{2n-3}{2} e_2 + \gamma_2 e_n, \\ [e_i, x] = -[x, e_i] = \dfrac{2n+2i-7}{2} e_i, & 3 \leqslant i \leqslant 2n-1, \\ [x, x] = \gamma_3 e_n, \end{cases}$$

where $(\gamma_1, \gamma_2, \gamma_3) \neq (0, 0, 0)$.

Theorem 3.2. *Any $(2n+1)$-dimensional solvable Leibniz algebra with nilradical Q_{2n} is isomorphic to one of the following pairwise non isomorphic algebras:*

$$R_{2n+1,1}(0, 0, 1), \quad R_{2n+1,1}(0, 1, 0), \quad R_{2n+1,1}(1, 1, 0), \quad R_{2n+1,1}(1, 0, 0).$$

Proof. The proof is carried out by applying similar arguments as in the proof of Theorem 3.1 $\square$

4. Solvable Leibniz algebras whose nilradical is a non-Lie Leibniz algebra

In the following proposition we describe the derivations of the algebra F_n^1.

Proposition 4.1. *Any derivation of the algebra F_n^1 has the following matrix form:*

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1+\alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_{n-1} & \beta \\ 0 & 0 & 2\alpha_1+\alpha_2 & \alpha_3 & \dots & \alpha_{n-2} & \alpha_{n-1} \\ 0 & 0 & 0 & 3\alpha_1+\alpha_2 & \dots & \alpha_{n-3} & \alpha_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & (n-1)\alpha_1+\alpha_2 \end{pmatrix}.$$

Proof. Let d be a derivation of the algebra. We set

$$d(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \quad d(e_2) = \sum_{i=1}^{n} \beta_i e_i.$$

From the equality

$$0 = d([e_1, e_2]) = [d(e_1), e_2] + [e_1, d(e_2)] = \beta_1 e_3,$$

we get $\beta_1 = 0$.

Further, we have

$$d(e_3) = d([e_1, e_1]) = [d(e_1), e_1] + [e_1, d(e_1)] = (2\alpha_1 + \alpha_2) e_3 + \sum_{i=3}^{n-1} \alpha_i e_{i+1}.$$

On the other hand,

$$d(e_3) = d([e_2, e_1]) = [d(e_2), e_1] + [e_2, d(e_1)] = (\alpha_1 + \beta_2) e_3 + \sum_{i=3}^{n-1} \beta_i e_{i+1}.$$

Therefore, $\beta_2 = \alpha_1 + \alpha_2,\ \beta_i = \alpha_i,\ 3 \leqslant i \leqslant n-1$.

With similar arguments applied on the products $[e_i, e_1] = e_{i+1}$ and with an induction on i, it is easy to check that the following identities hold for $3 \leqslant i \leqslant n$:

$$d(e_i) = ((i-1)\alpha_1 + \alpha_2) e_i + \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j, \quad 3 \leqslant i \leqslant n. \quad \square$$

From Proposition 4.1 we conclude that the number of nil-independent outer derivations of the algebra F_n^1 is equal to two. Therefore, by arguments after Proposition 2.15, we have that any solvable Leibniz algebra whose nilradical is F_n^1 has dimension either $n+1$ or $n+2$.

4.1. Solvable Leibniz algebras with nilradical F_n^1

Below we present the description of such Leibniz algebras when dimension is equal to $n+1$.

Theorem 4.2. *An arbitrary $(n+1)$-dimensional solvable Leibniz algebra with nilradical F_n^1 is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$R_1 : \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [x, e_1] = -e_1 - e_2, & \\ [e_1, x] = e_1, & \\ [e_i, x] = (i-1)e_i, & 2 \leqslant i \leqslant n. \end{cases}$$

$$R_2(\alpha) : \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [x, e_1] = -e_1, & \\ [e_1, x] = e_1, & \\ [e_i, x] = (i-1+\alpha) e_i, & 2 \leqslant i \leqslant n, \end{cases}$$

$$R_3 : \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [x, e_1] = -e_1, & \\ [e_1, x] = e_1, & \\ [e_i, x] = (i-n) e_i, & 2 \leqslant i \leqslant n, \\ [x, x] = e_n. & \end{cases}$$

$$R_4:\begin{cases}[e_i,e_1]=e_{i+1}, & 2\leqslant i\leqslant n-1,\\ [x,e_1]=-e_1, & \\ [e_1,x]=e_1+e_n, & \\ [e_i,x]=(i+1-n)\,e_i, & 2\leqslant i\leqslant n,\\ [x,x]=-e_{n-1}. & \end{cases}$$

$$R_5(\alpha_4,\dots,\alpha_{n-1}):\begin{cases}[e_1,e_1]=e_3, & \\ [e_i,e_1]=e_{i+1}, & 2\leqslant i\leqslant n-1,\\ [e_1,x]=e_2+\sum\limits_{i=4}^{n-1}\alpha_i\,e_i & \\ [e_2,x]=e_2+\sum\limits_{i=4}^{n-1}\alpha_i\,e_i, & \\ [e_i,x]=e_i+\sum\limits_{j=i+2}^{n}\alpha_{j-i+2}\,e_j, & 3\leqslant i\leqslant n.\end{cases}$$

$$R_6(\alpha_4,\dots,\alpha_{n-1}):\begin{cases}[e_1,e_1]=e_3, & \\ [e_i,e_1]=e_{i+1}, & 2\leqslant i\leqslant n-1,\\ [e_1,x]=e_2+\sum\limits_{i=4}^{n-1}\alpha_i\,e_i+e_n & \\ [e_2,x]=e_2+\sum\limits_{i=4}^{n-1}\alpha_i\,e_i, & \\ [e_i,x]=e_i+\sum\limits_{j=i+2}^{n}\alpha_{j-i+2}\,e_j, & 3\leqslant i\leqslant n.\end{cases}$$

$$R_7(\alpha_4,\dots,\alpha_{n-1}):\begin{cases}[e_1,e_1]=e_3, & \\ [e_i,e_1]=e_{i+1}, & 2\leqslant i\leqslant n-1,\\ [e_1,x]=e_2+\sum\limits_{i=4}^{n-1}\alpha_i\,e_i, & \\ [e_2,x]=e_2+\sum\limits_{i=4}^{n-1}\alpha_i\,e_i+e_n, & \\ [e_i,x]=e_i+\sum\limits_{j=i+2}^{n}\alpha_{j-i+2}\,e_j, & 3\leqslant i\leqslant n.\end{cases}$$

Moreover, the first non-vanishing parameter $\{\alpha_4,\dots,\alpha_{n-1}\}$ in the algebras $R_5(\alpha_4,\dots,\alpha_{n-1})$, $R_6(\alpha_4,\dots,\alpha_{n-1})$ and $R_7(\alpha_4,\dots,\alpha_{n-1})$ can be scaled to 1.

Proof. From Theorem 2.10 and arguments after Proposition 2.15 we know that there exists a basis $\{e_1,e_2,\dots,e_n,x\}$ such that the multiplication table of the algebra F_n^1 is completed with the products coming from $\mathcal{R}_{x|_{F_n^1}}(e_i),\ 1\leqslant i\leqslant n$, i.e.

$$[e_1, x] = \sum_{i=1}^{n} \alpha_i\, e_i, \qquad [e_2, x] = (\alpha_1 + \alpha_2)\, e_2 + \sum_{i=3}^{n-1} \alpha_i\, e_i + \beta\, e_n,$$
$$[e_i, x] = ((i-1)\alpha_1 + \alpha_2)\, e_i + \sum_{j=i+1}^{n} \alpha_{j-i+2}\, e_j, \quad 3 \leqslant i \leqslant n.$$

Finally, we consider the remaining products as follows:

$$[x, e_1] = \sum_{i=1}^{n} \beta_i\, e_i, \quad [x, e_2] = \sum_{i=1}^{n} \gamma_i\, e_i, \quad [x, x] = \sum_{i=1}^{n} \delta_i\, e_i.$$

From the chain of equalities

$$0 = [x, e_3] = [x, [e_2, e_1]] = [[x, e_2], e_1] - [[x, e_1], e_2] = [[x, e_2], e_1]$$
$$= (\gamma_1 + \gamma_2)\, e_3 + \sum_{i=4}^{n} \gamma_{i-1}\, e_i,$$

we conclude that $\gamma_2 = -\gamma_1,\ \gamma_i = 0,\ 3 \leqslant i \leqslant n-1$.

Since $\gamma_1 e_3 = [e_1, [x, e_2]] = [[e_1, x], e_2] - [[e_1, e_2], x] = 0$, then $\gamma_1 = 0$.

The identity

$$[e_1, [x, e_1]] = [[e_1, x], e_1] - [[e_1, e_1], x]$$

implies $\beta_1 = -\alpha_1$.

Applying the Leibniz identity to the elements of the form $\{x, x, e_2\}$ and $\{x, e_2, x\}$, we conclude that:

$$\begin{cases} ((n-1)\alpha_1 + \alpha_2)\gamma_n = 0, \\ (n-2)\alpha_1 \gamma_n = 0. \end{cases}$$

Note that $\gamma_n = 0$ (otherwise $\alpha_1 = \alpha_2 = 0$ and then we get a contradiction with the non-nilpotency of the derivation $\mathcal{R}_{x|_{F_n^1}}$ (see Proposition 4.1)).

Now we are going to discuss the possible cases of the parameters α_1 and α_2.

Case 1. $\alpha_1 \neq 0$.

Case 1.1. Let $\alpha_1 \neq \beta_2$. Then taking the following change of basis:

$$x' = -\frac{1}{\alpha_1} x, \quad e_1' = e_1 - \frac{1}{\alpha_1} \sum_{i=2}^{n} \beta_i\, e_i,$$
$$e_i' = -\frac{1}{\alpha_1} \left((-\alpha_1 + \beta_2)\, e_i + \sum_{j=i+1}^{n} \beta_{j-i+2}\, e_j \right), \quad 2 \leqslant i \leqslant n,$$

we obtain

$$[e_1, e_1] = e_3, \quad [e_1, x] = \sum_{i=1}^{n} \mu_i\, e_i, \quad [e_i, e_1] = e_{i+1}, \quad 2 \leqslant i \leqslant n-1,$$
$$[x, e_1] = e_1, \quad [e_2, x] = \sum_{i=1}^{n} \eta_i\, e_i, \quad [x, e_2] = 0, \quad [x, x] = \sum_{i=1}^{n} \theta_i\, e_i.$$

From the equalities

$$0 = [[e_1, e_2], x] = [e_1, [e_2, x]] + [[e_1, x], e_2] = \left[e_1, \sum_{i=1}^{n} \eta_i\, e_i \right] = \eta_1\, e_3,$$

we get $\eta_1 = 0$.

Consider

$$[e_3, x] = [[e_1, e_1], x] = [e_1, [e_1, x]] + [[e_1, x], e_1]$$
$$= \mu_1 e_3 + (\mu_1 + \mu_2) e_3 + \sum_{i=3}^{n-1} \mu_i e_{i+1} = (2\mu_1 + \mu_2) e_3 + \sum_{i=3}^{n-1} \mu_i e_{i+1}.$$

On the other hand,

$$[e_3, x] = [[e_2, e_1], x] = [e_2, [e_1, x]] + [[e_2, x], e_1] = \mu_1 e_3 + \eta_2 e_3 + \sum_{i=3}^{n-1} \eta_i e_{i+1}$$
$$= (\mu_1 + \eta_2) e_3 + \sum_{i=3}^{n-1} \eta_i e_{i+1}.$$

The comparison of both linear combinations implies that:

$$\eta_2 = \mu_1 + \mu_2, \quad \eta_i = \mu_i, \qquad 3 \leqslant i \leqslant n-1,$$

that it is to say:

$$[e_2, x] = (\mu_1 + \mu_2) e_2 + \sum_{i=3}^{n-1} \mu_i e_i + \eta_n e_n \text{ and } [e_3, x] = (2\mu_1 + \mu_2) e_3 + \sum_{i=3}^{n-1} \mu_i e_{i+1}.$$

Now we shall prove the following equalities by an induction on i:

$$[e_i, x] = ((i-1)\mu_1 + \mu_2) e_i + \sum_{j=i+1}^{n} \mu_{j-i+2} e_j, \quad 3 \leqslant i \leqslant n. \tag{1}$$

Obviously, the equality holds for $i = 3$. Let us assume that the equality holds for $3 < i < n$, and we shall prove it for $i + 1$:

$$[e_{i+1}, x] = [[e_i, e_1], x] = [e_i, [e_1, x]] + [[e_i, x], e_1]$$
$$= \mu_1 e_{i+1} + ((i-1)\mu_1 + \mu_2) e_{i+1} + \sum_{j=i+2}^{n} \mu_{j-i+1} e_j$$
$$= (i\mu_1 + \mu_2) e_{i+1} + \sum_{j=i+2}^{n} \mu_{j-i+1} e_j\,;$$

so the induction proves the equalities (1) for any $i,\ 3 \leqslant i \leqslant n$.

Applying the Leibniz identity to the elements $\{e_1, x, e_1\},\ \{e_1, x, x\},\ \{x, e_1, x\}$, we deduce that:

$$\mu_1 = -1, \quad \mu_2 = \theta_1 = 0, \quad \theta_i = \mu_{i+1}, \qquad 2 \leqslant i \leqslant n-1.$$

Below, we summarize the table of multiplication of the algebra

$$\begin{cases} [e_1, e_1] = e_3 & \\ [e_i, e_1] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [e_1, x] = -e_1 + \sum_{i=3}^{n} \mu_i e_i, & \\ [e_2, x] = -e_2 + \sum_{i=3}^{n-1} \mu_i e_i + \eta_n e_n, & \\ [e_i, x] = -(i-1) e_i + \sum_{j=i+1}^{n} \mu_{j-i+2} e_j, \quad 3 \leqslant i \leqslant n, & \\ [x, e_1] = e_1, & [x, x] = \sum_{i=2}^{n-1} \mu_{i+1} e_i + \theta_n e_n. \end{cases}$$

Let us take the change of basis in the following form:

$$e_1' = e_1 + \sum_{i=3}^{n} A_i e_i, \;\; e_2' = e_2 + \sum_{i=3}^{n} A_i e_i, \;\; e_i' = e_i + \sum_{j=i+1}^{n} A_{j-i+2} e_j, \;\; 3 \leqslant i \leqslant n,$$

$$x' = \sum_{i=2}^{n-1} A_{i+1} e_i + B e_n + x,$$

where

$$A_3 = \mu_3, \quad A_i = \frac{1}{(i-2)} \left(\mu_i + \sum_{j=3}^{i-1} A_j \mu_{i-j+2} \right), \quad 4 \leqslant i \leqslant n, \quad \text{and}$$

$$B = \frac{1}{n-1} \left(\theta_n + \sum_{j=3}^{n} A_j \mu_{n-j+3} \right).$$

Then

$$[x', e_1'] = \left[\sum_{i=2}^{n-1} A_{i+1} e_i + B e_n + x, e_1 \right] = e_1 + \sum_{i=3}^{n} A_i e_i = e_1',$$

$$\begin{aligned}
[e_1', x'] &= [e_1, x] + \sum_{i=3}^{n} A_i [e_i, x] \\
&= -e_1 + \sum_{i=3}^{n} \mu_i e_i + \sum_{i=3}^{n} A_i \left(-(i-1) e_i + \sum_{j=i+1}^{n} \mu_{j-i+2} e_j \right) \\
&= -e_1 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n} \mu_i e_i - \sum_{i=3}^{n} A_i (i-2) e_i + \sum_{i=3}^{n} A_i \left(\sum_{j=i+1}^{n} \mu_{j-i+2} e_j \right) \\
&= -e_1 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n} \mu_i e_i - \sum_{i=3}^{n} A_i (i-2) e_i + \sum_{i=4}^{n} \left(\sum_{j=3}^{i-1} A_j \mu_{i-j+2} \right) e_i \\
&= -e_1 - \sum_{i=3}^{n} A_i e_i + (\mu_3 - A_3) e_3 \\
&\quad + \sum_{i=4}^{n} \left(-A_i (i-2) + \mu_i + \sum_{j=3}^{i-1} A_j \mu_{i-j+2} \right) e_i \\
&= -e_1 - \sum_{i=3}^{n} A_i e_i = -e_1',
\end{aligned}$$

$$\begin{aligned}
[e_2', x'] &= [e_2, x] + \sum_{i=3}^{n} A_i [e_i, x] \\
&= -e_2 + \sum_{i=3}^{n-1} \mu_i e_i + \eta_n e_n + \sum_{i=3}^{n} A_i \left(-(i-1) e_i + \sum_{j=i+1}^{n} \mu_{j-i+2}\, e_j \right) \\
&= -e_2 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n-1} \mu_i e_i + \eta_n e_n - \sum_{i=3}^{n} A_i (i-2) e_i + \sum_{i=3}^{n} A_i \left(\sum_{j=i+1}^{n} \mu_{j-i+2}\, e_j \right)
\end{aligned}$$

$$= -e_2 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n-1} \mu_i e_i + \eta_n e_n - \sum_{i=3}^{n} A_i (i-2) e_i + \sum_{i=4}^{n} \left(\sum_{j=3}^{i-1} A_j \mu_{i-j+2} \right) e_i$$

$$= -e_2 - \sum_{i=3}^{n} A_i e_i + (\mu_3 - A_3) e_3 + \sum_{i=4}^{n-1} \left(-A_i (i-2) + \mu_i + \sum_{j=3}^{i-1} A_j \mu_{i-j+2} \right) e_i$$

$$+ \left(\eta_n - (n-2) A_n + \sum_{i=3}^{n-1} A_i \mu_{n-i+2} \right) e_n = -e'_2 + \eta' e'_n,$$

$$[x', x'] = \sum_{i=2}^{n-1} A_{i+1} [e_i, x] + B[e_n, x] + [x, x]$$

$$= \sum_{i=2}^{n-1} A_{i+1} \left(-(i-1) e_i + \sum_{j=i+1}^{n} \mu_{j-i+2} e_j \right) - B(n-1) e_n + \sum_{i=2}^{n-1} \mu_{i+1} e_i + \theta_n e_n$$

$$= -\sum_{i=2}^{n-1} A_{i+1} (i-1) e_i + \sum_{i=2}^{n-1} \mu_{i+1} e_i - B(n-1) e_n + \theta_n e_n$$

$$+ \sum_{i=2}^{n-1} A_{i+1} \left(\sum_{j=i+1}^{n} \mu_{j-i+2} e_j \right)$$

$$= -\sum_{i=2}^{n-1} A_{i+1} (i-1) e_i + \sum_{i=2}^{n-1} \mu_{i+1} e_i - B(n-1) e_n + \theta_n e_n$$

$$+ \sum_{i=3}^{n} \left(\sum_{j=3}^{i} A_j \mu_{i-j+3} \right) e_i$$

$$= (\mu_3 - A_3) e_2 + \sum_{i=3}^{n-1} \left(-A_{i+1} (i-1) + \mu_{i+1} + \sum_{j=3}^{i} A_j \mu_{i-j+3} \right) e_i$$

$$+ \left(-B(n-1) + \theta_n + \sum_{j=3}^{n} A_j \mu_{n-j+3} \right) e_n = 0.$$

With a similar induction as the given for Eq. (1), it is easy to check that the following equalities hold:

$$[e_i, x] = -(i-1)\, e_i, \qquad 3 \leqslant i \leqslant n.$$

Thus, we obtain the following table of multiplication:

$$\begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [x, e_1] = e_1, & [e_1, x] = -e_1, & \\ [e_2, x] = -e_2 + \eta e_n, & [e_i, x] = -(i-1) e_i, & 3 \leqslant i \leqslant n. \end{cases}$$

If $\eta \neq 0$ then by taking the change of basis

$$e'_2 = e_2 + \frac{\eta}{n-2} e_n,$$

we get $\eta' = 0$.

Finally, by applying the change of basis $x' = -x$ and $e'_1 = e_1 - e_2$, we get the algebra R_1.

Case 1.2. Let $\alpha_1 = \beta_2$. Then by taking the following change of basis:

$$e_1' = e_1 - e_2, \quad e_i' = e_i, \quad 2 \leqslant i \leqslant n,$$

we can assume that the table of multiplication is the following

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [x, e_1] = -\alpha_1 e_1 + \sum\limits_{i=3}^{n} \beta_i e_i, \quad [e_1, x] = \alpha_1 e_1 + (\alpha_n - \beta) e_n, & \\ [e_2, x] = (\alpha_1 + \alpha_2) e_2 + \sum\limits_{i=3}^{n-1} \alpha_i e_i + \beta e_n, & \\ [e_i, x] = ((i-1)\alpha_1 + \alpha_2) e_i + \sum\limits_{j=i+1}^{n} \alpha_{j-i+2} e_j, & 3 \leqslant i \leqslant n, \\ [x, x] = \sum\limits_{i=1}^{n} \delta_i e_i. & \end{cases}$$

Now, by taking

$$x' = \frac{1}{\alpha_1} x - \frac{1}{\alpha_1} \sum_{i=2}^{n-1} \beta_{i+1} e_i,$$

and renaming the parameters, we get

$$\mathcal{F} : \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [x, e_1] = -e_1, \quad [e_1, x] = e_1 + \beta e_n, & \\ [e_2, x] = (1 + \alpha_2) e_2 + \sum\limits_{i=3}^{n-1} \alpha_i e_i + \lambda e_n, & \\ [e_i, x] = (i - 1 + \alpha_2) e_i + \sum\limits_{j=i+1}^{n} \alpha_{j-i+2} e_j, & 3 \leqslant i \leqslant n, \\ [x, x] = \sum\limits_{i=1}^{n} \delta_i e_i. & \end{cases}$$

Making the change of basis

$$x' = x, \quad e_1' = e_1, \quad e_i' = e_i + \sum_{j=i+1}^{n} A_{j-i+2} e_j, \quad 2 \leqslant i \leqslant n,$$

where

$$A_3 = -\alpha_3, \quad A_i = -\frac{1}{i-1}\Big(\alpha_i + \sum_{j=3}^{i-1} A_j \alpha_{i-j+2}\Big), \quad 4 \leqslant i \leqslant n-1,$$

$$A_n = -\frac{1}{n-2}\left(\lambda + \sum_{j=3}^{n-1} A_j \alpha_{n-j+2}\right),$$

and applying the Leibniz identity, we obtain of $\mathcal{F}$ the family of algebras

$$F(\alpha,\beta,\gamma):\begin{cases}[e_i,e_1]=e_{i+1}, & 2\leqslant i\leqslant n-1,\\ [x,e_1]=-e_1, \\ [e_1,x]=e_1+\beta e_n, \\ [e_i,x]=(i-1+\alpha)e_i, & 2\leqslant i\leqslant n,\\ [x,x]=-\beta e_{n-1}+\gamma e_n.\end{cases}$$

Let us take the general change of basis elements in the family $F(\alpha,\beta,\gamma)$,

$$e_1'=\sum_{i=1}^{n}A_ie_i,\quad e_2'=\sum_{i=1}^{n}B_ie_i,\quad x'=Cx+\sum_{i=1}^{n}P_ie_i,$$

we obtain in the new basis $\{e_1',e_2',\dots,e_n',x'\}$ the behavior of the parameters with the following expressions:

$$\alpha'=\alpha,\quad \beta'=\frac{A_1\beta+(n-2+\alpha)A_n}{A_1^{n-2}B_2},\quad \gamma'=\frac{\gamma A_1+(n-1+\alpha)(P_nA_1-P_1A_n)}{A_1^{n-3}B_2}.$$

Case 1.2.1. $\alpha\neq 2-n$. Taking

$$A_n=-\frac{A_1\beta}{n-2+\alpha},$$

we get $\beta'=0$.
Case 1.2.1.1. $\alpha\neq 1-n$. Taking

$$P_n=\frac{-\gamma A_1+(n-1+\alpha)P_1A_n}{(n-1+\alpha)A_1},$$

we have $\gamma'=0$ and hence the family $R_2(\alpha)$ with $\alpha\in\mathbb{C}\backslash\{2-n,1-n\}$.
Case 1.2.1.2. $\alpha=1-n$. Then

$$\gamma'=\frac{\gamma}{A_1^{n-4}B_2}.$$

If $\gamma\neq 0$, then taking $B_2=\frac{\gamma}{A_1^{n-4}}$, we get $\gamma'=1$ and thus we obtain the algebra R_3.

If $\gamma=0$, then we have the algebra $R_2(\alpha)$ with $\alpha=1-n$.
Case 1.2.2. $\alpha=2-n$. Then we have

$$\beta'=\frac{\beta}{A_1^{n-3}B_2},\qquad \gamma'=\frac{\gamma A_1+P_nA_1-P_1A_n}{A_1^{n-3}B_2}.$$

Set $P_n=\frac{-\gamma A_1+P_1A_n}{A_1}$, we get $\gamma'=0$.

If $\beta\neq 0$, then taking $B_2=\frac{\beta}{A_1^{n-3}}$, we get $\beta'=1$ and hence we obtain the algebra R_4.

If $\beta=0$, then we have the algebra $R_2(\alpha)$ with $\alpha=2-n$.

It is easy to check that any algebra of the family $F(\alpha,\beta,\gamma)$ is not isomorphic to the algebra R_1 applying a general change of basis.
Case 2. Let $\alpha_1=0$, $\alpha_2\neq 0$. Then making the following change of basis

$$x'=x-\sum_{i=2}^{n-1}\beta_{i+1}\,e_i,$$

we can assume that $[x,e_1]=\beta_2\,e_2$.

From the identity

$$[x,[x,e_1]] = [[x,x],e_1] - [[x,e_1],x],$$

we derive

$$0 = \sum_{i=3}^{n} \delta_{i-1}\, e_i - \beta_2 [e_2, x] = \sum_{i=3}^{n} \delta_{i-1}\, e_i - \beta_2 \left(\sum_{i=2}^{n-1} \alpha_i\, e_i + \beta\, e_n \right),$$

consequently, $\beta_2 = 0,\ \delta_i = 0,\ 2 \leqslant i \leqslant n-1$.

Making the change of basis

$$x' = x - \frac{\delta_n}{\alpha_2}\, e_n,$$

we can assume that $[x,x] = 0$.

Summarizing, we obtain the following table of multiplication of the algebra in this case

$$\begin{cases} [e_1,e_1] = e_3, \quad [e_i,e_1] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [e_1,x] = \sum\limits_{i=2}^{n} \alpha_i\, e_i, \quad [e_2,x] = \sum\limits_{i=2}^{n-1} \alpha_i\, e_i + \beta\, e_n, & \\ [e_i,x] = \sum\limits_{j=i}^{n} \alpha_{j-i+2}\, e_j, & 3 \leqslant i \leqslant n. \end{cases}$$

Now we shall study the behavior of the parameters in this family of algebras under the general change of basis of the form

$$\begin{cases} e_1' = \sum\limits_{i=1}^{n} A_i e_i, & \\ e_i' = A_1^{i-2} \left((A_1 + A_2)\, e_i + \sum\limits_{j=i+1}^{n} A_{j-i+2}\, e_j \right), & 2 \leqslant i \leqslant n, \\ x' = \sum\limits_{i=1}^{n} B_i e_i + B_{n+1} x, & \text{where } A_1(A_1+A_2)B_{n+1} \neq 0. \end{cases}$$

Then the equalities

$$0 = [x', e_1'] = \left[\sum_{i=1}^{n} B_i e_i + B_{n+1} x, A_1 e_1 \right] = A_1 \left((B_1 + B_2) e_3 + \sum_{i=4}^{n} B_{i-1} e_i \right)$$

imply $B_1 = -B_2,\ B_i = 0,\ 3 \leqslant i \leqslant n-1$.

Now we shall express the product $[e_1', x']$ as a linear combination of the basis $\{e_1, e_2, \ldots, e_n, x\}$, namely:

$$\begin{aligned} [e_1', x'] &= \left[\sum_{i=1}^{n} A_i e_i, B_1 e_1 + B_{n+1} x \right] \\ &= B_1 \left((A_1 + A_2) e_3 + \sum_{i=4}^{n} A_{i-1} e_i \right) \\ &\quad + B_{n+1} \left(A_1 \sum_{i=2}^{n} \alpha_i e_i + A_2 \left(\sum_{i=2}^{n-1} \alpha_i e_i + \beta e_n \right) + \sum_{i=3}^{n} A_i \sum_{j=i}^{n} \alpha_{j-i+2} e_j \right) \end{aligned}$$

$$
\begin{aligned}
&= B_1(A_1+A_2)e_3 + \sum_{i=4}^{n} B_1A_{i-1}e_i + B_{n+1}A_1\sum_{i=2}^{n}\alpha_i e_i \\
&\quad + B_{n+1}A_2\sum_{i=2}^{n-1}\alpha_i e_i + B_{n+1}A_2\beta e_n + B_{n+1}\sum_{i=3}^{n}\sum_{j=3}^{i}A_j\alpha_{i-j+2}e_i \\
&= B_{n+1}(A_1+A_2)\alpha_2\, e_2 + \Big((A_1+A_2)(B_1+B_{n+1}\alpha_3) + B_{n+1}A_3\alpha_2\Big)\, e_3 \\
&\quad + \sum_{i=4}^{n-1}\left(B_1A_{i-1} + B_{n+1}(A_1+A_2)\alpha_i + \sum_{j=3}^{i}B_{n+1}A_j\alpha_{i-j+2}\right)\, e_i \\
&\quad + \left(B_1A_{n-1} + B_{n+1}\left(A_1\alpha_n + A_2\beta + \sum_{i=3}^{n}A_i\alpha_{n-i+2}\right)\right)\, e_n.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
[e_1', x'] &= \sum_{i=2}^{n}\alpha_i' e_i' = \sum_{i=2}^{n}\alpha_i' A_1^{i-2}\left((A_1+A_2)e_i + \sum_{j=i+1}^{n}A_{j-i+2}e_j\right) \\
&= \sum_{i=2}^{n}\alpha_i' A_1^{i-2}(A_1+A_2)e_i + \sum_{i=3}^{n}\sum_{j=3}^{i}A_1^{i-j}A_j\alpha'_{i-j+2}e_i \\
&= \alpha_2'(A_1+A_2)e_2 + \Big(A_1(A_1+A_2)\alpha_3' + A_3\alpha_2'\Big)\, e_3 \\
&\quad + \sum_{i=4}^{n}\left(\alpha_i' A_1^{i-2}(A_1+A_2) + \sum_{j=3}^{i}A_1^{i-j}A_j\alpha'_{i-j+2}\right) e_i.
\end{aligned}
$$

Comparing coefficients at the basis elements in both combinations, we obtain the following relations:

$$
\begin{aligned}
&\alpha_2'(A_1+A_2) = B_{n+1}(A_1+A_2)\alpha_2, \\
&A_1(A_1+A_2)\alpha_3' + A_3\alpha_2' = (A_1+A_2)(B_1+B_{n+1}\alpha_3) + B_{n+1}A_3\alpha_2, \\
&\alpha_i' A_1^{i-2}(A_1+A_2) + \sum_{j=3}^{i}A_1^{i-j}A_j\alpha'_{i-j+2} = B_1A_{i-1} + B_{n+1}(A_1+A_2)\alpha_i \\
&\qquad\qquad\qquad\qquad + \sum_{j=3}^{i}B_{n+1}A_j\alpha_{i-j+2}, \quad 4 \leqslant i \leqslant n-1, \\
&A_1^{n-2}\alpha_n'(A_1+A_2) + \sum_{j=3}^{n}A_1^{n-j}A_j\alpha'_{n-j+2} = B_1A_{n-1} + B_{n+1}\left(A_1\alpha_n + A_2\beta + \sum_{i=3}^{n}A_i\alpha_{n-i+2}\right).
\end{aligned}
$$

The simplification of these relations implies the following identities:

$$
\alpha_2' = B_{n+1}\alpha_2, \quad \alpha_3' = \frac{B_1+\alpha_3B_{n+1}}{A_1}, \quad \alpha_i' = \frac{B_{n+1}\alpha_i}{A_1^{i-2}}, \qquad 4 \leqslant i \leqslant n-1,
$$
$$
\alpha_n' = \frac{(\alpha_nA_1+\beta A_2)B_{n+1}}{A_1^{n-2}(A_1+A_2)}.
$$

Analogously, considering the product $[e_2', x']$, we get the relation:

$$
\beta' = \frac{\beta B_{n+1}}{A_1^{n-2}},
$$

and

$$[x', x'] = -\frac{(\beta B_1 - \alpha_n B_1 - \alpha_2 B_n)B_{n+1}}{A_1^{n-2}(A_1 + A_2)} e_n.$$

Since $[x', x'] = 0$, then $B_n = \dfrac{\beta B_1 - \alpha_n B_1}{\alpha_2}$.

Setting $B_{n+1} = 1/\alpha_2$ and $B_1 = -\alpha_3/\alpha_2$, then we derive that $\alpha'_2 = 1,\ \alpha'_3 = 0$.

If $\beta = 0$ and $\alpha_n = 0$, then $\beta' = 0$ and thus we obtain the algebra $R_5(\alpha_4, \ldots, \alpha_{n-1})$.

If $\beta = 0$ and $\alpha_n \neq 0$, then putting $A_2 = \dfrac{\alpha_n - \alpha_2 A_1^{n-2}}{\alpha_2 A_1^{n-3}}$, we have $\alpha'_n = 1$ and hence we obtain the algebra $R_6(\alpha_4, \ldots, \alpha_{n-1})$.

If $\beta \neq 0$, then choosing

$$A_1 = \sqrt[n-2]{\frac{\beta}{\alpha_2}}, \qquad A_2 = -\frac{A_1 \alpha_n}{\beta},$$

we obtain $\beta' = 1,\ \alpha'_n = 0$ and the algebra $R_7(\alpha_4, \ldots, \alpha_{n-1})$. $\square$

Now we shall consider the case when the dimension of a solvable Leibniz algebra with nilradical F_n^1 is equal to $n + 2$.

Theorem 4.3. *There does not exist any $(n + 2)$-dimensional solvable Leibniz algebra with nilradical F_n^1.*

Proof. From the conditions of the theorem, we have the existence of a basis $\{e_1, e_2, \ldots, e_n, x, y\}$ such that the table of multiplication of F_n^1 remains the same. The outer non-nilpotent derivations of F_n^1, denoted by $\mathcal{R}_{x_{|F_n^1}}$ and $\mathcal{R}_{y_{|F_n^1}}$, are of the form given in Proposition 4.1, with the set of entries $\{\alpha_i, \gamma\}$ and $\{\beta_i, \delta\}$, respectively, where $[e_i, x] = \mathcal{R}_{x_{|F_n^1}}(e_i)$ and $[e_i, y] = \mathcal{R}_{y_{|F_n^1}}(e_i)$.

Taking the following change of basis:

$$x' = \frac{\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1}x - \frac{\alpha_2}{\alpha_1\beta_2 - \alpha_2\beta_1}y, \quad y' = -\frac{\beta_1}{\alpha_1\beta_2 - \alpha_2\beta_1}x + \frac{\alpha_1}{\alpha_1\beta_2 - \alpha_2\beta_1}y, \tag{2}$$

we may assume that $\alpha_1 = \beta_2 = 1$ and $\alpha_2 = \beta_1 = 0$.

Therefore we have the products

$$[e_1, x] = e_1 + \sum_{i=3}^{n} \alpha_i e_i, \qquad [e_2, x] = e_2 + \sum_{i=3}^{n-1} \alpha_i e_i + \gamma e_n,$$

$$[e_i, x] = (i-1)e_i + \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j, \qquad 3 \leqslant i \leqslant n,$$

$$[e_1, y] = e_2 + \sum_{i=3}^{n} \beta_i e_i, \qquad [e_2, y] = e_2 + \sum_{i=3}^{n-1} \beta_i e_i + \delta e_n,$$

$$[e_i, y] = e_i + \sum_{j=i+1}^{n} \beta_{j-i+2} e_j, \qquad 3 \leqslant i \leqslant n.$$

Applying similar arguments as in Case 1 of Theorem 4.2 and taking into account that the products $[e_1, y],\ [e_2, y],\ [e_i, y]$ will not be changed under the transformations of bases which were used there, we obtain the products:

$$[e_1, x] = e_1, \quad [e_2, x] = e_2 + \gamma e_n, \quad [e_i, x] = (i-1)e_i, \quad 3 \leqslant i \leqslant n, \quad [x, e_1] = -e_1.$$

Let us introduce the notations:

$$[y, e_1] = \sum_{i=1}^{n} \eta_i e_i, \quad [y, e_2] = \sum_{i=1}^{n} \theta_i e_i, \quad [y, x] = \sum_{i=1}^{n} \rho_i e_i, \quad [y, y] = \sum_{i=1}^{n} \tau_i e_i, \quad [x, y] = \sum_{i=1}^{n} \sigma_i e_i.$$

From the Leibniz identity

$$[e_1, [y, e_1]] = [[e_1, y], e_1] - [[e_1, e_1], y],$$

we get $\eta_1 = 0$.

Note that we can assume $[y, e_1] = \eta_2 e_2$ (by changing $y' = y - \sum_{i=2}^{n-1} \eta_{i+1} e_i$).

Due to

$$[y, [e_1, e_2]] = [[y, e_1], e_2] - [[y, e_2], e_1],$$

we obtain $\theta_2 = -\theta_1, \ \theta_i = 0, \ 3 \leqslant i \leqslant n-1$.

Since $[e_1, [y, e_2]] = [[e_1, y], e_2] - [[e_1, e_2], y]$, then we have $\theta_1 = \theta_2 = 0$. Moreover, the Leibniz identity $[y, [y, e_2]] = [[y, y], e_2] - [[y, e_2], y]$ implies that $\theta_n = 0$, i.e., $[y, e_2] = 0$.

From the following chain of equalities

$$\begin{aligned} 0 &= \eta_2 [y, e_2] = [y, \eta_2 e_2] = [y, [y, e_1]] = [[y, y], e_1] - [[y, e_1], y] \\ &= (\tau_1 + \tau_2) e_3 + \sum_{i=4}^{n} \tau_{i-1} e_i - \eta_2 [e_2, y] \\ &= (\tau_1 + \tau_2) e_3 + \sum_{i=4}^{n} \tau_{i-1} e_i - \eta_2 \left(e_2 + \sum_{i=3}^{n-1} \beta_i e_i + \delta e_n \right), \end{aligned}$$

we derive that

$$\eta_2 = 0, \quad \tau_2 = -\tau_1, \quad \tau_i = 0, \qquad 3 \leqslant i \leqslant n-1.$$

Therefore, we have $[y, e_1] = 0$ and $[y, y] = \tau_1 e_1 - \tau_1 e_2 + \tau_n e_n$.

Considering the Leibniz identity

$$[x, [y, e_1]] = [[x, y], e_1] - [[x, e_1], y],$$

then we get

$$-e_2 - \sum_{i=3}^{n} \beta_i e_i = (\sigma_1 + \sigma_2) e_3 + \sum_{i=3}^{n-1} \sigma_i e_{i+1}.$$

Thus, we have a contradiction with the assumption of the existence of an algebra under the conditions of the theorem. $\square$

4.2. Solvable Leibniz algebras with nilradical F_n^2

In this section we describe solvable Leibniz algebras with nilradical F_n^2, i.e. solvable Leibniz algebras R which decompose in the form $R = F_n^2 \oplus Q$.

Proposition 4.4. *An arbitrary derivation of the algebra F_n^2 has the following matrix form:*

$$D=\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_{n-1} & \alpha_n \\ 0 & \beta & 0 & 0 & \dots & 0 & \gamma \\ 0 & 0 & 2\alpha_1 & \alpha_3 & \dots & \alpha_{n-2} & \alpha_{n-1} \\ 0 & 0 & 0 & 3\alpha_1 & \dots & \alpha_{n-3} & \alpha_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & (n-1)\alpha_1 \end{pmatrix}.$$

Proof. The proof follows by straightforward calculations in a similar way as the proof of Proposition 4.1. □

Remark 4.5. It is an easy task to check that the number of nil-independent derivations of the algebra F_n^2 is equal to 2.

Corollary 4.6. *The dimension of a solvable Leibniz algebra with nilradical F_n^2 is either $n+1$ or $n+2$.*

Theorem 4.7. *An $(n+1)$-dimensional solvable Leibniz algebra with nilradical F_n^2 is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$R_1(\alpha):\begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\leqslant i\leqslant n-1, \\ [e_1,x]=-e_1, & [e_i,x]=-(i-1)\,e_i, \quad 3\leqslant i\leqslant n, \\ [x,e_1]=e_1, & [x,x]=\alpha\, e_2, \quad \alpha\in\{0,1\}. \end{cases}$$

$$R_2(\alpha):\begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\leqslant i\leqslant n-1, \\ [e_1,x]=-e_1, & [e_i,x]=-(i-1)\,e_i, \quad 3\leqslant i\leqslant n, \\ [x,e_1]=e_1, & [e_2,x]=\alpha\, e_2, \quad \alpha\neq 0. \end{cases}$$

$$R_3:\begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\leqslant i\leqslant n-1, \\ [e_1,x]=-e_1, & [e_i,x]=-(i-1)\,e_i, \quad 3\leqslant i\leqslant n, \\ [x,e_1]=e_1, & [e_2,x]=(1-n)\,e_2+e_n. \end{cases}$$

$$R_4(\alpha):\begin{cases} [e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\leqslant i\leqslant n-1, \\ [e_1,x]=-e_1, & [e_i,x]=-(i-1)\,e_i, \quad 3\leqslant i\leqslant n, \\ [x,e_1]=e_1, & [e_2,x]=-\alpha\, e_2, \quad \alpha\neq 1, \\ [x,e_2]=\alpha\, e_2. \end{cases}$$

$$R_5(\alpha):\begin{cases} [e_1,e_1]=e_3, \\ [e_1,x]=-e_1-\alpha\, e_2, \quad \alpha\in\{0,1\}, \quad [e_2,x]=-e_2, \\ [e_i,e_1]=e_{i+1}, \quad 3\leqslant i\leqslant n-1, \\ [e_i,x]=-(i-1)\,e_i, \quad 3\leqslant i\leqslant n, \\ [x,e_1]=e_1+\alpha\, e_2, \quad [x,e_2]=e_2. \end{cases}$$

$$R_6(\alpha_3, \alpha_4, \ldots, \alpha_n, \lambda, \delta): \begin{cases} [e_1, e_1] = e_3, \\ [e_1, x] = \sum\limits_{i=3}^{n} \alpha_i e_i, \qquad [e_2, x] = e_2, \\ [e_i, e_1] = e_{i+1}, \qquad 3 \leqslant i \leqslant n-1, \\ [e_i, x] = \sum\limits_{j=i+1}^{n} \alpha_{j-i+2} e_j, \qquad 3 \leqslant i \leqslant n-1, \\ [x, x] = \lambda e_n, \qquad [x, e_2] = \delta e_2, \quad \delta \in \{0, -1\}. \end{cases}$$

In the algebra $R_6(\alpha_3, \alpha_4, \ldots, \alpha_n, \lambda, \delta)$ the first non vanishing parameter $\{\alpha_3, \alpha_4, \ldots, \alpha_n, \lambda\}$ can be scaled to 1.

Proof. Let R be a solvable Leibniz algebra satisfying the conditions of the theorem, then there exists a basis $\{e_1, e_2, \ldots, e_n, x\}$, such that $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of F_n^2, and for non nilpotent outer derivations of the algebra F_n^2, we have that $[e_i, x] = \mathcal{R}_{x_{|F_n^2}}(e_i), \ 1 \leqslant i \leqslant n$,

Due to Proposition 4.4 we can assume that

$$[e_1, x] = \sum_{i=1}^{n} \alpha_i e_i, \qquad [e_2, x] = \beta_2 e_2 + \beta_n e_n,$$

$$[e_i, x] = (i-1)\alpha_1 e_i + \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j, \quad 3 \leqslant i \leqslant n.$$

Let us introduce the following notations:

$$[x, e_1] = \sum_{i=1}^{n} \gamma_i e_i, \quad [x, e_2] = \sum_{i=1}^{n} \delta_i e_i, \quad [x, x] = \sum_{i=1}^{n} \lambda_i e_i.$$

Considering the Leibniz identity for the elements $\{e_1, x, x\}$, $\{e_1, x, e_1\}$, $\{x, e_2, e_1\}$, we obtain $\lambda_1 = 0, \ \gamma_1 = -\alpha_1$ and $[x, e_2] = \delta_2 e_2 + \delta_n e_n$. By setting $e_2' = \delta_2 e_2 + \delta_n e_n$, we can assume that $[x, e_2] = \delta e_2$.

Now we distinguish the following possible cases:

Case 1. Let $\alpha_1 \neq 0$. Then the following change of basis

$$e_1' = e_1 + \frac{1}{\gamma_1} \sum_{j=3}^{n} \gamma_j e_j, \quad e_2' = e_2, \quad e_i' = e_i + \frac{1}{\gamma_1} \sum_{j=i+1}^{n} \gamma_{j-i+2} e_j, \quad 3 \leqslant i \leqslant n,$$

$$x' = \frac{1}{\gamma_1} x,$$

implies that $[x', e_1'] = e_1' + \gamma e_2'$ (where $\gamma = \frac{\gamma_2}{\gamma_1}$) and the rest of products remains unchanging.

From the equalities:

$$\begin{aligned} e_1 + \gamma(1+\delta) e_2 = [x, [x, e_1]] &= [[x, x], e_1] - [[x, e_1], x] \\ &= \sum_{i=4}^{n} \lambda_{i-1} e_i - \sum_{i=1}^{n} \alpha_i e_i - \gamma \beta_2 e_2 - \gamma \beta_n e_n, \end{aligned}$$

we deduce that

$$\alpha_1 = -1, \ \alpha_3 = 0, \ \alpha_2 = -\gamma(1 + \delta + \beta_2), \ \lambda_i = \alpha_{i+1}, \ 3 \leqslant i \leqslant n-2, \quad \text{and}$$
$$\lambda_{n-1} = \alpha_n + \gamma \beta_n.$$

In addition, if we take the following change of basis:

$$e_1' = e_1 + \sum_{i=4}^{n} A_i e_i, \quad e_2' = e_2, \quad e_i' = e_i + \sum_{j=i+2}^{n} A_{j-i+2} e_j, \quad 3 \leqslant i \leqslant n,$$
$$x' = \sum_{i=3}^{n-1} A_{i+1} e_i + B e_n + x,$$

where $A_j = \frac{1}{2}\alpha_j,\ j = 4, 5,\quad A_i = \frac{1}{i-2}(\alpha_i + \sum_{j=4}^{i-2} A_j \alpha_{i-j+2}),\quad 6 \leqslant i \leqslant n$ and $B = \frac{1}{n-1}(\lambda_n + \sum_{j=4}^{n-1} A_j \alpha_{i-j+3})$, then we have

$$[e_1', x'] = -e_1' + \alpha_2 e_2', \qquad [e_2', x'] = \beta_2 e_2' + \beta_n e_n',$$
$$[e_i', x'] = -(i-1) e_i', \quad 3 \leqslant i \leqslant n, \qquad [x', x'] = \lambda_2 e_2' + \gamma\beta_n e_{n-1}'.$$

Finally, we obtain the following table of multiplication of the algebra R:

$$\begin{cases} [e_1, x] = -e_1 - \gamma(1+\delta+\beta_2) e_2, & [e_2, x] = \beta_2 e_2 + \beta_n e_n, \\ [x, e_1] = e_1 + \gamma e_2, & [e_i, x] = -(i-1) e_i, \qquad 3 \leqslant i \leqslant n, \\ [x, e_2] = \delta e_2, & [x, x] = \lambda_2 e_2 + \gamma\beta_n e_{n-1}. \end{cases}$$

Considering the Leibniz identity for the elements $\{x, x, e_2\},\ \{x, x, x\},\ \{x, e_1, x\}$, we obtain:

$$\delta\beta_n = \delta(\delta + \beta_2) = \delta\lambda_2 = \gamma\delta(\delta + \beta_2) = 0.$$

Notice that if $e_2 \in \mathrm{Ann}_r(R)$, then $\dim \mathrm{Ann}_r(R) = n-1$ and if $e_2 \notin \mathrm{Ann}_r(R)$, then $\dim \mathrm{Ann}_r(R) = n-2$.

Now we analyze the following possible subcases:

Case 1.1. Let $e_2 \in \mathrm{Ann}_r(R)$. Then $\delta = 0$ and making the change $e_1' = e_1 + \gamma e_2$ we can assume that $[x, e_1] = e_1$.

In this case, we must consider two new subcases:

Case 1.1.1. Let $e_2 \in \mathrm{Center}(R)$. Then $\dim \mathrm{Center}(R) = 1$ and $\beta_2 = \beta_n = 0$. Then we have two options: if $\lambda_2 = 0$, then we get the split algebra $R_1(0)$; if $\lambda_2 \neq 0$, then we obtain the algebra $R_1(1)$ by scaling the basis.

Case 1.1.2. Let $e_2 \notin \mathrm{Center}(R)$. Then $\dim \mathrm{Center}(R) = 0$ and $(\beta_2, \beta_n) \neq (0, 0)$.

Let us take the following general change of basis:

$$e_1' = \sum_{i=1}^{n} A_i e_i,\ e_2' = \sum_{i=1}^{n} B_i e_i,\ e_i' = A_1^{i-2}\left(A_1 e_i + \sum_{j=i+1}^{n} A_{j-i+2} e_i\right),\ 3 \leqslant i \leqslant n,$$
$$x' = \sum_{i=1}^{n} C_i e_i + C_{n+1} x,$$

where $(A_1B_2 - A_2B_1)C_{n+1} \neq 0$.

From $0 = [e_2', e_1'] = [e_2', e_2']$, we obtain that $B_1 = 0,\ B_i = 0,\ 3 \leqslant i \leqslant n-1$, i.e. $e_2' = B_2 e_2 + B_n e_n$ and $A_1B_2 \neq 0$.

The equalities

$$e_1' = [x', e_1'] = A_1C_1 e_3 + \sum_{i=4}^{n} A_1C_{i-1} e_i + A_1C_{n+1} e_1,$$

imply that

$$C_{n+1} = 1, \qquad A_2 = 0, \qquad A_3 = A_1C_1, \qquad A_i = A_1C_{i-1}, \quad 4 \leqslant i \leqslant n.$$

Similarly, from

$$B_2\beta_2'\,e_2+(B_n\beta_2'+\beta_n'A_1^{n-1})\,e_n=\beta_2'\,e_2'+\beta_n'\,e_n'=[e_2',x']$$
$$=B_2\beta_2\,e_2+\big(B_2\beta_n-(n-1)B_n\big)\,e_n,$$

and

$$\lambda_2'B_2\,e_2+\lambda_2'B_n\,e_n=\lambda_2'\,e_2'=[x',x']=(\lambda_2+C_2\beta_2)\,e_2+(C_1^2-2C_3)\,e_3$$
$$+\sum_{i=4}^{n-1}\big(C_1C_{i-1}-(i-1)C_i\big)\,e_i+\big(C_1C_{n-1}-(n-1)C_n+C_2\beta_n\big)\,e_n,$$

we obtain $C_i=\dfrac{1}{(i-1)!}C_1^{i-1},\ 3\leqslant i\leqslant n-1$ and

$$\beta_2'=\beta_2,\quad \beta_n'=\frac{B_2\beta_n-B_n(\beta_2+n-1)}{A_1^{n-1}},\quad \lambda_2'=\frac{\lambda_2+\beta_2C_2}{B_2},$$
$$\lambda_2'B_n=C_1C_{n-1}-(n-1)C_n+C_2\beta_n.$$

Now we must distinguish two subcases:

Case 1.1.2.1. Let $\beta_2=1-n$. Putting $C_2=-\frac{\lambda_2}{1-n},\ C_n=\frac{C_1C_{n-1}+C_2\beta_n}{n-1}$, then we get $\lambda_2'=0$ and $\beta_n'=\frac{B_2\beta_n}{A_1^{n-1}}$.

If $\beta_n=0$, then we get the algebra $R_2(\alpha)$ for $\alpha=1-n$.

If $\beta_n\neq 0$, then making $A_1=\sqrt[n-1]{\beta_nB_2}$, we obtain $\beta_n'=1$ and the algebra R_3.

Case 1.1.2.2. Let $\beta_2\neq 1-n$. Taking the change $B_n=\frac{B_2\beta_n}{\beta_2+n-1}$, we obtain $\beta_n=0$. Since $\beta_2\neq 0$, we set $C_2=-\frac{\lambda_2}{\beta_2},\ C_n=\frac{C_1C_{n-1}+C_2\beta_n}{n-1}$ and we get $\lambda_2=0$, i.e., the algebra $R_2(\alpha)$ is obtained, for $\alpha\notin\{1-n,0\}$.

Case 1.2. Let $e_2\notin \mathrm{Ann}_r(R)$. Then $\delta\neq 0$ and $\beta_2=-\delta,\ \beta_n=\lambda_2=0$.

Let us consider the general change of basis in the following form:

$$e_1'=\sum_{i=1}^{n}A_i\,e_i,\quad e_2'=\sum_{i=1}^{n}B_i\,e_i,$$
$$e_i'=A_1^{i-2}\left(A_1\,e_i+\sum_{j=i+1}^{n}A_{j-i+2}\,e_j\right),\ 3\leqslant i\leqslant n,\quad x'=\sum_{i=1}^{n}C_i\,e_i+C_{n+1}\,x,$$

where $(A_1B_2-A_2B_1)C_{n+1}\neq 0$.

Then from $0=[e_2',e_1']=[e_2',e_2']$, we derive that $B_1=0,\ B_i=0,\ 3\leqslant i\leqslant n-1$, i.e. $e_2'=B_2\,e_2+B_n\,e_n$ and $A_1B_2\neq 0$.

Similarly, from the equations:

$$e_1'+\gamma'\,e_2'=[x',e_1']=A_1C_{n+1}\,e_1+C_{n+1}(A_1\gamma+A_2\delta)\,e_2+A_1C_1\,e_3+\sum_{i=4}^{n}A_1C_{i-1}\,e_i,$$

and

$$\delta'(B_2\,e_2+B_n\,e_n)=\delta'\,e_2'=[x',e_2']=B_2\delta\,e_2,$$

we obtain

$$C_{n+1}=1,\quad A_3=A_1C_1,\quad A_i=A_1C_{i-1},\qquad 4\leqslant i\leqslant n-1,$$
$$\gamma'=\frac{A_1\gamma+A_2(\delta-1)}{B_2},\quad A_1C_{n-1}=A_n+\gamma'B_n,\qquad \delta'=\delta,\ \delta'B_n=0.$$

Now we distinguish the following two subcases:

Case 1.2.1. Let $\delta \neq 1$. Then by the substitution $A_2 = -\dfrac{A_1\gamma}{\delta - 1},\ A_n = A_1 C_{n-1}$ into the above conditions, we get $\gamma' = 0$ and the algebra $R_4(\alpha)$.

Case 1.2.2. Let $\delta = 1$. Then $B_n = 0$. In the case $\gamma = 0$, we get $\gamma' = 0$. In the case $\gamma \neq 0$, by putting $B_2 = A_1\gamma$ and $A_n = A_1 C_{n-1} - B_n$, we get $\gamma' = 1$. Thus, the algebras $R_5(\alpha),\ \alpha \in \{0, 1\}$, are obtained.

Case 2. Let $\alpha_1 = 0$. Then $\beta_2 \neq 0$ and by replacing x by $x' = \dfrac{1}{\beta_2}x$, we can assume $[e_2, x'] = e_2 + \beta_n e_n$.

Under these conditions, the table of multiplication of the solvable algebra R has the form:

$$\begin{cases} [e_1, x] = \sum\limits_{i=2}^{n} \alpha_i e_i, \quad [e_2, x] = e_2 + \beta_n e_n, \\ [x, e_1] = \sum\limits_{i=2}^{n} \gamma_i e_i, \quad [e_i, x] = \sum\limits_{j=i+1}^{n} \alpha_{j-i+2} e_j, \quad 3 \leqslant i \leqslant n-1, \\ [x, e_2] = \delta e_2, \qquad [x, x] = \sum\limits_{i=2}^{n} \lambda_i e_i. \end{cases}$$

Making the transformation $x' = x - \gamma_3 e_1 - \sum\limits_{i=3}^{n-1} \gamma_{i+1} e_i$, we can assume that $[x, e_1] = \gamma e_2$.

Similarly as above, we obtain the conditions:

$$\gamma(\delta + 1) = \alpha_2\delta - \gamma = \beta_n\delta = \delta(\delta + 1) = \lambda_2\delta = 0.$$

Now we distinguish the following subcases depending on the possible values of the parameter δ:

Case 2.1. Let $\delta \neq 0$. Then $\dim \mathrm{Ann}_r(R) = n - 2$ and $\beta_n = \lambda_2 = 0,\ \delta = -1,\ \alpha_2 = -\gamma$. By means of the change of the basis element $e_1' = e_1 + \gamma e_2$, we can suppose that $[x', e_1] = 0$.

Taking the general change of basis as in the above considered cases, we derive the following conditions for the parameters

$$\alpha_i' = \frac{\alpha_i}{A_1^{i-2}}, \quad 3 \leqslant i \leqslant n, \qquad \lambda_n' = \frac{\lambda_n}{A_1^{n-1}}.$$

Consequently, we deduce the algebra $R_6(\alpha_3, \alpha_4, \ldots, \alpha_n, \lambda, -1)$.

Case 2.2. Let $\delta = 0$. Then $\dim \mathrm{Ann}_r(R) = n-1$ and $\gamma = 0$. Taking the change of basis $e_2' = e_2 + \beta_n e_n$, we can assume that $[e_2, x] = e_2$ and by the change $x' = x - \lambda_2 e_2$, we can also suppose that $[x, x] = \lambda_n e_n$. Therefore, we have the products

$$[e_1, x] = \sum_{i=2}^{n} \alpha_i e_i, \quad [e_2, x] = e_2, \quad [e_i, x] = \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j, \quad 3 \leqslant i \leqslant n-1,$$
$$[x, x] = \lambda_n e_n.$$

Applying similar arguments to general transformation of bases, we have

$$\alpha_2' = 0, \qquad \alpha_i' = \frac{\alpha_i}{A_1^{i-2}}, \ 3 \leqslant i \leqslant n, \qquad \lambda_n' = \frac{\lambda_n}{A_1^{n-1}}.$$

Thus, we obtain the algebra $R_6(\alpha_3, \alpha_4, \ldots, \alpha_n, \lambda, 0)$. $\square$

Theorem 4.8. *An arbitrary $(n+2)$-dimensional solvable Leibniz algebra with nilradical F_n^2 is isomorphic to one of the following non isomorphic algebras:*

$$L_1:\begin{cases}[e_1,e_1]=e_3, & [e_i,e_1]=e_{i+1}, \quad 3\leqslant i\leqslant n-1,\\ [e_1,x]=e_1, & [x,e_1]=-e_1,\\ [e_2,y]=-[y,e_2]=e_2, & [e_i,x]=(i-1)e_i, \quad 3\leqslant i\leqslant n,\end{cases}$$

$$L_2:\begin{cases}[e_1,e_1]=e_3, \quad [e_i,e_1]=e_{i+1}, & 3\leqslant i\leqslant n-1,\\ [e_1,x]=e_1, \quad [x,e_1]=-e_1,\\ [e_2,y]=e_2, \quad [e_i,x]=(i-1)e_i, & 3\leqslant i\leqslant n.\end{cases}$$

Proof. Let

$$\mathcal{R}_{x_{|F_n^2}}=\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3&\alpha_4&\dots&\alpha_{n-1}&\alpha_n\\ 0&\beta&0&0&\dots&0&\gamma\\ 0&0&2\alpha_1&\alpha_3&\dots&\alpha_{n-2}&\alpha_{n-1}\\ 0&0&0&3\alpha_1&\dots&\alpha_{n-3}&\alpha_{n-2}\\ \vdots&\vdots&\vdots&\vdots&\dots&\vdots&\vdots\\ 0&0&0&0&\dots&0&(n-1)\alpha_1\end{pmatrix}$$

and

$$\mathcal{R}_{y_{|F_n^2}}=\begin{pmatrix}\lambda_1&\lambda_2&\lambda_3&\lambda_4&\dots&\lambda_{n-1}&\lambda_n\\ 0&\mu&0&0&\dots&0&\nu\\ 0&0&2\lambda_1&\lambda_3&\dots&\lambda_{n-2}&\lambda_{n-1}\\ 0&0&0&3\lambda_1&\dots&\lambda_{n-3}&\lambda_{n-2}\\ \vdots&\vdots&\vdots&\vdots&\dots&\vdots&\vdots\\ 0&0&0&0&\dots&0&(n-1)\lambda_1\end{pmatrix}$$

be two nil independent outer derivations of the algebra F_n^2.

Taking the change of the basis elements x, y similar to (2), we can assume that $\alpha_1=\mu=1,\ \lambda_1=\beta=0$.

Thus, we have the products:

$$[e_1,x]=e_1+\sum_{i=2}^{n}\alpha_i e_i, \qquad [e_2,x]=\gamma e_n,$$

$$[e_i,x]=(i-1)e_i+\sum_{j=i+1}^{n}\alpha_{j-i+2}e_j, \qquad 3\leqslant i\leqslant n,$$

$$[e_1,y]=\sum_{i=2}^{n}\lambda_i e_i, \qquad [e_2,y]=e_2+\nu e_n,$$

$$[e_i,y]=\sum_{j=i+1}^{n}\lambda_{j-i+2}e_j, \qquad 3\leqslant i\leqslant n.$$

Applying similar reasonings and changes of bases which we have used in Theorem 4.7, we obtain isomorphism classes of algebras whose representative elements are L_1 and L_2. □

Remark 4.9. In fact, the algebra $L_1 = I_1 \oplus J_1$, where $I_1 = NF_{n-1} + \langle x \rangle$ and $J_1 = \langle e_2, y \rangle$, verifies that I_1 is a solvable Leibniz algebra with nilradical NF_{n-1} and J_1 is a two-dimensional solvable Lie algebra. The algebra $L_2 = I_2 \oplus J_2$, where $I_2 = NF_{n-1} + \langle x \rangle$ and $J_2 = \langle e_2, y \rangle$, verifies that J_2 is a two-dimensional solvable non-Lie Leibniz algebra. Thus, from Theorem 4.8, we conclude that any $(n+2)$-dimensional solvable Leibniz algebra with nilradical F_n^2 is split.

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