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Classification of solvable Leibniz algebras with null-filiform nilradical

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In this article we classify solvable Leibniz algebras whose nilradical is a null-filiform algebra. We extend the obtained classification to the case when the solvable Leibniz algebra is decomposed as a direct sum of its nilradical, which is a direct sum of null-filiform ideals and a one-dimensional complementary subspace. Moreover, in this case we establish that these ideals are ideals of the algebra as well.

Keywords: Leibniz algebra; null-filiform algebra; solvability; nilpotence; nilradical

AMS Subject Classifications: 17A32; 17A65; 17B30

1. Introduction

The notion of Leibniz algebra was first introduced by Loday [10] as a non-antisymmetric generalization of Lie algebras. During the last 20 years, the theory of Leibniz algebras has been actively studied and many results of the theory of Lie algebras have been extended to Leibniz algebras. For instance, the classical results on Cartan subalgebras, regular elements and others from the theory of Lie algebras are also true for Leibniz algebras [1,14].

From the classical theory of finite-dimensional Lie algebras it is known that an arbitrary Lie algebra is a semidirect sum of the solvable radical and a semisimple subalgebra (Levi's theorem). In addition, the semisimple part is a direct sum of simple ideals, which is completely classified [9]. Thanks to Malcev's results [11], the study of solvable Lie algebras is reduced to the study of nilpotent ones. Thus, the description of finite-dimensional Lie algebras is reduced to the description of nilpotent algebras.

In the case of Leibniz algebras, the analogue of Levi's theorem was proved in [6]. Namely, a Leibniz algebra is a semidirect sum of the solvable radical and a semisimple Lie algebra. As the semisimple part can be described by simple Lie ideals, the main problem is to understand the solvable radical. Thus, it is important to study

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solvable Leibniz algebras. The inherent properties of non-Lie Leibniz algebras imply that the subspace spanned by squares of elements of the algebra is a non-trivial Abelian ideal. In fact, this ideal is the minimal one such that the quotient algebra is a Lie algebra. Thus, we also restrict our study of Leibniz algebras to the solvable ones.

The investigation of solvable Lie algebras with some special types of nilradical comes from different problems in Physics and was the subject of various papers [2,3,7,8,13,15–17,19] and many other references given therein. Also, it is natural to add restrictions to the index of nilpotency and graduation on the nilradical. For example, the cases where the nilradical of a solvable Lie algebra is filiform, the quasi-filiform and abelian were considered [3,8,13,19]. We recall that the maximal index of nilpotency of an n-dimensional Lie algebra is n (such algebras were called filiform in [18]). However, the maximal index of nilpotency of an n-dimensional Leibniz algebra is equal to $n+1$ (such algebras were called null-filiform in [5]).

Our goal in this article is to classify solvable Leibniz algebras with null-filiform nilradical. Moreover, this classification is extended to the case when the nilradical is a direct sum of null-filiform ideals and the complementary vector space of the nilradical is one-dimensional.

This article is organized as follows. In Section 2 we recall some needed notions and properties of Leibniz algebras. We start Section 3 by establishing that the dimension of a solvable Leibniz algebra whose nilradical is an n-dimensional null-filiform Leibniz algebra is exactly $n+1$; after that, we present our main results: the classification of solvable Leibniz algebras that can be decomposed as a direct sum of their nilradical and a complementary vector space, where the nilradical is a direct sum of null-filiform Leibniz algebras. First, we study the case when the solvable Leibniz algebra is a direct sum of its nilradical and a one-dimensional complementary vector space, where the nilradical is null-filiform; after that we consider the case where the nilradical decomposes in a direct sum of two null-filiform ideals. Finally, we consider the general situation where the nilradical decomposes as a direct sum of null-filiform ideals.

Throughout this article we consider finite-dimensional vector spaces and algebras over the field of the complex numbers. Moreover, in the multiplication table of an algebra the omitted products are assumed to be zero and if it is not noted we shall consider non-nilpotent solvable algebras.

2. Preliminaries

In this section we give necessary definitions and preliminary results.

Definition 2.1 An algebra $(L, [-, -])$ over the field $\mathbb{C}$ is said to be a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds.

A subalgebra H of a Leibniz algebra L is said to be a *two-sided ideal* if $[L, H] \subseteq L$ and $[H, L] \subseteq L$. Let H and K be two-sided ideals of a Leibniz algebra L. The *commutator ideal* of H and K, denoted by $[H, K]$, is the two-sided ideal of L spanned by the brackets $[h, k]$, $[k, h]$, $h \in H, K \in K$. Obviously, $[H, K] \subset H \cap K$.

From the Leibniz identity we conclude that the elements of the form $[x, x]$ and $[x, y]+[y, x]$, for any x, y, lie in the *right annihilator* $\mathrm{Ann}_r(L)=\{x \in L\colon [y, x]=0$ for all $y \in L\}$ of the Leibniz algebra. Moreover, we also get that $\mathrm{Ann}_r(L)$ is a two-sided ideal of the Leibniz algebra.

For a given Leibniz algebra L, we define the *lower central* and *derived series* to the sequences of two-sided ideals defined recursively as follows:

$$L^1 = L,\ L^{k+1} = [L^k, L],\quad k \geq 1; \qquad L^{[1]} = L,\ L^{[s+1]} = [L^{[s]}, L^{[s]}],\quad s \geq 1.$$

Definition 2.2 A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists an $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^n=0$ (respectively, $L^{[m]}=0$). The minimal number n (respectively, m) with such property is said to be the index of nilpotency (respectively, of solvability) of the algebra L.

Remark 1 Obviously, the index of nilpotency of an n-dimensional nilpotent Leibniz algebra is not greater than $n+1$.

Definition 2.3 An n-dimensional Leibniz algebra is said to be null-filiform if $\dim L^i=n+1-i$, $1 \leq i \leq n+1$.

Remark 2 Obviously, a null-filiform Leibniz algebra has a maximal index of nilpotency.

THEOREM 2.4 [5] *An arbitrary n-dimensional null-filiform Leibniz algebra is isomorphic to the algebra*:

$$NF_n: \quad [e_i, e_1] = e_{i+1},\ 1 \leq i \leq n-1,$$

where $\{e_1, e_2, \ldots, e_n\}$ *is a basis of the algebra* NF_n.

From this theorem it is easy to see that a nilpotent Leibniz algebra is null-filiform if and only if it is a one-generated algebra, i.e. an algebra generated by a simple element. Note that this notion has no sense in the Lie algebra case, because they are at least two-generated.

Definition 2.5 The maximal nilpotent ideal of a Leibniz algebra is said to be the nilradical of the algebra.

Definition 2.6 For a Leibniz algebra L, a linear map $d\colon L \to L$ is said to be a derivation if

$$d[x, y] = [d(x), y] + [x, d(y)]$$

for all $x, y \in L$.

For a fixed $x \in L$, the map $\mathcal{R}_x\colon L \to L$, $\mathcal{R}_x(y)=[y, x]$ is a derivation. We call this kind of derivations as *inner derivations* and we denote the set of all inner derivations of L by $\mathrm{Inn}(L)$. Derivations that are not inner are said to be *outer derivations*.

Definition 2.7 [12] Let $d_1, d_2, \ldots, d_n$ be derivations of a Leibniz algebra L. The derivations $d_1, d_2, \ldots, d_n$ are said to be nil-independent if

$$\alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_n$$

is not nilpotent for any scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$, which are not all zero.

In other words, if for any $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ there exists a natural number k such that $(\alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_2)^k=0$, then $\alpha_1=\alpha_2=\cdots=\alpha_n=0$.

3. Main results

Let R be a solvable Leibniz algebra. Then it can be decomposed into the form $R = N \oplus Q$, where N is the nilradical and Q is the complementary vector space. Since the square of a solvable algebra is a nilpotent ideal and the finite sum of nilpotent ideals is a nilpotent ideal too [4], then the ideal R^2 is nilpotent, i.e. $R^2 \subseteq N$ and consequently, $Q^2 \subseteq N$.

LEMMA 3.1 *Let $x \in Q$ be such that the operator $\mathcal{R}_{x|_N}$ is nilpotent. Then the subspace $V = \langle x + N \rangle$ is a nilpotent ideal of the algebra R.*

Proof Since $R^2 \subseteq N$, V is an ideal. We argue that it is nilpotent. If $a \in N$, then $\mathcal{R}_{a|_N}$ is a nilpotent operator. Let us suppose that there exists a $k \in \mathbb{N}$ such that $\left(\mathcal{R}_{a|_N}\right)^k = 0$, then $\left(\mathcal{R}_{a|_V}\right)^{k+1} = 0$. Hence $\mathcal{R}_{a|_V}$ is nilpotent. If V is an ideal of the solvable Leibniz algebra R, then $\mathrm{Inn}(V)$ is a solvable Lie algebra of $\mathrm{End}(V)$, and so by Lie's theorem [9] there exists a basis such that $\mathcal{R}_{a|_V}$ and $\mathcal{R}_{x|_V}$ are upper triangular; moreover, $\mathcal{R}_{a|_V}$ is nilpotent, which means that $\mathcal{R}_{a|_V}$ has zero diagonal elements. On the other hand, by assumption, $\mathcal{R}_{x|_N}$ is nilpotent, then with a similar argument as the previous one, there exists an $s \in \mathbb{N}$ such that $\left(\mathcal{R}_{x|_N}\right)^s = 0$, then $\left(\mathcal{R}_{x|_V}\right)^{s+1} = 0$. Summarizing, $\mathcal{R}_{a|_V}$ and $\mathcal{R}_{x|_V}$ are nilpotent and upper triangular, hence $\mathcal{R}_{a|_V} + \mathcal{R}_{x|_V}$ is nilpotent. Thus, by Engel's theorem [4], V is a nilpotent ideal. ■

THEOREM 3.2 *Let R be a solvable Leibniz algebra and N its nilradical. Then the dimension of the complementary vector space to N is not greater than the maximal number of nil-independent derivations of N.*

Proof We assert that every $\mathcal{R}_{x|_N}$, $x \in Q$, is a non-nilpotent outer derivation of N. Indeed, if there exists some $x \in Q$ such that the operator $\mathcal{R}_{x|_N}$ is nilpotent, then the subspace $V = \langle x + N \rangle$ is a nilpotent ideal of the algebra R by Lemma 3.1, contradicting the maximality condition of N.

Let $\{x_1, \ldots, x_m\}$ be a basis of Q. Then the operators $\mathcal{R}_{x_1|_N}, \ldots, \mathcal{R}_{x_m|_N}$ are nil-independent, since if for some scalars $\{\alpha_1, \ldots, \alpha_m\} \in \mathbb{C}\backslash\{0\}$ we have that $(\sum_{i=1}^m \alpha_i \mathcal{R}_{x_i|_N})^k = 0$, then $\mathcal{R}^k_{y|_N}$, where $y = \sum_{i=1}^m \alpha_i x_i$. Hence $y = 0$, and so $\alpha_i = 0$ for $i = 1, \ldots, m$.

Therefore, we see that the dimension of Q is bounded by the maximal number of nil-independent derivations of the nilradical N. Moreover, similar to the case of Lie algebras, for a solvable Leibniz algebra R we also have the inequality $\dim N \geq \frac{\dim R}{2}$. ■

From Theorem 3.2 we conclude the following properties of derivations of null-filiform Leibniz algebras.

PROPOSITION 3.3 *Any derivation of the algebra NF_n has the following matrix form*:

$$\begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ 0 & 2a_1 & a_2 & \ldots & a_{n-1} \\ 0 & 0 & 3a_1 & \ldots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & na_1 \end{pmatrix}.$$

Proof The proof is carried out by checking the derivation property on algebra NF_n. ■

COROLLARY 3.4 *The maximal number of nil-independent derivations of the n-dimensional null-filiform Leibniz algebra NF_n is* 1.

Proof Let

$$D_i = \begin{pmatrix} a_1^i & a_2^i & a_3^i & \dots & a_n^i \\ 0 & 2a_1^i & a_2^i & \dots & a_{n-1}^i \\ 0 & 0 & 3a_1^i & \dots & a_{n-2}^i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & na_1^i \end{pmatrix}, \quad i = 1, 2, \dots, p,$$

be derivations of NF_n. If $p > 1$, then $(D_i - \frac{a_1^i}{a_1^1} D_1)^n = 0$ with non-trivial scalars. Hence $\{D_1, D_2, \dots, D_p\}$ is not nil-independent. ■

COROLLARY 3.5 *The dimension of a solvable Leibniz algebra with nilradical NF_n is equal to $n+1$.*

Proof Let us assume that the solvable Leibniz algebra is decomposed as $R = NF_n \oplus Q$. By Corollary 3.4 and Theorem 3.2 we have $1 \le \dim Q \le 1$. ■

THEOREM 3.6 *Let R be a solvable Leibniz algebra whose nilradical is NF_n. Then there exists a basis $\{e_1, e_2, \dots, e_n, x\}$ of the algebra R such that the multiplication table of R with respect to this basis has the following form*:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-1, \\ [x, e_1] = e_1, & \\ [e_i, x] = -ie_i, & 1 \le i \le n. \end{cases}$$

Proof According to Theorem 2.4 and Corollary 3.5 there exists a basis $\{e_1, e_2, \dots, e_n, x\}$ such that all products of elements of the basis, except for the products $[e_i, x]$ which can be derived from the equalities $[e_{i+1}, x] = [[e_i, e_1], x] = [e_i, [e_1, x]] + [[e_i, x], e_1]$, $1 \le i \le n-1$, have the following form:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-1, \\ [x, e_1] = \sum\limits_{i=1}^{n} \alpha_i e_i, & \\ [e_1, x] = \sum\limits_{i=1}^{n} \beta_i e_i, & \\ [x, x] = \sum\limits_{i=1}^{n} \gamma_i e_i, & \end{cases}$$

where $\{e_1, e_2, \dots, e_n\}$ is a basis of NF_n and $\{x\}$ is a basis of Q.

Now we consider the following two possible cases.

Case 1 Let $\alpha_1 \ne 0$. Then taking the change of basis:

$$e_i' = \frac{1}{\alpha_1} \sum_{j=i}^{n} \alpha_{j-i+1} e_j, \quad 1 \le i \le n, \quad x' = \frac{1}{\alpha_1} x,$$

we can assume that $[x, e_1] = e_1$ and other products can be assumed as not changed by redesignation of parameters.

From the products

$$0=[x,[x,x]]=\left[x,\ \sum_{i=1}^{n}\gamma_i e_i\right]=\sum_{i=1}^{n}\gamma_i[x,e_i]=\gamma_1 e_1,$$

we can deduce that $\gamma_1=0$.

On the other hand, from the Leibniz identity

$$\big[x,[e_1,x]\big]=[[x,e_1],x]-\big[[x,x],e_1\big]$$

we get $\beta_1[x,e_1]=[e_1,x]-\sum_{i=3}^{n}\gamma_{i-1}e_i$, i.e. $\beta_1 e_1=\sum_{i=1}^{n}\beta_i e_i-\sum_{i=3}^{n}\gamma_{i-1}e_i$.

Comparing the coefficients at the elements of the basis, we obtain $\beta_2=0$ and $\gamma_i=\beta_{i+1}$ for $2\le i\le n-1$. From the equality $\big[e_1,[e_1,x]\big]=-\big[e_1,[x,e_1]\big]$, we derive that $\beta_1=-1$.

Thus, we have

$$[e_1,x]=-e_1+\sum_{i=3}^{n}\beta_i e_i,\qquad [x,x]=\sum_{i=2}^{n-1}\beta_{i+1}e_i+\gamma_n e_n.$$

Now we are going to prove the following identity:

$$[e_i,x]=-ie_i+\sum_{j=i+2}^{n}\beta_{j-i+1}e_j, \tag{3.1}$$

for $1\le i\le n$. We have seen that (3.1) is true for $i=1$. Assume that (3.1) holds for each i, $1\le i<k\le n$. Then

$$\begin{aligned}
[e_k,x]&=\big[[e_{k-1},e_1],x\big]=[e_{k-1},[e_1,x]]+\big[[e_{k-1},x],e_1\big]\\
&=[e_{k-1},\ -e_1]+\left[-(k-1)e_{k-1}+\sum_{j=k+1}^{n}\beta_{j-k+2}e_j,e_1\right]\\
&=-e_k-(k-1)e_k+\sum_{j=k+1}^{n}\beta_{j-k+2}[e_j,e_1]=-ke_k+\sum_{j=k+2}^{n}\beta_{j-k+1}e_j.
\end{aligned}$$

By induction, we see that indeed (3.1) holds for all i, $1\le i\le n$.

Thus, the multiplication table of the algebra R has the form:

$$\begin{cases}
[e_i,e_1]=e_{i+1}, & 1\le i\le n-1,\\
[x,e_1]=e_1, & \\
[e_i,x]=-ie_i+\sum_{j=i+2}^{n}\beta_{j-i+1}e_j, & 1\le i\le n,\\
[x,x]=\sum_{i=2}^{n-1}\beta_{i+1}e_i+\gamma_n e_n. &
\end{cases} \tag{3.2}$$

Let us take the change of basis:

$$e_i'=e_i+\sum_{j=i+2}^{n}A_{j-i+1}e_j,\ \ 1\le i\le n,\quad x'=\sum_{i=2}^{n-1}A_{i+1}e_i+B_n e_n+x,$$

where parameters A_i, B_n are as follows

$$A_3 = \frac{1}{2}\beta_3, \quad A_4 = \frac{1}{3}\beta_4, \quad A_i = \frac{1}{i-1}\left(\sum_{j=3}^{i-2} A_{i-j+1}\beta_j + \beta_i\right), \quad (5 \le i \le n),$$

$$B_n = \frac{1}{n}\left(\sum_{j=3}^{n-1} A_{n-j+2}\beta_j + \gamma_n\right).$$

Then taking into account the multiplication table (3.2), we compute the products in the new basis

$$[e_i', e_1'] = \left[e_i + \sum_{j=i+2}^{n} A_{j-i+1}e_j, e_1\right] = e_{i+1} + \sum_{j=i+3}^{n} A_{j-i}e_j = e_{i+1}', \quad 1 \le i \le n-1,$$

$$[x', e_1'] = \left[\sum_{i=2}^{n-1} A_{i+1}e_i + B_n e_n + x, e_1\right] = \sum_{i=3}^{n} A_i e_i + [x, e_1] = e_1 + \sum_{i=3}^{n} A_i e_i = e_1',$$

$$\begin{aligned}
[x', x'] &= \left[\sum_{i=2}^{n-1} A_{i+1}e_i + B_n e_n + x, x\right] = \sum_{i=2}^{n-1} A_{i+1}[e_i, x] + B_n[e_n, x] + [x, x] \\
&= \sum_{i=2}^{n-1} A_{i+1}\left(-ie_i + \sum_{j=i+2}^{n} \beta_{j-i+1}e_j\right) - nB_n e_n + \sum_{i=2}^{n-1} \beta_{i+1}e_i + \gamma_n e_n \\
&= -\sum_{i=2}^{n-1} iA_{i+1}e_i + \sum_{i=2}^{n-3} A_{i+1}\sum_{j=i+2}^{n-1} \beta_{j-i+1}e_j + \sum_{i=2}^{n-1} \beta_{i+1}e_i \\
&\quad + \sum_{i=2}^{n-2} A_{i+1}\beta_{n-i+1}e_n - B_n e_n + \gamma_n e_n \\
&= \sum_{i=2}^{n-1}(-iA_{i+1} + \beta_{i+1})e_i + \sum_{i=4}^{n-1}\sum_{j=3}^{i-1} A_{i-j+2}\beta_j e_i \\
&\quad + \left(-nB_n + \gamma_n + \sum_{i=2}^{n-1} A_{i+1}\beta_{n-i+1}\right)e_n \\
&= (-2A_3 + \beta_3)e_2 + (-3A_4 + \beta_4)e_3 \\
&\quad + \sum_{i=4}^{n-1}\sum_{j=3}^{i-1}(-iA_{i+1} + \beta_{i+1} + A_{i-j+2}\beta_j)e_i = 0,
\end{aligned}$$

$$
\begin{aligned}
[e_1', x'] &= \left[e_1 + \sum_{i=3}^{n} A_i e_i, x \right] = [e_1, x] + \sum_{i=3}^{n} A_i [e_i, x] \\
&= -e_1 + \sum_{i=3}^{n} \beta_i e_i + \sum_{i=3}^{n} A_i \left(-i e_i + \sum_{j=i+2}^{n} \beta_{j-i+1} e_j \right) \\
&= -e_1 + \sum_{i=3}^{n} \beta_i e_i - \sum_{i=3}^{n} i A_i e_i + \sum_{i=3}^{n} A_i \sum_{j=i+2}^{n} \beta_{j-i+1} e_j \\
&= -e_1 - \sum_{i=3}^{n} A_i e_i - \sum_{i=3}^{n} (i-1) A_i e_i + \sum_{i=3}^{n} \beta_i e_i + \sum_{i=3}^{n} \left(\sum_{j=3}^{i-2} A_{i-j+1} b_j \right) e_i \\
&= -e_1 - \sum_{i=3}^{n} A_i e_i + \sum_{i=3}^{n} (-(i-1) A_i + \beta_i) e_i + \sum_{i=5}^{n} \sum_{j=3}^{i-2} A_{i-j+1} \beta_j e_i \\
&= -e_1 - \sum_{i=3}^{n} A_i e_i + (-2A_3 + \beta_3) e_3 + (-3A_4 + \beta_4) e_4 \\
&\quad + \sum_{i=5}^{n} \sum_{j=3}^{i-2} \big(-(i-1) A_i + \beta_i + A_{i-j+1} \beta_j \big) e_i = -e_1 - \sum_{i=3}^{n} A_i e_i = -e_1'.
\end{aligned}
$$

By means of similar computations as in Equation (3.1), we deduce that $[e_i', x'] = -i e_i', \ 1 \leq i \leq n.$

Finally, we obtain the multiplication table of the algebra R given in the assertion of the theorem.

Case 2 Let $\alpha_1 = 0$. Then from the equalities $[e_1, [e_1, x]] = -[e_1, [x, e_1]]$ and $0 = [x, [x, x]]$ we get $\beta_1 = 0$ and $\gamma_1 = 0$, respectively.

Thus, we have the following products:

$$
\begin{cases}
[e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-1, \\
[x, e_1] = \sum_{i=2}^{n} \alpha_i e_i, & \\
[e_1, x] = \sum_{i=2}^{n} \beta_i e_i, & \\
[x, x] = \sum_{i=2}^{n} \gamma_i e_i. &
\end{cases}
$$

In a similar way as for Equation (3.1), we can prove the equality: $[e_i, x] = \sum_{j=i+1}^{n} \beta_{j-i+1} e_j$. Consequently, we have $[e_i, x] \in \langle \{e_{i+1}, e_{i+2}, \ldots, e_n\} \rangle$, i.e. $R^i \subseteq \langle \{e_i, e_{i+1}, \ldots, e_n\} \rangle$. Thus $R^{n+1} = 0$, which contradicts the assumption of non-nilpotency of the algebra R. This implies that, in the case of $\alpha_1 = 0$, there is no non-nilpotent solvable Leibniz algebra with nilradical NF_n. ■

Now we are going to clarify the situation when the nilradical is a direct sum of two null-filiform ideals of the nilradical.

THEOREM 3.7 *Let R be a solvable Leibniz algebra such that $R = NF_k \oplus NF_s + Q$, where $NF_k \oplus NF_s$ is the nilradical of R, NF_k and NF_s are ideals of the nilradical and* $\dim Q = 1$. *Then NF_k and NF_s are also ideals of the algebra R.*

Proof Let $\{e_1, e_2, \dots, e_k\}$ be a basis of NF_k, $\{f_1, f_2, \dots, f_s\}$ a basis of NF_s and $\{x\}$ a basis of Q. We can assume, without loss of generality, that $k \geq s$.

By Theorem 2.4 we have that $\{e_2, e_3, \dots, e_k, f_2, f_3, \dots, f_s\} \subseteq \mathrm{Ann}_r(R)$ and the following products:

$$[e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq k-1, \quad [f_i, f_1] = f_{i+1}, \quad 1 \leq i \leq s-1.$$

Let us introduce the notations:

$$\begin{cases} [x, e_1] = \sum\limits_{i=1}^{k} \alpha_i e_i + \sum\limits_{i=1}^{s} \beta_i f_i, & [x, f_1] = \sum\limits_{i=1}^{k} \delta_i e_i + \sum\limits_{i=1}^{s} \gamma_i f_i, \\ [e_1, x] = \sum\limits_{i=1}^{k} \lambda_i e_i + \sum\limits_{i=1}^{s} \sigma_i f_i, & [f_1, x] = \sum\limits_{i=1}^{k} \tau_i e_i + \sum\limits_{i=1}^{s} \mu_i f_i, \\ [x, x] = \sum\limits_{i=1}^{k} \rho_i e_i + \sum\limits_{i=1}^{s} \xi_i f_i. & \end{cases}$$

From the products

$$0 = [x, [e_1, f_1]] = [[x, e_1], f_1] - [[x, f_1], e_1] = \sum_{i=2}^{s} \beta_{i-1} f_i - \sum_{i=2}^{k} \delta_{i-1} e_i,$$

we obtain $\beta_i = 0$, $1 \leq i \leq s-1$ and $\delta_i = 0$, $1 \leq i \leq k-1$.

The equalities $[e_1, [e_1, x]] = -[e_1, [x, e_1]]$ and $[f_1, [f_1, x]] = -[f_1, [x, f_1]]$ imply that $\lambda_1 = -\alpha_1$, $\mu_1 = -\gamma_1$.

From the equalities $0 = [e_1, [x, x]] = \rho_1 e_2$ and $0 = [f_1, [x, x]] = \xi_1 f_2$, we get $\rho_1 = \xi_1 = 0$.

In a similar way as in the proof of Theorem 3.6, the following equalities can be proved:

$$\begin{aligned} [e_i, x] &= -i\alpha_1 e_i + \sum_{j=i+1}^{k} \lambda_{j-i+1} e_j, \quad 2 \leq i \leq k, \\ [f_i, x] &= -i\gamma_1 f_i + \sum_{j=i+1}^{s} \mu_{j-i+1} f_j, \quad 2 \leq i \leq s. \end{aligned}$$

Summarizing, we have obtained the following multiplication table for the algebra R:

$$\begin{cases} [e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq k-1, & [f_i, f_1] = f_{i+1}, \quad 1 \leq i \leq s-1, \\ [x, e_1] = \sum\limits_{i=1}^{k} \alpha_i e_i + \beta_s f_s, & [x, f_1] = \delta_k e_k + \sum\limits_{i=1}^{s} \gamma_i f_i, \\ [e_1, x] = -\alpha_1 e_1 + \sum\limits_{i=2}^{k} \lambda_i e_i + \sum\limits_{i=1}^{s} \sigma_i f_i, & [f_1, x] = \sum\limits_{i=1}^{k} \tau_i e_i - \gamma_1 f_1 + \sum\limits_{i=2}^{s} \mu_i f_i, \\ [e_i, x] = -i\alpha_1 e_i + \sum\limits_{j=i+1}^{k} \lambda_{j-i+1} e_j, \quad 2 \leq i \leq k, & \\ [f_i, x] = -i\gamma_1 e_i + \sum\limits_{j=i+1}^{s} \mu_{j-i+1} f_j, \quad 2 \leq i \leq s, & \\ [x, x] = \sum\limits_{i=2}^{k} \rho_i e_i + \sum\limits_{i=2}^{s} \xi_i f_i. & \end{cases} \tag{3.3}$$

Below, we analyse the different cases that can appear in terms of the possible values of α_1 and γ_1.

Case 1 Let $\alpha_1=\gamma_1=0$. Then the multiplication table (3.3) implies $[e_i, x] \in \langle\{e_{i+1}, e_{i+2}, \dots, e_k\}\rangle$, $[f_i, x] \in \langle\{f_{i+1}, f_{i+2}, \dots, f_s\}\rangle$, $[e_1, x] \in \langle\{e_2, e_3, \dots, e_k, f_1, f_2, \dots, f_s\}\rangle$ and $[f_1, x] \in \langle\{e_1, e_2, \dots, e_k, f_2, f_3, \dots, f_s\}\rangle$. The above facts mean that the algebra R is nilpotent, so we get a contradiction with the assumption of non-nilpotency of R. Therefore, this case is impossible.

Case 2 Let $\alpha_1 \neq 0$ and $\gamma_1=0$. Using the following change of basis:

$$e_1' = \frac{1}{\alpha_1}\left(\sum_{i=1}^{k}\alpha_i e_i + \beta_s f_s\right), \quad e_i' = \frac{1}{\alpha_1}\sum_{j=i}^{k}\alpha_{j-i+1}e_j, \quad 2 \le i \le k, \quad x' = \frac{1}{\alpha_1}x,$$

we assume that

$$[x, e_1] = e_1.$$

From the identity

$$[x,[x,e_1]] = [[x,x],e_1] - [[x,e_1],x]$$

we have that

$$e_1 = \sum_{i=2}^{k}\rho_i[e_i,e_1] - [e_1,x] = \sum_{i=3}^{k}\rho_{i-1}e_i + e_1 - \sum_{i=2}^{k}\lambda_i e_i - \sum_{i=1}^{s}\sigma_i f_i.$$

Consequently, $\lambda_2=\sigma_i=0$ for $1 \le i \le s$ and $\rho_i=\lambda_{i+1}$ for $2 \le i \le k-1$.

From the identity

$$[f_1,[x,e_1]] = [[f_1,x],e_1] - [[f_1,e_1],x]$$

we conclude that $0 = [[f_1,x],e_1] = \sum_{i=2}^{k}\tau_{i-1}e_i \Rightarrow \tau_i = 0,\ 1 \le i \le k-1.$

From the identity

$$[x,[x,f_1]] = [[x,x],f_1] - [[x,f_1],x],$$

we obtain

$$0 = \sum_{i=3}^{s}\xi_{i-1}f_i - \sum_{i=2}^{s}\gamma_i[f_i,x] + \delta_k[e_k,x] = \sum_{i=3}^{s}\xi_{i-1}f_i - \sum_{i=2}^{s}\gamma_i\left(\sum_{j=i+1}^{s}\mu_{j-i+1}f_j\right) - k\delta_k e_k$$

$$= \sum_{i=3}^{s}\xi_{i-1}f_i - \sum_{i=3}^{s}\left(\sum_{j=3}^{i}\gamma_{j-1}\mu_{i-j+2}\right)f_i - k\delta_k e_k = \sum_{i=3}^{s}\left(\xi_{i-1} - \sum_{j=3}^{i}\gamma_{j-1}\mu_{i-j+2}\right)f_i - k\delta_k e_k.$$

By comparison of coefficients at the elements of the basis, we deduce that

$$\xi_i = \sum_{j=3}^{i+1}\gamma_{j-1}\mu_{i-j+3}, \quad 2 \le i \le s-1 \text{ and } \delta_k = 0.$$

Now we consider the following change of basis:

$$f_1' = f_1 + \frac{\tau_k}{k}e_k, \qquad f_i' = f_i,\ 2 \le i \le s.$$

Then we obtain

$$[f_1', x] = \left[f_1 + \frac{\tau_k}{k} e_k, x\right] = \sum_{i=2}^{s} \mu_i f_i + \tau_k e_k - \tau_k e_k = \sum_{i=2}^{s} \mu_i f_i = \sum_{i=2}^{s} \mu_i f_i'$$

and

$$[x, f_1'] = \left[x, f_1 + \frac{\tau_k}{k} e_k\right] = [x, f_1] = \sum_{i=2}^{s} \gamma_i f_i = \sum_{i=2}^{s} \gamma_i f_i'.$$

Thus, we have the following multiplication table of the algebra R:

$$\begin{cases} [e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq k-1, & [f_i, f_1] = f_{i+1}, \quad 1 \leq i \leq s-1, \\ [x, e_1] = e_1, & [x, f_1] = \sum\limits_{i=2}^{s} \gamma_i f_i, \\ [e_1, x] = -e_1 + \sum\limits_{i=2}^{k} \lambda_i e_i, & [f_1, x] = \sum\limits_{i=2}^{s} \mu_i f_i, \\ [e_i, x] = -ie_i + \sum\limits_{j=i+2}^{k} \lambda_{j-i+1} e_j, \ 2 \leq i \leq k, & [f_i, x] = \sum\limits_{j=i+1}^{s} \mu_{j-i+1} f_j, \ 2 \leq i \leq s, \\ [x, x] = \sum\limits_{i=2}^{k} \rho_i e_i + \sum\limits_{i=2}^{s} \xi_i f_i. & \end{cases}$$

From the above multiplication table the following inclusions can be immediately derived:

$$[x, NF_k] \subseteq NF_k, \quad [NF_k, x] \subseteq NF_k, \quad [x, NF_s] \subseteq NF_s, \quad [NF_s, x] \subseteq NF_s.$$

This completes the proof of the assertion established in the theorem for this case.

Case 3 Let $\alpha_1 = 0$ and $\gamma_1 \neq 0$. Due to the symmetry of Cases 2 and 3, the proof of the assertion of the theorem follows similar arguments as in Case 2.

Case 4 Let $\alpha_1 \neq 0$ and $\gamma_1 \neq 0$. Consider the following change of basis:

$$\begin{aligned} e_1' &= \tfrac{1}{\alpha_1}\left(\sum_{i=1}^{k} \alpha_i e_i + \beta_s f_s\right), \quad e_i' = \frac{1}{\alpha_1} \sum_{j=i}^{k} \alpha_{j-i+1} e_j, \qquad 2 \leq i \leq k, \\ f_1' &= \tfrac{1}{\gamma_1}\left(\sum_{i=1}^{s} \gamma_i f_i + \delta_k e_k\right), \quad f_i' = \frac{1}{\gamma_1} \sum_{j=i}^{k} \gamma_{j-i+1} f_j, \qquad 2 \leq i \leq s, \qquad x' = \frac{1}{\alpha_1} x. \end{aligned}$$

Then we derive

$$[x', e_1'] = \left[\frac{1}{\alpha_1} x, \frac{1}{\alpha_1}\left(\sum_{i=1}^{k} \alpha_i e_i + \beta_s f_s\right)\right] = \frac{1}{\alpha_1^2} \alpha_1 [x, e_1] = \frac{1}{\alpha_1}[x, e_1] = e_1',$$

$$[x', f_1'] = \left[\frac{1}{\alpha_1} x, \frac{1}{\gamma_1}\left(\sum_{i=1}^{s} \gamma_i f_i + \delta_k e_k\right)\right] = \frac{1}{\alpha_1 \gamma_1} \gamma_1 [x, f_1] = \frac{\gamma_1}{\alpha_1} f_1'.$$

From the identity $[x,[x,e_1]] = [[x,x],e_1] - [[x,e_1],x]$ we deduce

$$e_1 = \sum_{i=2}^{k} \rho_i [e_i, e_1] - [e_1, x] = \sum_{i=3}^{k} \rho_{i-1} e_i + \alpha_1 e_1 - \sum_{i=2}^{k} \lambda_i e_i - \sum_{i=1}^{s} \sigma_i f_i.$$

Therefore, $\alpha_1 = 1$, $\lambda_1 = -1$, $\lambda_2 = \sigma_i = 0$, $1 \le i \le s$ and $\rho_i = \lambda_{i+1}$, $2 \le i \le k-1$.

Expanding the identity $[x,[x,f_1]] = [[x,x],f_1] - [[x,f_1],x]$, we derive the equalities

$$\left(\frac{\gamma_1}{\alpha_1}\right)^2 f_1 = \sum_{i=2}^{s} \xi_i [f_i, f_1] - \frac{\gamma_1}{\alpha_1}[f_1, x] = \sum_{i=3}^{s} \xi_{i-1} f_i - \frac{\gamma_1}{\alpha_1} \sum_{i=1}^{s} \mu_i f_i - \frac{\gamma_1}{\alpha_1} \sum_{i=1}^{k} \tau_i e_i$$

from which we have $\mu_1 = -\frac{\gamma_1}{\alpha_1}$, $\mu_2 = \tau_i = 0$, $1 \le i \le k$ and $\xi_i = \frac{\gamma_1}{\alpha_1}\mu_{i+1}$, $2 \le i \le s-1$.

Finally, we obtain the following products of basis elements in the algebra R:

$$\begin{cases} [e_i, e_1] = e_{i+1}, \quad 1 \le i \le k-1, & [f_i, f_1] = f_{i+1}, \quad 1 \le i \le s-1, \\ [x, e_1] = e_1, & [x, f_1] = \dfrac{\gamma_1}{\alpha_1} f_1, \\ [e_1, x] = -e_1 + \displaystyle\sum_{i=3}^{k} \lambda_i e_i, & [f_1, x] = -\dfrac{\gamma_1}{\alpha_1} f_1 + \displaystyle\sum_{i=3}^{s} \mu_i f_i, \\ [x, x] = \displaystyle\sum_{i=2}^{k} \rho_i e_i + \sum_{i=2}^{s} \xi_i f_i. & \end{cases}$$

These products are sufficient in order to check the inclusions

$$[x, NF_k] \subseteq NF_k, \quad [NF_k, x] \subseteq NF_k, \quad [x, NF_s] \subseteq NF_s, \quad [NF_s, x] \subseteq NF_s.$$

Thus, the ideals NF_k and NF_s of the nilradical are also ideals of the algebra. ■

Now we are going to study solvable Leibniz algebras with nilradical $NF_k \oplus NF_s$ and with the one-dimensional complementary vector space. Due to Theorem 3.7, we can assume that NF_k and NF_s are ideals of the algebra.

THEOREM 3.8 *Let R be a solvable Leibniz algebra such that $R = NF_k \oplus NF_s + Q$, where $NF_k \oplus NF_s$ is the nilradical of R and $\dim Q = 1$. Let us assume that $\{e_1, e_2, \ldots, e_k\}$ is a basis of NF_k, $\{f_1, f_2, \ldots, f_s\}$ is a basis of NF_s and $\{x\}$ is a basis of Q. Then the algebra R is isomorphic to one of the following pairwise non-isomorphic algebras*:

$$R(\alpha): \begin{cases} [e_i, e_1] = e_{i+1}, \quad 1 \le i \le k-1, & [f_i, f_1] = f_{i+1}, \quad 1 \le i \le s-1, \\ [x, e_1] = e_1, & [x, f_1] = \alpha f_1, \quad \alpha \ne 0, \\ [e_i, x] = -i e_i, \quad 1 \le i \le k, & [f_i, x] = -i\alpha f_i, \quad 1 \le i \le s, \end{cases}$$

$$R(\beta_2, \beta_3, \ldots, \beta_s, \gamma): \begin{cases} [e_i, e_1] = e_{i+1}, \quad 1 \le i \le k-1, & [f_i, f_1] = f_{i+1}, \quad 1 \le i \le s-1, \\ [x, e_1] = e_1, & [f_i, x] = \displaystyle\sum_{j=i+1}^{s} \beta_{j-i+1} f_j, \ 1 \le i \le s, \\ [e_i, x] = -i e_i, \quad 1 \le i \le k, & [x, x] = \gamma f_s. \end{cases}$$

In the second family of algebras the first non-zero element of the vector $(\beta_2, \beta_3, \ldots, \beta_s, \gamma)$ *can be assumed to be equal to* 1.

Proof First, we note that the algebras $NF_k + Q$ and $NF_s + Q$ are not simultaneously nilpotent. Indeed, if they are both nilpotent, then we have

$$\begin{array}{llll}
[e_i, e_1] \in \langle\{e_{i+1}, \ldots, e_k\}\rangle, & 1 \le i \le k-1, & [f_i, f_1] \in \langle\{f_{i+1}, \ldots, f_s\}\rangle, & 1 \le i \le s-1, \\
[x, e_1] \in \langle\{e_2, e_3, \ldots, e_k\}\rangle, & & [x, f_1] \in \langle\{f_2, f_3, \ldots, f_s\}\rangle, & \\
[e_i, x] \in \langle\{e_{i+1}, \ldots, e_k\}\rangle, & 1 \le i \le k-1, & [f_j, x] \in \langle\{f_{j+1}, \ldots, f_s\}\rangle, & 2 \le i \le s-1.
\end{array}$$

From the equalities $0 = [e_1, [x, x]],\ 0 = [f_1, [x, x]]$ we conclude that

$$[x, x] \in \langle\{e_2, e_3, \ldots, e_k, f_2, f_3, \ldots, f_s\}\rangle.$$

Therefore, $R^2 \subseteq \{e_2, e_3, \ldots, e_k, f_2, f_3, \ldots, f_s\}$. Moreover, we have $R^i \subseteq \{e_i, e_{i+1}, \ldots, e_k, f_i, f_{i+1}, \ldots, f_s\}$, which implies that $R^{\max k, s+1} = \{0\}$. Thus, we have a contradiction to the assumption that R is not nilpotent. Hence, the algebras $NF_k + Q$ and $NF_s + Q$ cannot be both nilpotent.

Without loss of generality, we can assume that algebra $NF_k + Q$ is non-nilpotent.

We take the quotient algebra by an ideal NF_s, then $R/NF_s \cong \overline{NF_k} + \overline{Q}$. Thanks to Theorem 3.6, the structure of the algebra $\overline{NF_k} + \overline{Q}$ is known. Namely,

$$\begin{cases}
[\overline{e_i}, \overline{e_1}] = \overline{e_{i+1}}, & 1 \le i \le k-1, \\
[\overline{x}, \overline{e_1}] = \overline{e_1}, & \\
[\overline{e_i}, \overline{x}] = -i\overline{e_i}, & 1 \le i \le k.
\end{cases} \tag{3.4}$$

Using the fact that NF_k and NF_s are ideals of R and having in mind the multiplication table (3.4), we have

$$\begin{cases}
[e_i, e_1] = e_{i+1}, \quad 1 \le i \le k-1, & [f_i, f_1] = f_{i+1}, \quad 1 \le i \le s-1, \\
[x, e_1] = e_1, & [x, f_1] = \sum_{i=1}^{s} \alpha_i f_i, \\
[e_i, x] = -ie_i, \quad 1 \le i \le k, & [f_1, x] = \sum_{i=1}^{s} \beta_i f_i, \\
 & [x, x] = \sum_{i=1}^{s} \gamma_i f_i.
\end{cases} \tag{3.5}$$

If $\alpha_1 \neq 0$, then in a similar way as the Case 1 of Theorem 3.6 we obtain the family of algebras $R(\alpha)$, where $\alpha \neq 0$.

The fact that two algebras in the family $R(\alpha)$ with different values of parameter α are not isomorphic can be easily determined by a general change of basis and considering the expansion of the product $[x', f_1']$ in both bases.

Now consider $\alpha_1 = 0$. Then by the change of basis

$$x' = x - (\alpha_2 f_1 + \alpha_3 f_2 + \cdots + \alpha_s f_{s-1})$$

we can suppose $[x, f_1] = 0$.

From the identity $[f_1, [f_1, x]] = [[f_1, f_1], x] - [[f_1, x], f_1]$ we get $\beta_1 = 0$.

Similarly to the proof of Equation (3.1), we can prove that $[f_i, x] = \sum_{m=i+1}^{s} \beta_{m-i+1} f_j,\ 1 \le i \le s.$

The identity $[x, [f_1, x]] = [[x, f_1], x] - [[x, x], f_1]$ implies the following chain of equalities:

$$0 = -[[x, x], f_1] = -\sum_{m=3}^{s} \gamma_{m-1} f_m.$$

Consequently, $\gamma_i = 0,\ 2 \le i \le s-1.$

Thus, we obtain the products of the family $R(\beta_2, \beta_3, \ldots, \beta_s, \gamma)$

$$\begin{cases} [f_i, f_1] = f_{i+1}, & 1 \le i \le s-1, \\ [f_i, x] = \sum\limits_{m=i+1}^{s} \beta_{m-i+1} f_m, & 1 \le i \le s, \\ [x, x] = \gamma_s f_s. \end{cases}$$

Now we are going to study the isomorphism inside the family $R(\beta_2, \beta_3, \ldots, \beta_s, \gamma)$.

Taking into account that, under general basis transformation, the products (3.5) should not be changed, we conclude that it is sufficient to take the following change of basis:

$$f'_i = A_1^{i-1} \sum_{j=i}^{s} A_{j-i+1} f_j, \quad (A_1 \ne 0), \quad 1 \le i \le s, \quad x' = x.$$

Then we have

$$[f'_1, x'] = \sum_{i=1}^{s} A_i [f_i, x] = \sum_{i=1}^{s-1} A_i \left(\sum_{j=i+1}^{s} \beta_{j-i+1} f_j \right) = \sum_{i=2}^{s} \left(\sum_{j=1}^{i-1} A_j B_{i-j+1} \right) f_i.$$

On the other hand,

$$[f'_1, x'] = \sum_{i=2}^{s} \beta'_i f'_i = \sum_{i=1}^{s-1} A_1^i \beta'_{i+1} \left(\sum_{j=1}^{s-i} A_j f_{i+j} \right) = \sum_{i=2}^{s} \left(\sum_{j=1}^{i-1} A_1^j A_{i-j} \beta'_{j+1} \right) f_i.$$

Comparing coefficients at the elements of the basis, we deduce that

$$\sum_{i=1}^{k-1} A_i \beta_{k-i+1} = \sum_{i=1}^{k-1} A_1^i A_{k-i} \beta'_{i+1}, \quad k = 2, 3, \ldots, s.$$

From these systems of equations it follows that

$$\beta'_i = \frac{\beta_i}{A_1^{i-1}}, \qquad 2 \le i \le s.$$

If we consider

$$\gamma'_s A_1^s f_s = \gamma'_s f'_s = [x', x'] = [x, x] = \gamma_s f_s,$$

then we obtain

$$\gamma'_s = \frac{\gamma_s}{A_1^s}.$$

It is easy to see that by choosing an adequate value for the parameter A_1, the first non-zero element of the vector $(\beta_2, \beta_3, \ldots, \beta_s, \gamma)$ can be assumed to be equal to 1.

Therefore, two algebras $R(\beta_2, \beta_3, \ldots, \beta_s, \gamma)$ and $R(\beta'_2, \beta'_3, \ldots, \beta'_s, \gamma')$ with different set of parameters are not isomorphic.

For given parameters α and $\beta_2, \beta_3, \ldots, \beta_s, \gamma$, the algebras $R(\alpha)$ and $R(\beta_2, \beta_3, \ldots, \beta_s, \gamma)$ are not isomorphic because

$$k + s = \dim R(\alpha)^2 \neq \dim R(\beta_2, \beta_3, \ldots, \beta_s, \gamma)^2 = k + s - 1. \qquad \blacksquare$$

Remark 1 In the case when all coefficients $(\beta_2, \beta_3, \ldots, \beta_s, \gamma)$ are equal to zero we have the split algebra $(NF_k + Q) \oplus NF_s$. Therefore, in the non-split case, we can always assume that $(\beta_2, \beta_3, \ldots, \beta_s, \gamma) \neq (0, 0, 0, \ldots, 0)$.

Now, by an induction process, we are going to generalize Theorem 3.8 to the case when the nilradical is a direct sum (greater than 2) of several copies of null-filiform ideals.

THEOREM 3.9 *Let R be a solvable Leibniz algebra such that $R = NF_{n_1} \oplus NF_{n_2} \oplus \cdots \oplus NF_{n_s} + Q$, where $NF_{n_1} \oplus NF_{n_2} \oplus \cdots \oplus NF_{n_s}$ is the nilradical of R and $\dim Q = 1$. There exist $p, q \in \mathbb{N}$ with $p \neq 0$ and $p + q = s$, a basis $\{e^i_1, e^i_2, \ldots, e^i_{n_i}\}$ of NF_{n_i}, for $1 \le i \le p$, a basis $\{f^k_1, f^k_2, \ldots, f^k_{n_k}\}$ of $NF_{n_{p+k}}$, for $1 \le k \le q$, and a basis $\{x\}$ of Q such that the multiplication table of the algebra is given by*

$$R_{p,q}: \begin{cases} [e^j_i, e^j_1] = e^j_{i+1}, & 1 \le i \le n_j - 1, \quad [f^k_i, f^k_1] = f^k_{i+1}, \quad 1 \le i \le n_k - 1, \\ [x, e^j_1] = \delta^j e^j_1, & \delta^j \neq 0 \quad\quad\quad\; [f^k_i, x] = \displaystyle\sum_{m=i+1}^{n_k} \beta^k_{m-i+1} f^k_m, \quad 1 \le i \le n_k, \\ [e^j_i, x] = -i\delta^j e^j_i, & 1 \le i \le n_j, \quad\quad [x, x] = \displaystyle\sum_{m=1}^{k} \gamma^m f_{n_m}, \end{cases} \tag{3.6}$$

where $1 \le j \le p$, $1 \le k \le q$ and $\delta^1 = 1$. Moreover, the first non-zero component of the vectors $(\beta^k_2, \beta^k_3, \ldots, \beta^k_{n_k}, \gamma^k)$ can be assumed to be equal to 1. *Moreover, the algebras are pairwise non-isomorphic.*

Proof By induction on s:

If $s = 1$, then $p = 1$, $q = 0$, so $R_{1,0}$ is the algebra given in Theorem 3.6.

If $s = 2$, then we have two cases: either $p = 2$, $q = 0$ or $p = 1$, $q = 1$, which were considered in Theorem 3.7. Namely, we have two families of algebras: $R(\alpha)$, which corresponds to $R_{2,0}$, and $R(\beta_2, \beta_3, \ldots, \beta_s, \gamma)$, which corresponds to $R_{1,1}$.

Let us assume that the theorem is true for s and we shall prove it for $s + 1$.

Let $R = NF_{n_1} \oplus NF_{n_2} \oplus \cdots \oplus NF_{n_s} \oplus NF_{n_{s+1}} + Q$. We consider the quotient algebra by $NF_{n_{s+1}}$, i.e. $R/NF_{n_{s+1}} \cong \overline{NF_{n_1}} \oplus \overline{NF_{n_2}} \oplus \cdots \oplus \overline{N_{nF_s}} + \overline{Q}$. Then we get the multiplication table given in (3.6).

Note that the multiplication table for the algebra R can be obtained from (3.6) by adding the products

$$\begin{aligned} &[e^{s+1}_i, e^{s+1}_1] = e^{s+1}_{i+1}, \qquad 1 \le i \le n_{s+1} - 1, \\ &[x, e^{s+1}_1] = \sum_{m=1}^{n_{s+1}} \alpha^{s+1}_m e^{s+1}_m, \\ &[e^{s+1}_1, x] = \sum_{m=1}^{n_{s+1}} \beta^{s+1}_m e^{s+1}_m, \\ &[x, x] = \sum_{m=1}^{n_{s+1}} \gamma^{s+1}_m e^{s+1}_m. \end{aligned}$$

If $\alpha_1^{s+1} \neq 0$, then in an analogous way as in the proof of Theorem 3.6, we deduce that

$$\begin{aligned}
&[e_i^{s+1}, e_1^{s+1}] = e_{i+1}^{s+1}, && 1 \leq i \leq n_{s+1} - 1,\\
&[x, e_1^{s+1}] = \alpha_{s+1}^{s+1} e_1^{s+1}, &&\\
&[e_i^{s+1}, x] = -i\alpha^{s+1} e_i^{s+1}, && 1 \leq i \leq n_{s+1}.
\end{aligned}$$

Therefore we get the algebra $R_{p+1,\,q}$.

If $\alpha_1^{s+1} = 0$, then by similar arguments as in Theorem 3.8, we obtain

$$\begin{aligned}
&[e_i^{s+1}, e_1^{s+1}] = e_{i+1}^{s+1}, && 1 \leq i \leq n_{s+1} - 1,\\
&[e_i^{s+1}, x] = \sum_{m=i+1}^{n_{s+1}} \beta_{m-i+1}^{s+1} f_m^{s+1}, && 1 \leq i \leq n_{s+1},\\
&[x, x] = \sum_{m=1}^{k} \gamma^m f_{n_m} + \gamma^{s+1} f_{n_{s+1}}^{s+1}. &&
\end{aligned}$$

Setting $f_{i-1}^{q+1} = e_{i-1}^{s+1}$, we get the family of algebras $R_{p,\,q+1}$.

The proof that two algebras of the family $R_{p,q}$ with different values of parameters are not isomorphic can be carried out in a similar way as in the proof of Theorem 3.8. ■

In fact, due to Theorem 3.2, the complementary vector space, in the case when the nilradical of a solvable Leibniz algebra is a direct sum of s copies of null-filiform ideals, has dimension not grater than s. By taking direct sum of ideals $NF_i + Q_i$ and $NF_k \oplus \cdots \oplus NF_s$, where $1 \leq i \leq k-1$, $k \leq s$, we can construct a solvable Leibniz algebra whose nilradical is $NF_1 \oplus \cdots \oplus NF_s$ and whose complementary vector space is k-dimensional for each k $(k \leq s)$.

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