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A linear map x on V is represented by a diagonal matrix $\Leftrightarrow$ V has a basis of eigenvectors for x

$\Leftrightarrow$ $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_r}$ $\quad \lambda_1, .., \lambda_r$ distinct eigenvalues

$\overset{\text{def}}{\Longleftrightarrow}$ x is <u>diagonalizable</u> $\quad V_{\lambda_i} = \ker(x - \lambda_i 1_V)$

Eigenvalues of x are the roots of its characteristic polynomial $c_x(\lambda) = \det(x - \lambda I_V)$.

The <u>minimal polynomial</u> of x is the monic polynomial of least degree which vanishes at x:

$$m(x) = x^d + a_{d-1} x^{d-1} + \cdots + a_0 1_V = 0, \quad d \text{ as small as possible.}$$

Exercise If f is a polynomial with $f(x)=0$, then m divides f.

[If $f(\lambda) = a(\lambda) m(\lambda) + r(\lambda)$ $\quad$ degree $r(\lambda) <$ degree $m(\lambda)$

If $r \neq 0$ then $r(x) = 0$, a contradiction]

By the Cayley-Hamilton theorem $c_x(x) = 0$ so m divides c_x

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Primary Decomposition Theorem Given x, If $m(\lambda) = (\lambda-\lambda_1)^{a_1}\cdots(\lambda-\lambda_r)^{a_r}$

$\lambda_1,\dots,\lambda_r$ distinct, $a_i \geq 1$, then

$$V = V_1 \oplus \cdots \oplus V_r \quad \text{where } V_i = \ker(x-\lambda_i 1_V)^{a_i}$$

Lemma Given x on V. If f, g are coprime polynomials with $f(x)\,g(x) = 0$ then

(0) $\operatorname{im} f(x)$ and $\operatorname{im} g(x)$ are x-invariant subspaces of V

(1) $V = \operatorname{im} f(x) \oplus \operatorname{im} g(x)$

(2) $\operatorname{im} f(x) = \ker g(x)$, $\operatorname{im} g(x) = \ker f(x)$

Proof of Lemma If $v = f(x)w$ then $x(v) = x\,f(x)w = f(x)\,xw \in \operatorname{im} f(x)$ proving (0)

Since f, g are relatively prime $\exists$ polynomials a, b s.t. $af + bg \equiv 1$

$$a(\lambda)f(\lambda) + b(\lambda)g(\lambda) = 1$$

$$\alpha_0 + \alpha_1\lambda + \cdots + \alpha_n\lambda^n = 1$$

$$\alpha_1 = \alpha_2 \cdots = \alpha_n = 0$$

$$\alpha_0 = 1$$

So $\forall v \in V$

$$f(x)\,(a(x)v) + g(x)\,(b(x)v) = v$$

$\therefore$ $V = \operatorname{im} f(x) + \operatorname{im} g(x)$

If $v = g(x)w \in \operatorname{im} g(x)$ then $f(x)v = f(x)\,g(x)w = 0$

So $\operatorname{im} g(x) \subset \ker f(x)$. If $v \in \ker f(x)$, then by

$v = g(x)\,b(x)\,v \in \operatorname{im} g(x)$ proving (2). Finally if

$v \in \operatorname{im} f(x) \cap \operatorname{im} g(x)$ then $v =$ ~~$a f(x)$~~ $\in \ker g(x) \cap \ker f(x)$, $v = 0$

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Corollary of Primary Dec. Theorem

Theorem x is diagonalizable $\Leftrightarrow$ m is a product of distinct linear factors.

$V = V_1 + \cdots + V_r \quad V_i = \ker (x - \lambda_i 1_V)^{a_i}$

$V = V_{\lambda_1} + \cdots + V_{\lambda_r} \quad V_{\lambda_i} = \ker (x - \lambda_i 1_V) \subset V_i$

so $V_{\lambda_i} = V_i$ This implies $a_i = 1$ for if any

$a_i > 1$ the degree of $m(\lambda)$ would be $> r$

and $m_0(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_r)$ has degree r & $m_0(x) = 0$

Corollary If $x : V \to V$ is diagonalizable & $U \subset V$ $x(U) \subset U$, then

(a) $x|U$ is diagonalizable

(b) ~~Any~~ Any basis for U consisting of eigenvectors of x extends to a basis of V consisting of eigenvectors of x.

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Lemma Let x have minimal polynomial $f(X) = \prod_{i=1}^{r} (X-\lambda_i)^{a_i}$

$\lambda_1, \ldots, \lambda_r$ distinct and let $V = V_1 \oplus \cdots \oplus V_r$ be the primary decomp. ($V_i = \ker(x - \lambda_i 1_V)^{a_i}$)

Then for any $\mu_1, \ldots, \mu_r \in \mathbb{C}$ there is a polynomial $p(X) \in \mathbb{C}[X]$ with $p(x) = \mu_1 1_{V_1} + \mu_2 1_{V_2} + \cdots + \mu_r 1_{V_r}$

Proof. Consider the ring $\mathbb{C}[X]$ of all complex polynomials. The polynomials $(X-\lambda_1)^{a_1}, \ldots, (X-\lambda_r)^{a_r}$ are relatively prime $i \neq j$

So $1 = a(x)(X-\lambda_i)^{a_i} + b(x)(X-\lambda_j)^{a_j}$ & $\mathbb{C}[X] = I_i + I_j$

(need : • The division algorithm for polynomials

• The Chinese remainder theorem)

Stay tuned

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where $I_i = \mathbb{C}[X] / (X-\lambda_i)^{a_i}$. By the Chinese Remainder Theorem (Corollary 1) the map $f(X) \to (f(X) \pmod{(X-\lambda_1)^{a_1}}, \ldots, f(X) \pmod{(X-\lambda_r)^{a_r}})$ is surjective. Consider the constant polynomials $\mu_1, \ldots, \mu_r$

Then there is a polynomial $p(X) \equiv \mu_i \pmod{(X-\lambda_i)^{a_i}}$ $i = 1, \ldots, r$.

Take $v \in V_i = \ker(x - \lambda_i 1_V)^{a_i}$. Since $p(X) = \mu_i + a(X)(X-\lambda_i)^{a_i}$ for some polynomial $a(X)$ we have

$$p(x)v = \mu_i 1_{V_i} v + a(x)(x-\lambda_i)^{a_i} v = \mu_i v, \text{ that is}$$

$$p(x) = \begin{bmatrix} \mu_1 I_{n_1} & 0 & & 0 \\ 0 & \mu_2 I_{n_2} & & \\ & & \ddots & \\ 0 & & & \mu_r I_{n_r} \end{bmatrix}$$ where $n_i = \dim V_i$.

$\blacksquare$

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<u>Theorem</u> Any linear transf.^{x} of a complex vector space V can be represented by a matrix in Jordan Canonical form.

$J_1(\lambda) = (\lambda)_{1\times 1}$ $\quad J_2(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}_{2\times 2}$ $\quad J_3(\lambda) = \begin{matrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{matrix}$

$$X \sim \begin{bmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_r \end{bmatrix} \qquad A_i = J_{t_i}(\lambda_i) \qquad \begin{matrix} t_i \geq 1 \\ \lambda_i \in \mathbb{C} \end{matrix}$$

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Jordan decomposition

Any linear transf. $x : V \to V$ has a Jordan decomposition

$x = d + n$ d diagonalizable n nilpotent

$$x = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{bmatrix} = \begin{bmatrix} D_1+N_1 & & 0 \\ & \ddots & \\ 0 & & D_r+N_r \end{bmatrix} = \begin{bmatrix} D_1 & & 0 \\ & \ddots & \\ 0 & & D_r \end{bmatrix} + \begin{bmatrix} N_1 & & 0 \\ & \ddots & \\ 0 & & N_r \end{bmatrix}$$

$dn = nd$

$$dn = \begin{bmatrix} D_1 & & 0 \\ & \ddots & \\ 0 & & D_r \end{bmatrix}\begin{bmatrix} N_1 & & 0 \\ & \ddots & \\ 0 & & N_r \end{bmatrix} = \begin{bmatrix} D_1N_1 & & 0 \\ & \ddots & \\ 0 & & D_rN_r \end{bmatrix} = \begin{bmatrix} N_1D_1 & & 0 \\ & \ddots & \\ 0 & & N_rD_r \end{bmatrix}$$

Lemma Let $x = d + n$ as above.

(a) $\exists\, p(X) \in \mathbb{C}[X]$, $p(x) = d$

(b) Fix a basis in which d is diagonal. Then

$\exists\, q(X) \in \mathbb{C}[X]$, $q(x) = \bar{d}$ (if $d \sim \mathrm{diag}(a_1, \dots, a_n)$ then $\bar{d} \sim \mathrm{diag}(\bar{a}_1, \dots, \bar{a}_n)$ w/r that basis)

Proof. Let $\lambda_1, \dots, \lambda_r$ be the distinct eigenvalues and let $m(X) = (X - \lambda_1)^{a_1} \cdots (X - \lambda_r)^{a_r}$ be the minimal polynomial of x. By the lemma on p. 4 $\exists$ polynomial p with $p(x) = \lambda_1 1_{V_1} + \dots + \lambda_r I_{V_r} \stackrel{!}{=} d$

Similarly $\exists$ polynomial q with $q(x) = \bar{\lambda}_1 1_{V_1} + \dots + \bar{\lambda}_r I_{V_r} \stackrel{!}{=} \bar{d}$. ∎