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On classification of 5-dimensional solvable Leibniz algebras

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ABSTRACT

In the paper we describe 5-dimensional solvable Leibniz algebras with three-dimensional nilradical. Since those 5-dimensional solvable Leibniz algebras whose nilradical is three-dimensional Heisenberg algebra have been classified before we focus on the rest cases. The result of the paper together with Heisenberg nilradical case gives complete classification of all 5-dimensional solvable Leibniz algebras with three-dimensional nilradical.

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1. Introduction

According to the structural theory of Lie algebras a finite-dimensional Lie algebra is written as a semidirect sum of its semisimple subalgebra and the solvable radical (Levi's theorem). The semisimple part is a direct sum of simple Lie algebras which were completely classified in the fifties of the last century. At the same period the essential progress has been made in the solvable part by Mal'cev reducing the problem of classification of solvable Lie algebras to that of nilpotent Lie algebras. Since then all the classification results have been related to the nilpotent part.

Leibniz algebras, a "noncommutative version" of Lie algebras, were first introduced in the mid-1960's by Blokh [4] under the name of *D*-algebras. They came in again in the 1990's after Loday's work [13], where he introduced calling them Leibniz algebras. During the last 20 years the theory of Leibniz algebras has been actively studied and many results on Lie algebras have been extended to Leibniz algebras (see, e.g. [10,16–18]). Particularly, in 2011 the analogue of Levi's theorem has been proven by D. Barnes [3]. He showed that any finite-dimensional complex Leibniz algebra is decomposed into a semidirect sum of the solvable radical and a semisimple Lie algebra. As above, the semisimple part can be composed by simple Lie algebras and the main issue in the classification problem of finite-dimensional complex Leibniz algebras is to study the solvable part. Therefore the classification of solvable Leibniz algebras is important to construct finite-dimensional Leibniz algebras.

Owing to a result of [14], a new approach for studying the solvable Lie algebras by using their nilradicals was developed [2,6,15,19,20], etc. The analogue of Mubarakzjanov's [14] results has been applied for Leibniz algebras case in [8] which shows the importance of the consideration of their nilradicals in Leibniz algebras case as well. The papers [5,8,9,11] are also devoted to the study of solvable Leibniz algebras by considering their nilradicals.

The classification, up to isomorphism, of any class of algebras is a fundamental and a very difficult problem. It is one of the first problems that one encounters when trying to understand the structure of a member of this class of algebras. Due to results of [5,7] there are complete lists of isomorphism classes of complex Leibniz algebras in dimensions less then five.

The focus of the present paper is on classification of Leibniz algebras in dimension five. Since the description of the whole of isomorphism classes in 5-dimensional Leibniz algebras seems to be hard we deal with the study of 5-dimensional solvable Leibniz algebras with three-dimensional nilradical. It should be noted that the description of solvable Leibniz algebras with three-dimensional Heisenberg nilradical has been given in [12]. Moreover, it was shown that a 5-dimensional solvable Leibniz algebra with three-dimensional Heisenberg nilradical is a Lie algebra. Therefore, in this paper we don't consider this case.

Throughout the paper all the algebras (vector spaces) considered are finite-dimensional and over the field of complex numbers. Also in tables of multiplications of algebras we give nontrivial products only.

2. Preliminaries

This section is devoted to recalling some basic notions and concepts used throughout the paper.

Definition 2.1. A vector space with bilinear bracket $(L, [\cdot,\cdot])$ is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity

$$\big[x,[y,z]\big] = \big[[x,y],z\big] - \big[[x,z],y\big],$$

holds.

Here, we adopt the right Leibniz identity; since the bracket is not skew-symmetric, there exists the version corresponding to the left Leibniz identity,

$$\big[[x,y],z\big] = \big[x,[y,z]\big] - \big[y,[x,z]\big].$$

The sets $Ann_r(L) = \{x \in L : [y,x] = 0,\ \forall y \in L\}$ and $Ann_l(L) = \{x \in L : [x,y] = 0,\ \forall y \in L\}$ are called *the right and left annihilators of* L, respectively. It is observed that for any $x, y \in L$ the elements $[x,x]$ and $[x,y] + [y,x]$ are always in $Ann_r(L)$, and that is $Ann_r(L)$ is a two-sided ideal of L.

The set $C(L) = \{z \in L : [x,z] = [z,x] = 0,\ \forall x \in L\}$ is called *the Center of* L.

For a given Leibniz algebra $(L, [\cdot,\cdot])$ the sequences of two-sided ideals defined recursively as follows:

$$L^1 = L, \quad L^{k+1} = \big[L^k, L\big], \quad k \geq 1, \qquad L^{[1]} = L, \quad L^{[s+1]} = \big[L^{[s]}, L^{[s]}\big], \quad s \geq 1$$

are said to be the lower central and the derived series of L, respectively.

Definition 2.2. A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^{[m]} = 0$). The minimal number n (respectively, m) with such property is said to be the index of nilpotency (respectively, solvability) of the algebra L.

Evidently, the index of nilpotency of an n-dimensional Leibniz algebra is not greater than $n + 1$.

Definition 2.3. An ideal of a Leibniz algebra is called nilpotent if it is nilpotent as subalgebra.

It is easy to see that the sum of any two nilpotent ideals is nilpotent. Therefore the maximal nilpotent ideal always exists.

Definition 2.4. The maximal nilpotent ideal of a Leibniz algebra is said to be a nilradical of the algebra.

Definition 2.5. A linear map $d: L \to L$ of a Leibniz algebra $(L, [\cdot,\cdot])$ is said to be a derivation if for all $x, y \in L$, the following condition holds:

$$d\big([x,y]\big) = \big[d(x), y\big] + \big[x, d(y)\big].$$

The set of all derivations of L is denoted by $Der(L)$. The $Der(L)$ is a Lie algebra with respect to the commutator.

For a given element x of a Leibniz algebra L, the right multiplication operator $R_x: L \to L$, defined by $R_x(y) = [y, x]$, $y \in L$ is a derivation. In fact, a Leibniz algebra is characterized by this property of the right multiplication operators (remind that the left Leibniz algebras are characterized the same property of the left multiplication operators). As in Lie case these kinds of derivations are said to be *inner derivations*. Let the set of all inner derivations of a Leibniz algebra L denote by $R(L)$, i.e. $R(L) = \{R_x \mid x \in L\}$. The set $R(L)$ inherits the Lie algebra structure from $Der(L)$:

$$[R_x, R_y] = R_x \circ R_y - R_y \circ R_x = R_{[y,x]}.$$

Here is the definition of nil-independency imitated from Lie case (see [14]).

Definition 2.6. Let $d_1, d_2, \dots, d_n$ be derivations of a Leibniz algebra L. The derivations $d_1, d_2, \dots, d_n$ are said to be a linearly nil-independent if for $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ and a natural number k

$$(\alpha_1 d_1 + \alpha_2 d_2 + \dots + \alpha_n d_n)^k = 0 \quad \text{implies} \quad \alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$$

Note that in the definition above the power is understood with respect to the composition.

Let L be a solvable Leibniz algebra. Then it can be written in the form $L = N + Q$, where N is the nilradical and Q is the complementary subspace. The following is a result from [8] on the dimension of Q which we make use in the paper.

Theorem 2.7. *Let L be a solvable Leibniz algebra and N be its nilradical. Then the dimension of Q is not greater than the maximal number of nil-independent derivations of N.*

In this paper we classify the class of 5-dimensional solvable Leibniz algebras with 3-dimensional nilradical. To do this we need to know their nilradicals and the maximal number of linearly nil-independent derivations of the nilradicals. Below we present the list of all the three dimensional nilpotent Leibniz algebras from [1].

Theorem 2.8. *Let L be a 3-dimensional nilpotent Leibniz algebra. Then L is isomorphic to one of the following pairwise nonisomorphic algebras:*

$$\begin{aligned} \lambda_1&: && [e_1,e_1]=e_2,\ [e_2,e_1]=e_3,\\ \lambda_2(\alpha)&: && [e_2,e_1]=e_3,\ [e_1,e_2]=\alpha e_3,\ \alpha\neq\alpha^{-1},\\ \lambda_3&: && [e_1,e_1]=e_3,\ [e_2,e_1]=e_3,\ [e_1,e_2]=-e_3,\\ \lambda_4&: && [e_1,e_2]=e_3,\ [e_2,e_1]=-e_3,\\ \lambda_5&: && [e_1,e_1]=e_3,\\ \lambda_6&: && \textit{abelian.} \end{aligned}$$

Note that the list of isomorphism classes of all three-dimensional Leibniz algebras has been given in [7]. For some conveniences we change the bases of the algebras $\lambda_2(\alpha)$ and λ_3, therefore their tables of multiplications are a slightly different those are in [1] and [7].

We declare the following subsidiary result. The proof can be given by direct computations.

Proposition 2.9. *The matrix forms of the derivations of λ_1, $\lambda_2(\alpha)$, λ_3, λ_4, λ_5 and λ_6 are represented as follows*

$$Der(\lambda_1)=\left\{\begin{pmatrix} a_1 & a_2 & a_3\\ 0 & 2a_1 & a_2\\ 0 & 0 & 3a_1\end{pmatrix}\middle| a_i\in\mathbb{C}\right\},$$

$$Der\bigl(\lambda_2(\alpha)\bigr)=\left\{\begin{pmatrix} a_1 & 0 & a_3\\ 0 & b_2 & b_3\\ 0 & 0 & a_1+b_2\end{pmatrix}\middle| a_i,b_j\in\mathbb{C}\right\},$$

$$Der(\lambda_3)=\left\{\begin{pmatrix} a_1 & a_2 & a_3\\ 0 & 2a_1 & b_3\\ 0 & 0 & 3a_1\end{pmatrix}\middle| a_i,b_j\in\mathbb{C}\right\},$$

$$Der(\lambda_4)=\left\{\begin{pmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ 0 & 0 & a_1+b_2\end{pmatrix}\middle| a_i,b_j\in\mathbb{C}\right\},$$

$$Der(\lambda_5)=\left\{\begin{pmatrix} a_1 & a_2 & a_3\\ 0 & b_2 & b_3\\ 0 & 0 & 2a_1\end{pmatrix}\middle| a_i,b_j\in\mathbb{C}\right\},$$

$$Der(\lambda_6) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \middle| a_i, b_j, c_k \in \mathbb{C} \right\}.$$

It is observed that due to Proposition 2.9 the number of maximal linearly nil-independent derivations of the algebras λ_1 and λ_3 equals one, the algebra λ_4 is Heisenberg algebra and the number of maximal linearly nil-independent derivations of the algebras $\lambda_2(\alpha)$, λ_5, λ_6 is two.

3. Main result

In this section we give the list of isomorphism classes of those five-dimensional solvable Leibniz algebras with three-dimensional nilradical which is not Heisenberg's algebra, the latter case, i.e., for five-dimensional solvable Leibniz algebras with three-dimensional Heisenberg's nilradical, the result is known from [12]. So we deal with the classification of 5-dimensional solvable Leibniz algebras with the 3-dimensional nilradical having at least two nil-independent derivations. These are the algebras $\lambda_2(\alpha)$, λ_5 and λ_6 from the list above. Therefore, it remains to describe 5-dimensional solvable Leibniz algebras with these nilradicals cases one by one.

Note that in constructing the multiplication tables we simplify them applying base changes. To simplify notations after each of this kind changes we keep writing vectors in the tables without "prime" although the basis vectors should be written with "primes". To describe five-dimensional solvable Leibniz algebras with nilradical N which is one of $\lambda_2(\alpha)$, λ_5 and λ_6 first we extend the basis $\{e_1, e_2, e_3\}$ of N to a basis $\{e_1, e_2, e_3, x_1, x_2\}$ of five-dimensional space and keep track the products of basis vectors under the base changes.

Under this circumstances one has

Lemma 3.1. *The restrictions of the right multiplication operators R_{x_1} and R_{x_2} to N are nil-independent derivations.*

Proof. Let us assume that there exists k such that $(\alpha_1 R_{x_1} + \alpha_2 R_{x_2})^k = R^k_{\alpha_1 x_1 + \alpha_2 x_2} = 0$. Consider $y = \alpha_1 x_1 + \alpha_2 x_2$, and the subspace K spanned by $\{e_1, e_2, e_3, y\}$. Since L is solvable the derived subalgebra L^2 is nilpotent, i.e., $L^2 \subseteq N$. Therefore, K is an ideal of L. Moreover, the operators R_{e_1}, R_{e_2}, R_{e_3} also are nilpotent on K. Hence, due to Engel's Theorem K is nilpotent. But K contains N which contradicts to the maximality of N. This means $\alpha_1 = 0$, $\alpha_2 = 0$ which shows that R_{x_1} and R_{x_2} are linearly nil-independent. □

3.1. Nonabelian nilradical case

We start with $N = \lambda_2(0)$.

Proposition 3.2. *Let L be a 5-dimensional solvable Leibniz algebra, whose nilradical is isomorphic to $\lambda_2(0)$. Then there exists a basis $\{e_1, e_2, e_3, x_1, x_2\}$ such that the L on this basis is represented by the table of multiplication as follows:*

$$\begin{array}{lll} [e_2, e_1] = e_3, & [e_1, x_1] = e_1, & [e_2, x_2] = e_2, \\ [x_1, e_1] = -e_1, & [e_3, x_1] = e_3, & [e_3, x_2] = e_3. \end{array}$$

Proof. The required basis of L is constructed as follows. First we choose a basis $\{e_1, e_2, e_3, x_1, x_2\}$ of L such that $\{e_1, e_2, e_3\}$ is a basis of $\lambda_2(0)$ chosen in Theorem 2.8. By using the fact that the nilradical of L is $\lambda_2(0)$ we define the products of the basis vectors. Since the nilradical of the algebra L is three-dimensional, the restriction of the right multiplication operators R_{x_1} and R_{x_2} to $\lambda_2(0)$ are nil-independent derivations of $\lambda_2(0)$ (see Lemma 3.1). Then owing to Proposition 2.9 we have a part of the table of multiplication of L on this basis as follows

$$\begin{array}{lll} [e_2, e_1] = e_3, & & \\ [e_1, x_1] = a_1 e_1 + a_2 e_3, & [e_2, x_1] = a_3 e_2 + a_4 e_3, & [e_3, x_1] = (a_1 + a_3) e_3, \\ [e_1, x_2] = b_1 e_1 + b_2 e_3, & [e_2, x_2] = b_3 e_2 + b_4 e_3, & [e_3, x_2] = (b_1 + b_3) e_3, \end{array}$$

where $a_1 b_3 - a_3 b_1 \neq 0$, since R_{x_1} and R_{x_2} are linearly nil-independent.

The base change

$$x_1' = \frac{b_3}{a_1 b_3 - a_3 b_1} x_1 - \frac{a_3}{a_1 b_3 - a_3 b_1} x_2, \qquad x_2' = -\frac{b_1}{a_1 b_3 - a_3 b_1} x_1 + \frac{a_1}{a_1 b_3 - a_3 b_1} x_2,$$

brings the table to

$$\begin{array}{lll} [e_2, e_1] = e_3, & & \\ [e_1, x_1] = e_1 + a_2 e_3, & [e_2, x_1] = a_4 e_3, & [e_3, x_1] = e_3, \\ [e_1, x_2] = b_2 e_3, & [e_2, x_2] = e_2 + b_4 e_3, & [e_3, x_2] = e_3. \end{array}$$

Here we can suppose that $b_2 = a_4 = b_4 = 0$ since the base change

$$e_1' = e_1 - b_2 e_3, \qquad e_2' = e_2 - a_4 e_3, \qquad x_2' = x_2 - b_4 e_1$$

yields the result.

Note that these changes don't effect the other products in the table.

Let us to form the other products. First of all taking into account the fact that $e_1 \notin Ann_r(L)$ and applying the properties

$$[x, x] \in Ann_r(L) \quad \text{and} \quad [x, y] + [y, x] \in Ann_r(L)$$

of the right annihilator we can write:

$$\begin{array}{lll}
[x_1,e_1]=-e_1+\alpha_2e_2+\alpha_3e_3, & [x_1,e_2]=\alpha_4e_2+\alpha_5e_3, & [x_1,e_3]=0,\\
[x_2,e_1]=\beta_2e_2+\beta_3e_3, & [x_2,e_2]=\beta_4e_2+\beta_5e_3, & [x_2,e_3]=0,\\
[x_1,x_1]=\gamma_1e_2+\gamma_2e_3, & [x_2,x_2]=\gamma_3e_2+\gamma_4e_3, & \\
[x_2,x_1]=\delta e_1+\gamma_5e_2+\gamma_6e_3, & [x_1,x_2]=-\delta e_1+\gamma_7e_2+\gamma_8e_3. &
\end{array}$$

Indeed, the coefficient -1 of e_1 in the expansion of $[x_1,e_1]$ is derived as follows: let $[x_1,e_1]=\alpha_1e_1+\alpha_2e_2+\alpha_3e_3$ be the expansion of $[x_1,e_1]$. Then

$$\begin{aligned}
&[x_1,e_1]+[e_1,x_1]=(1+\alpha_1)e_1+(a_2+\alpha_2)e_2+\alpha_3e_3\in Ann_r(L),\\
&\text{i.e.,}\quad \big[L,(1+\alpha_1)e_1+(a_2+\alpha_2)e_2+\alpha_3e_3\big]=0.
\end{aligned}$$

Particularly, $[e_2,(1+\alpha_1)e_1+(a_2+\alpha_2)e_2+\alpha_3e_3]=(1+\alpha_1)e_3=0$. This gives $\alpha_1=-1$.

The coefficient of e_1 in the expansion of $[x_2,e_1]$, $[x_1,e_2]$, $[x_2,e_2]$, $[x_1,x_1]$ and $[x_2,x_2]$ to be zero also can be easily derived by the same manner.

The products $[x_1,e_3]=0$ and $[x_2,e_3]=0$ are obtained from the fact that $[e_1,e_2]+[e_2,e_1]=e_3$, i.e., $e_3\in Ann_r(L)$.

Now we simplify this table by using the Leibniz identity.

Applying the Leibniz identity to the triples e_1,x_2,x_1 and e_2,x_2,x_1 as follows

$$\begin{aligned}
0=\big[e_1,[x_2,x_1]\big]&=\big[[e_1,x_2],x_1\big]-\big[[e_1,x_1],x_2\big]=-[e_1+a_2e_3,x_2]=-a_2e_3,\\
\delta e_3=\big[e_2,[x_2,x_1]\big]&=\big[[e_2,x_2],x_1\big]-\big[[e_2,x_1],x_2\big]=[e_2,x_1]=0,
\end{aligned}$$

we get

$$a_2=0,\qquad \delta=0.$$

The identities

$$\begin{aligned}
\big[x_1,[e_1,x_1]\big]&=\big[[x_1,e_1],x_1\big]-\big[[x_1,x_1],e_1\big]=[-e_1+\alpha_2e_2+\alpha_3e_3,x_1]-[\gamma_1e_2+\gamma_2e_3,e_1]\\
&=-e_1+(\alpha_3-\gamma_1)e_3,\\
\big[x_1,[e_1,x_1]\big]&=[x_1,e_1]=-e_1+\alpha_2e_2+\alpha_3e_3
\end{aligned}$$

give

$$\alpha_2=\gamma_1=0.$$

Similarly applying the Leibniz identity to $[x_1,[x_1,e_2]]$; $[x_2,[e_1,x_1]]$; $[x_2,[e_1,x_1]]$; $[x_2,[e_2,x_1]]$; $[x_2,[e_2,e_1]]$; $[x_1,[e_1,x_2]]$; $[x_2,[e_1,x_2]]$; $[x_1,[x_2,x_1]]$ and $[x_2,[x_2,x_1]]$ we get $\alpha_4=\alpha_5=0$; $\beta_2=\gamma_5=0$; $\beta_4=0$; $\beta_5=0$; $\alpha_3=\gamma_7$; $\beta_3=\gamma_3$; $\gamma_8=\gamma_2$ and $\gamma_6=\gamma_4$, respectively.

Finally we change the basis as follows

$$x_1' = x_1 - \gamma_7 e_2 - \gamma_2 e_3, \qquad x_2' = x_2 - \gamma_3 e_2 - \gamma_4 e_3,$$

to obtain the required table of multiplication. □

The 5-dimensional solvable Leibniz algebra from Proposition 3.2 we denote by L_1. Next we prove the following

Proposition 3.3. *There is no a five-dimensional solvable Leibniz algebra with three-dimensional nilradical $\lambda_2(\alpha)$ with $\alpha \neq 0$.*

Proof. Let us assume the contrary and L be a 5-dimensional Leibniz algebra with nilradical $\lambda_2(\alpha)$, $\alpha \neq 0$. We choose a basis $\{e_1, e_2, e_3, x_1, x_2\}$ of L such a way that $\{e_1, e_2, e_3\}$ is a basis of $\lambda_2(\alpha)$ chosen in Theorem 2.8. According to Lemma 3.1 the restriction of the right multiplication operators R_{x_1} and R_{x_2} to $\lambda_2(\alpha)$ are linearly nil-independent derivations of $\lambda_2(\alpha)$. Then using Proposition 2.9 we get

$$\begin{array}{lll} [e_2, e_1] = e_3, & [e_1, e_2] = \alpha e_3, & \\ [e_1, x_1] = a_1 e_1 + a_2 e_3, & [e_2, x_1] = a_3 e_2 + a_4 e_3, & [e_3, x_1] = (a_1 + a_3) e_3, \\ [e_1, x_2] = b_1 e_1 + b_2 e_3, & [e_2, x_2] = b_3 e_2 + b_4 e_3, & [e_3, x_2] = (b_1 + b_3) e_3, \end{array}$$

where $a_1 b_3 - a_3 b_1 \neq 0$, since R_{x_1} and R_{x_2} are linearly nil-independent.

Taking the change

$$x_1' = \frac{b_3}{a_1 b_3 - a_3 b_1} x_1 - \frac{a_3}{a_1 b_3 - a_3 b_1} x_2, \qquad x_2' = -\frac{b_1}{a_1 b_3 - a_3 b_1} x_1 + \frac{a_1}{a_1 b_3 - a_3 b_1} x_2,$$

we obtain

$$\begin{array}{lll} [e_2, e_1] = e_3, & [e_1, e_2] = \alpha e_3, & \\ [e_1, x_1] = e_1 + a_2 e_2, & [e_2, x_1] = a_4 e_3, & [e_3, x_1] = e_3, \\ [e_1, x_2] = b_2 e_3, & [e_2, x_2] = e_2 + b_4 e_3, & [e_3, x_2] = e_3. \end{array}$$

Since $\alpha \neq 0$, then it is easy to see that the right annihilator of L consists of only $\{e_3\}$. Therefore,

$$\begin{array}{ll} [x_1, e_1] = -e_1 + \alpha_2 e_2, & [x_1, e_2] = \alpha_4 e_3, \\ [x_2, e_1] = \beta_2 e_3, & [x_2, e_2] = -e_2 + \beta_4 e_3. \end{array}$$

Then considering the Leibniz identity

$$0 = \big[x_1, [e_1, e_2]\big] = \big[[x_1, e_1], e_2\big] - \big[[x_1, e_2], e_1\big] = [-e_1 + \alpha_2 e_3, e_2] - [\alpha_4 e_3, e_2] = -\alpha e_3,$$

we get a contradiction. □

Proposition 3.4. *Let L be a 5-dimensional solvable Leibniz algebra, whose nilradical is isomorphic to λ_5. Then L is isomorphic to one of the following two nonisomorphic algebras:*

$$L_2\colon\quad \begin{cases} [e_1,e_1]=e_3, \\ [e_1,x_1]=e_1, \\ [x_1,e_1]=-e_1, \\ [e_3,x_1]=2e_3, \\ [e_2,x_2]=e_2, \end{cases} \qquad L_3\colon\quad \begin{cases} [e_1,e_1]=e_3, \\ [e_1,x_1]=e_1, \\ [x_1,e_1]=-e_1, \\ [e_3,x_1]=2e_3, \\ [e_2,x_2]=e_2, \\ [x_2,e_2]=-e_2. \end{cases}$$

Proof. Let L be a 5-dimensional Leibniz algebra with nilradical λ_5. Similar to those of previous propositions we take a basis $\{e_1,e_2,e_3,x_1,x_2\}$ of L as an extension of the basis $\{e_1,e_2,e_3\}$ of λ_2 chosen in Theorem 2.8. Taking into account Lemma 3.1 and applying Proposition 2.9 for $N=\lambda_5$ case we get

$$\begin{array}{lll} [e_1,e_1]=e_3, & & \\ [e_1,x_1]=a_1e_1+a_2e_2+a_3e_3, & [e_2,x_1]=a_4e_2+a_5e_3, & [e_3,x_1]=2a_1e_3, \\ [e_1,x_2]=b_1e_1+b_2e_2+b_3e_3, & [e_2,x_2]=b_4e_2+b_5e_3, & [e_3,x_2]=2b_1e_3, \end{array}$$

where $a_1b_4-a_4b_1\neq 0$, since R_{x_1} and R_{x_2} are linearly nil-independent.

Taking the same base change as in the proof of Proposition 3.2 and due to the fact that $e_1\notin Ann_r(L)$, we can write:

$$\begin{array}{ll} [e_1,e_1]=e_3, & [e_1,x_1]=e_1, \\ [e_2,x_1]=a_5e_3, & [e_3,x_1]=2e_3, \\ [e_1,x_2]=b_2e_2, & [e_2,x_2]=e_2, \\ [x_1,e_1]=-e_1+\alpha_2e_2+\alpha_3e_3, & [x_1,e_2]=\alpha_4e_2+\alpha_5e_3, \\ [x_2,e_1]=\beta_2e_2+\beta_3e_3, & [x_2,e_2]=\beta_4e_2+\beta_5e_3, \\ [x_1,x_1]=\gamma_1e_2+\gamma_2e_3, & [x_2,x_2]=\gamma_3e_2+\gamma_4e_3, \\ [x_2,x_1]=\delta e_1+\gamma_5e_2+\gamma_6e_3, & [x_1,x_2]=-\delta e_1+\gamma_7e_2+\gamma_8e_3. \end{array}$$

Applying sequentially to triples of basis vectors from $\{e_1,e_2,e_3,x_1,x_2\}$ the Leibniz identity together with the table above we obtain the following relations for the structure constants

$$a_5=b_2=\delta=\alpha_2=\alpha_3=\alpha_4=\alpha_5=0,$$

$$\beta_2=\beta_3=\beta_5=\gamma_1=\gamma_4=\gamma_8=0,$$

$$\beta_4(1+\beta_4)=0, \qquad \beta_4\gamma_3=0, \qquad \gamma_5=\gamma_7\beta_4.$$

Owing to $\beta_4(1+\beta_4)=0$ we have the following two choices: $\beta_4=0$ and $\beta_4=-1$.

If $\beta_4 = 0$, then taking the basis transformation of the form

$$x_1' = x_1 - \gamma_7 e_2 - \frac{\gamma_2}{2} e_3, \qquad x_2' = x_2 - \gamma_3 e_2 - \frac{\gamma_6}{2} e_3,$$

we obtain L_2.

But if $\beta_4 = -1$, then $\gamma_3 = 0$ and taking the basis transformation of the form

$$x_1' = x_1 - \gamma_7 e_2 - \frac{\gamma_2}{2} e_3, \qquad x_2' = x_2 - \frac{\gamma_6}{2} e_3,$$

we get L_3.

Since $Ann_r(L_2) = Span\{e_2, e_3\}$ and $Ann_r(L_3) = Span\{e_3\}$ the algebras L_2 and L_3 are not isomorphic. □

3.2. Abelian nilradical case

Let L be a five-dimensional solvable Leibniz algebra with a basis $\{x_1, x_2, e_1, e_2, e_3\}$, where $\{e_1, e_2, e_3\}$ is the basis of three-dimensional abelian nilradical λ_6 chosen in Theorem 2.8. Due to Lemma 3.1 the operators R_{x_1} and R_{x_2} are linearly nil-independent derivations of the nilradical N. Further we need the description of the actions of R_{x_1} and R_{x_2} on N.

Proposition 3.5. *The basis $\{x_1, x_2, e_1, e_2, e_3\}$ of L can be chosen such a way that the actions of the right multiplication operators R_{x_1} and R_{x_2} on the basis $\{e_1, e_2, e_3\}$ of N are expressed as follows:*

$$\begin{array}{lllll}
A. & R_{x_1}(e_1) = e_1, & R_{x_1}(e_3) = \mu_1 e_3, & R_{x_2}(e_2) = e_2, & R_{x_2}(e_3) = \mu_2 e_3, \\
B. & R_{x_1}(e_1) = e_1, & R_{x_1}(e_2) = e_2, & R_{x_2}(e_1) = e_2, & R_{x_2}(e_3) = e_3, \\
C. & R_{x_1}(e_1) = e_1 + e_2, & R_{x_1}(e_2) = e_2, & R_{x_2}(e_1) = \mu e_2, & R_{x_2}(e_3) = e_3,
\end{array}$$

where nonwritten actions are zero.

Proof. First of all we have some freedom of choosing the matrix of R_{x_1} depending on multiplicity of eigenvalues of R_{x_1}. The following three cases may occur: the matrix of R_{x_1} is congruent to

$$\begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 1 \\ 0 & 0 & \mu_1 \end{pmatrix}.$$

Let us now search the possibilities for the matrix of R_{x_2}. Put

$$R_{x_2}(e_i) = \alpha_{i,1} e_1 + \alpha_{i,2} e_2 + \alpha_{i,3} e_3, \quad 1 \le i \le 3.$$

Since L is solvable and its nilradical $N = \lambda_6$ is abelian this implies $R_{[x_1,x_2]}(y) = 0$ for any $y \in \lambda_6$. Now we make a case by case consideration according to the above matrix view of R_{x_1}.

Case 1. Let the matrix of R_{x_1} be congruent to

$$\begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}.$$

Then we have

$$R_{x_1}(e_1) = \mu_1 e_1, \qquad R_{x_1}(e_2) = \mu_2 e_2, \qquad R_{x_1}(e_3) = \mu_3 e_3.$$

By the use of the identities $R_{[x_1,x_2]}(e_i) = 0$ for $1 \le i \le 3$ we obtain the following constraints:

$$\begin{aligned} &(\mu_1 - \mu_2)\alpha_{1,2} = 0, \qquad (\mu_1 - \mu_3)\alpha_{1,3} = 0, \\ &(\mu_2 - \mu_1)\alpha_{2,1} = 0, \qquad (\mu_2 - \mu_3)\alpha_{2,3} = 0, \\ &(\mu_3 - \mu_1)\alpha_{3,1} = 0, \qquad (\mu_3 - \mu_2)\alpha_{3,2} = 0. \end{aligned} \tag{3.1}$$

Case 1.1. Let $\mu_1 \neq \mu_2$, $\mu_1 \neq \mu_3$, $\mu_2 \neq \mu_3$. Owing to the constraints (3.1) we have

$$\alpha_{1,2} = 0, \qquad \alpha_{1,3} = 0, \qquad \alpha_{2,1} = 0, \qquad \alpha_{2,3} = 0, \qquad \alpha_{3,1} = 0, \qquad \alpha_{3,2} = 0.$$

Since R_{x_1} and R_{x_2} are linearly nil-independent, without loss of generality we can assume that $\mu_1\alpha_{2,2} - \mu_2\alpha_{1,1} \neq 0$. Then applying the transformation

$$\begin{aligned} x_1' &= \frac{\alpha_{2,2}}{\mu_1\alpha_{2,2} - \mu_2\alpha_{1,1}} x_1 - \frac{\mu_2}{\mu_1\alpha_{2,2} - \mu_2\alpha_{1,1}} x_2, \\ x_2' &= -\frac{\alpha_{1,1}}{\mu_1\alpha_{2,2} - \mu_2\alpha_{1,1}} x_1 + \frac{\mu_1}{\mu_1\alpha_{2,2} - \mu_2\alpha_{1,1}} x_2 \end{aligned}$$

we get $\mu_1 = \alpha_{2,2} = 1$, $\mu_2 = \alpha_{1,1} = 0$, that means the operators R_{x_1} and R_{x_2} have the form A in the proposition.

Case 1.2. Let any two of μ_1, μ_2, μ_3 be equal. Then, without loss of generality, we can assume that $\mu_1 = \mu_2 \neq \mu_3$. Then due to the constraints (3.1) we get

$$\alpha_{1,3} = 0, \qquad \alpha_{2,3} = 0, \qquad \alpha_{3,1} = 0, \qquad \alpha_{3,2} = 0.$$

Changing the basis we bring the matrix $\left(\begin{smallmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{smallmatrix}\right)$ to one of the following Jordan's forms

$$\begin{pmatrix} \alpha_{1,1} & 0 \\ 0 & \alpha_{2,2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha_{1,1} & 1 \\ 0 & \alpha_{1,1} \end{pmatrix}.$$

In the former case the actions of R_{x_1} and R_{x_2} have the form A.

In the later case we use the base change

$$x_1' = \frac{\alpha_{3,3}}{\mu_1\alpha_{3,3} - \mu_3\alpha_{1,1}}x_1 - \frac{\mu_3}{\mu_1\alpha_{3,3} - \mu_3\alpha_{1,1}}x_2,$$
$$x_2' = -\frac{\alpha_{1,1}}{\mu_1\alpha_{3,3} - \mu_3\alpha_{1,1}}x_1 + \frac{\mu_1}{\mu_1\alpha_{3,3} - \mu_3\alpha_{1,1}}x_2$$

to get

$$R_{x_1}(e_1) = e_1 + \alpha e_2, \qquad R_{x_1}(e_2) = e_2, \qquad R_{x_2}(e_1) = \beta e_2, \qquad R_{x_2}(e_3) = e_3.$$

- if $\alpha = 0$, $\beta = 0$, then the actions of R_{x_1} and R_{x_2} have the form A with $\mu_1 = 1$, $\mu_2 = 0$;
- if $\alpha = 0$, $\beta \neq 0$, then by the change $e_2' = \beta e_2$ we see that R_{x_1} and R_{x_2} act like B;
- if $\alpha \neq 0$, then applying $e_2' = \alpha e_2$ we obtain that the actions of R_{x_1} and R_{x_2} have the form C.

Case 1.3. Let $\mu_1 = \mu_2 = \mu_3$. Then the operator R_{x_1} acts as the identity operator on N. Let us consider Jordan's form of R_{x_2}. Since the operators R_{x_1} and R_{x_2} are linearly nil-independent the following two possibilities may occur:

If

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix} \quad \text{is congruent to} \quad \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}$$

then similar to Case 1.1 we obtain R_{x_1} and R_{x_2} in the form A.

But if

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix} \quad \text{is congruent to} \quad \begin{pmatrix} \beta_1 & 1 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}$$

then similar to Case 1.2 the R_{x_1} and R_{x_2} have the form B.

Case 2. Let the matrix of the operator R_{x_1} be congruent to

$$\begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}.$$

Then we have

$$R_{x_1}(e_1) = \mu_1 e_1 + e_2, \qquad R_{x_1}(e_2) = \mu_1 e_2, \qquad R_{x_1}(e_3) = \mu_2 e_3.$$

Using the identities $R_{[x_1,x_2]}(e_i) = 0$ for $1 \leq i \leq 3$ we get the following constraints:

$$\begin{array}{lll} \alpha_{2,1} = 0, & \alpha_{2,3} = 0, & \alpha_{3,1} = 0, \\ \alpha_{2,2} = \alpha_{1,1}, & (\mu_2 - \mu_1)\alpha_{1,3} = 0, & (\mu_2 - \mu_1)\alpha_{3,2} = 0. \end{array} \tag{3.2}$$

Similarly to Case 1, considering all possibilities for parameters μ_1 and μ_2, we obtain the operators R_{x_1} and R_{x_2} in one the forms A, B, C.

Case 3. Let the matrix of the operator R_{x_1} be congruent to

$$\begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 1 \\ 0 & 0 & \mu_1 \end{pmatrix}.$$

Then

$$R_{x_1}(e_1) = \mu_1 e_1 + e_2, \qquad R_{x_1}(e_2) = \mu_1 e_2 + e_3, \qquad R_{x_1}(e_3) = \mu_1 e_3.$$

Again due to the identities $R_{[x_1,x_2]}(e_i) = 0$ for $1 \leq i \leq 3$, it is easy to obtain

$$\alpha_{2,1} = \alpha_{3,1} = \alpha_{3,2} = 0, \qquad \alpha_{1,1} = \alpha_{2,2} = \alpha_{3,3},$$

which shows that R_{x_1} and R_{x_2} are nil-dependent. However, it contradicts to the hypothesis of the proposition. This contradiction completes Case 3. □

Theorem 3.6. *Let L be a 5-dimensional solvable Leibniz algebra, whose nilradical is 3-dimensional abelian algebra. Then there exists a basis $\{e_1, e_2, e_3, x_1, x_2\}$ of L such that on $\{e_1, e_2, e_3, x_1, x_2\}$ the L is represented as one of the following pairwise nonisomorphic algebras*

$M_1(\mu_1, \mu_2), \mu_1 \neq 0$:

$$\begin{cases} [e_1, x_1] = e_1 & [e_3, x_1] = \mu_1 e_3, \\ [e_2, x_2] = e_2, & [e_3, x_2] = \mu_2 e_3, \\ [x_1, e_1] = -e_1, & [x_1, e_3] = -\mu_1 e_3, \\ [x_2, e_2] = -e_2, & [x_2, e_3] = -\mu_2 e_3, \end{cases}$$

$M_2(\mu_1, \mu_2)$:

$$\begin{cases} [e_1, x_1] = e_1 & [e_3, x_1] = \mu_1 e_3, \\ [e_2, x_2] = e_2, & [e_3, x_2] = \mu_2 e_3, \\ [x_1, e_1] = -e_1, & [x_2, e_2] = -e_2, \end{cases}$$

$M_3(\mu), \mu \neq 0$:

$$\begin{cases} [e_1, x_1] = e_1 & \\ [e_2, x_2] = e_2, & [e_3, x_2] = \mu e_3, \\ [x_2, e_2] = -e_2, & [x_2, e_3] = -\mu e_3, \end{cases}$$

$M_4(\mu_1, \mu_2)$:

$$\begin{cases} [e_1, x_1] = e_1 & [e_3, x_1] = \mu_1 e_3, \\ [e_2, x_2] = e_2, & [e_3, x_2] = \mu_2 e_3, \\ [x_2, e_2] = -e_2, & \end{cases}$$

$M_5(\mu_1, \mu_2)$:

$$\begin{cases} [e_1, x_1] = e_1 & [e_3, x_1] = \mu_1 e_3, \\ [e_2, x_2] = e_2, & [e_3, x_2] = \mu_2 e_3, \end{cases}$$

$M_6(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$:

$$\begin{cases} [e_1, x_1] = e_1 & [e_2, x_2] = e_2, \\ [x_1, e_1] = -e_1, & [x_2, e_2] = -e_2, \\ [x_1, x_1] = \lambda_1 e_3, & [x_2, x_1] = \lambda_2 e_3, \\ [x_1, x_2] = \lambda_3 e_3, & [x_2, x_2] = \lambda_4 e_3, \end{cases}$$

$M_7(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$:

$$\begin{cases} [e_1,x_1]=e_1, & \\ [e_2,x_2]=e_2, & [x_2,e_2]=-e_2 \\ [x_1,x_1]=\lambda_1 e_3, & [x_2,x_1]=\lambda_2 e_3, \\ [x_1,x_2]=\lambda_3 e_3, & [x_2,x_2]=\lambda_4 e_3, \end{cases}$$

$M_8(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$:

$$\begin{cases} [e_1,x_1]=e_1 & [e_2,x_2]=e_2, \\ [x_1,x_1]=\lambda_1 e_3, & [x_2,x_1]=\lambda_2 e_3, \\ [x_1,x_2]=\lambda_3 e_3, & [x_2,x_2]=\lambda_4 e_3, \end{cases}$$

M_9:

$$\begin{cases} [e_1,x_1]=e_1 & [e_2,x_2]=e_2, \\ [e_3,x_1]=e_3 & \\ [x_1,e_1]=-e_1, & [x_2,e_1]=-e_3, \end{cases}$$

M_{10}:

$$\begin{cases} [e_1,x_1]=e_1, & [e_3,x_1]=e_3, \\ [x_1,e_1]=-e_1, & [x_2,e_1]=e_3, \\ [e_2,x_2]=e_2 & [x_2,e_2]=-e_2, \end{cases}$$

P_1:

$$\begin{cases} [e_1,x_1]=e_1, & [e_2,x_1]=e_2, \\ [e_1,x_2]=e_2, & [e_3,x_2]=e_3, \end{cases}$$

P_2:

$$\begin{cases} [e_1,x_1]=e_1, & [e_2,x_1]=e_2, \\ [e_1,x_2]=e_2, & [e_3,x_2]=e_3, \\ [x_2,e_3]=-e_3, & \end{cases}$$

P_3:

$$\begin{cases} [e_1,x_1]=e_1, & [e_2,x_1]=e_3, \\ [e_1,x_2]=e_2, & [e_3,x_2]=e_3, \\ [x_1,e_1]=-e_1, & [x_1,e_2]=-e_2, \\ [x_2,e_1]=-e_2, & \end{cases}$$

P_4:

$$\begin{cases} [e_1,x_1]=e_1, & [e_2,x_1]=e_2, \\ [e_1,x_2]=e_2, & [e_3,x_2]=e_3, \\ [x_1,e_1]=-e_1, & [x_1,e_2]=-e_2, \\ [x_2,e_1]=-e_2, & [x_2,e_3]=-e_3, \end{cases}$$

$Q_1(\mu)$:

$$\begin{cases} [e_1,x_1]=e_1+e_2, & [e_2,x_1]=e_2, \\ [e_1,x_2]=\mu e_2, & [e_3,x_2]=e_3, \end{cases}$$

$Q_2(\mu)$:

$$\begin{cases} [e_1,x_1]=e_1+e_2, & [e_2,x_1]=e_2, \\ [e_1,x_2]=\mu e_2, & [e_3,x_2]=e_3, \\ [x_2,e_3]=-e_3, & \end{cases}$$

$Q_3(\mu)$:

$$\begin{cases} [e_1,x_1]=e_1+e_2, & [e_2,x_1]=e_2, \\ [e_1,x_2]=\mu e_2, & [e_3,x_2]=e_3, \\ [x_1,e_1]=-e_1-e_2, & [x_1,e_2]=-e_2, \\ [x_2,e_1]=-\mu e_2, & \end{cases}$$

$Q_4(\mu)$:

$$\begin{cases} [e_1,x_1]=e_1+e_2, & [e_2,x_1]=e_2, \\ [e_1,x_2]=\mu e_2, & [e_3,x_2]=e_3, \\ [x_1,e_1]=-e_1-e_2, & [x_1,e_2]=-e_2, \\ [x_2,e_1]=-\mu e_2, & [x_2,e_3]=-e_3, \end{cases}$$

where $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\neq(0,0,0,0)$.

Moreover,

- $M_1(\mu_1,\mu_2)\cong M_1(\mu_2,\mu_1)\cong M_1(\frac{1}{\mu_1},-\frac{\mu_2}{\mu_1})$,
- $M_2(\mu_1,\mu_2)\cong M_2(\mu_2,\mu_1)$,
- $M_5(\mu_1,\mu_2)\cong M_5(\mu_2,\mu_1)$,
- $M_6(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\cong M_6(\lambda_4,\lambda_3,\lambda_2,\lambda_1)$,
- $M_8(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\cong M_8(\lambda_4,\lambda_3,\lambda_2,\lambda_1)$.

Proof. Let L be a 5-dimensional solvable Leibniz algebra, whose nilradical is 3-dimensional abelian algebra. The products $[e_i, x_j]$ are due to Proposition 3.5. For the other products we let

$$\begin{cases} [x_1, e_i] = \alpha_{i,1}e_1 + \alpha_{i,2}e_2 + \alpha_{i,3}e_3, & [x_2, e_i] = \beta_{i,1}e_1 + \beta_{i,2}e_2 + \beta_{i,3}e_3, \quad 1 \le i \le 3, \\ [x_1, x_1] = \gamma_{1,1}e_1 + \gamma_{1,2}e_2 + \gamma_{1,3}e_3, & [x_2, x_1] = \gamma_{2,1}e_1 + \gamma_{2,2}e_2 + \gamma_{2,3}e_3, \\ [x_1, x_2] = \gamma_{3,1}e_1 + \gamma_{3,2}e_2 + \gamma_{3,3}e_3, & [x_2, x_2] = \gamma_{4,1}e_1 + \gamma_{4,2}e_2 + \gamma_{4,3}e_3. \end{cases}$$

Case 1. Let $\{e_1, e_2, e_3, x_1, x_2\}$ be the basis corresponding to part A in Proposition 3.5. Therefore

$$[e_1, x_1] = e_1, \qquad [e_3, x_1] = \mu_1 e_3, \qquad [e_2, x_2] = e_2, \qquad [e_3, x_2] = \mu_2 e_3.$$

We use the Leibniz identity for the products

$$\bigl[x_1, [e_1, x_1]\bigr], \ \bigl[x_1, [e_1, x_2]\bigr], \ \bigl[x_1, [e_2, x_1]\bigr], \ \bigl[x_1, [e_2, x_2]\bigr], \ \bigl[x_1, [e_3, x_1]\bigr], \ \bigl[x_1, [e_3, x_2]\bigr],$$

$$\bigl[x_2, [e_1, x_1]\bigr], \ \bigl[x_2, [e_1, x_2]\bigr], \ \bigl[x_2, [e_2, x_1]\bigr], \ \bigl[x_2, [e_2, x_2]\bigr], \ \bigl[x_2, [e_3, x_1]\bigr], \ \bigl[x_2, [e_3, x_2]\bigr],$$

to obtain the following constraints for the structure constants

$$\begin{cases} \alpha_{1,2} = 0, & \alpha_{2,3}\mu_1 = 0, & \alpha_{1,3}\mu_2 = 0, & \alpha_{1,3}(\mu_1 - 1) = 0, & \alpha_{2,3}(\mu_2 - 1) = 0, \\ \alpha_{2,1} = 0, & \alpha_{3,2}\mu_1 = 0, & \alpha_{3,1}\mu_2 = 0, & \alpha_{3,1}(\mu_1 - 1) = 0, & \alpha_{3,2}(\mu_2 - 1) = 0, \\ \beta_{1,2} = 0, & \beta_{2,3}\mu_1 = 0, & \beta_{1,3}\mu_2 = 0, & \beta_{1,3}(\mu_1 - 1) = 0, & \beta_{2,3}(\mu_2 - 1) = 0, \\ \beta_{2,1} = 0, & \beta_{3,2}\mu_1 = 0, & \beta_{3,1}\mu_2 = 0, & \beta_{3,1}(\mu_1 - 1) = 0, & \beta_{3,2}(\mu_2 - 1) = 0. \end{cases} \tag{3.3}$$

Case 1.1. Let $(\mu_1, \mu_2) \notin \{(0, 1), (1, 0)\}$. Then by virtue of (3.3) one has

$$\alpha_{1,2} = \alpha_{2,1} = \alpha_{1,3} = \alpha_{2,3} = \alpha_{3,1} = \alpha_{3,2} = 0,$$
$$\beta_{1,2} = \beta_{2,1} = \beta_{1,3} = \beta_{2,3} = \beta_{3,1} = \beta_{3,2} = 0.$$

The Leibniz identities

$$0 = \bigl[[x_1, x_1], e_2\bigr] = \bigl[x_1, [x_1, e_2]\bigr] + \bigl[[x_1, e_2], x_1\bigr] = [x_1, \alpha_{2,2}e_2] + [\alpha_{2,2}e_2, x_1] = \alpha_{2,2}^2 e_2,$$
$$0 = \bigl[[x_2, x_2], e_1\bigr] = \bigl[x_2, [x_2, e_1]\bigr] + \bigl[[x_2, e_1], x_2\bigr] = [x_2, \beta_{1,1}e_1] + [\beta_{1,1}e_1, x_2] = \beta_{1,1}^2 e_1,$$

give

$$\alpha_{2,2} = 0, \qquad \beta_{1,1} = 0.$$

Considering the Leibniz identity for the products

$$\bigl[x_1, [x_1, x_2]\bigr], \qquad \bigl[x_2, [x_2, x_1]\bigr], \qquad \bigl[x_1, [x_2, x_1]\bigr], \qquad \bigl[x_2, [x_1, x_2]\bigr]$$

we get

$$\begin{array}{lll} \gamma_{1,2} = 0, & \gamma_{3,1}(\alpha_{1,1}+1) = 0, & \gamma_{3,3}\alpha_{3,3} = \gamma_{1,3}\mu_2 - \gamma_{3,3}\mu_1, \\ \gamma_{4,1} = 0, & \gamma_{2,2}(\beta_{2,2}+1) = 0, & \gamma_{2,3}\beta_{3,3} = \gamma_{4,3}\mu_1 - \gamma_{2,3}\mu_2, \\ & \gamma_{3,1} = \gamma_{2,1}\alpha_{1,1}, & \gamma_{2,3}\alpha_{3,3} = \gamma_{3,3}\mu_1 - \gamma_{1,3}\mu_2, \\ & \gamma_{2,2} = \gamma_{3,2}\beta_{2,2}, & \gamma_{3,3}\beta_{3,3} = \gamma_{2,3}\mu_2 - \gamma_{4,3}\mu_1. \end{array} \tag{3.4}$$

As well as sequentially applying the Leibniz identity to $[x_1,[x_1,e_1]]$, $[x_1,[x_1,e_3]]$, $[x_1,[x_1,x_1]]$, $[x_1,[x_2,e_3]]$, $[x_1,[x_2,x_2]]$, $[x_2,[x_1,e_3]]$, $[x_2,[x_1,x_1]]$, $[x_2,[x_2,x_2]]$, $[x_2,[x_2,e_3]]$ and $[x_2,[x_2,x_2]]$ we obtain the constraints

$$\begin{gathered} \alpha_{1,1}(\alpha_{1,1}+1) = 0, \\ \alpha_{3,3}(\alpha_{3,3}+\mu_1) = 0, \\ \alpha_{1,1}\gamma_{1,1} = 0, \qquad \alpha_{3,3}\gamma_{1,3} = 0, \\ \alpha_{3,3}(\beta_{3,3}+\mu_2) = 0, \\ \alpha_{3,3}\gamma_{4,3} = 0, \\ \beta_{3,3}(\alpha_{3,3}+\mu_1) = 0, \\ \beta_{3,3}\gamma_{1,3} = 0, \\ \beta_{2,2}(\beta_{2,2}+1) = 0, \\ \beta_{3,3}(\beta_{3,3}+\mu_2) = 0, \quad \text{and} \\ \beta_{2,2}\gamma_{4,2} = 0, \qquad \beta_{3,3}\gamma_{4,3} = 0, \end{gathered} \tag{3.5}$$

respectively.

Therefore the table of multiplication of L is written

$$\begin{cases} [e_1,x_1] = e_1, & [x_1,e_1] = \alpha_{1,1}e_1, & [x_1,x_1] = \gamma_{1,1}e_1 + \gamma_{1,3}e_3, \\ [e_3,x_1] = \mu_1 e_3, & [x_1,e_3] = \alpha_{3,3}e_3, & [x_2,x_1] = \gamma_{2,1}e_1 + \gamma_{2,2}e_2 + \gamma_{2,3}e_3, \\ [e_2,x_2] = e_2, & [x_2,e_2] = \beta_{2,2}e_2, & [x_1,x_2] = \gamma_{3,1}e_1 + \gamma_{3,2}e_2 + \gamma_{3,3}e_3, \\ [e_3,x_2] = \mu_2 e_3, & [x_2,e_3] = \beta_{3,3}e_3, & [x_2,x_2] = \gamma_{4,2}e_2 + \gamma_{4,3}e_3, \end{cases}$$

with the conditions (3.4) and (3.5).

It is observed that if $\mu_1 = \mu_2 = 0$, then $\alpha_{3,3} = \beta_{3,3} = 0$ and $C(L) = \mathit{Span}\{e_3\}$, otherwise $C(L)$ is trivial. Thus, we distinguish following two cases $(\mu_1,\mu_2) \neq (0,0)$ and $(\mu_1,\mu_2) = (0,0)$, which correspond to $C(L) = \mathit{Span}\{e_3\} \neq 0$ and $C(L) = 0$, respectively.

Case 1.1.1. Let $(\mu_1,\mu_2) \neq (0,0)$ (i.e., $C(L) = 0$).

- Let $\alpha_{1,1} = -1$, $\beta_{2,2} = -1$ (i.e., $e_1, e_2 \notin \mathit{Ann}_r(L)$). Then one has $\gamma_{1,1} = 0$, $\gamma_{4,2} = 0$, $\gamma_{3,1} = -\gamma_{2,1}$, $\gamma_{3,2} = -\gamma_{2,2}$.

Taking the base change

$$x_1 = x_1 + \gamma_{2,2}e_2, \qquad x_2 = x_2 - \gamma_{2,1}e_1,$$

we can assume that $\gamma_{2,1} = \gamma_{2,2} = 0$.

Note that, due to the symmetricity of the basis vectors e_1, e_2 and x_1, x_2, without loss of generality we can assume that $\mu_1 \neq 0$.

- If $\alpha_{3,3} = -\mu_1$, $\beta_{3,3} = -\mu_2$ (i.e. $Ann_r(L) = 0$), then we obtain $\gamma_{1,3} = 0$, $\gamma_{4,3} = 0$, $\gamma_{3,3} = -\gamma_{2,3}$ and taking the base change $x_2 = x_2 - \frac{\gamma_{2,3}}{\mu_1}e_1$ we get the algebra $M_1(\mu_1, \mu_2)$.
- If $\alpha_{3,3} = 0$, $\beta_{3,3} = 0$, (i.e. $Ann_r(L) = Span\{e_3\}$), then one has $\gamma_{3,3} = \frac{\gamma_{1,3}\mu_2}{\mu_1}$, $\gamma_{4,3} = \frac{\gamma_{2,3}\mu_2}{\mu_1}$ and considering the base change $x_1' = x_1 - \frac{\gamma_{1,3}}{\mu_1}e_3$, $x_2' = x_2 - \frac{\gamma_{2,3}}{\mu_1}e_3$, we derive $M_2(\mu_1, \mu_2)$.

- Let $\alpha_{1,1} = 0$, $\beta_{2,2} = -1$, or $\alpha_{1,1} = -1$, $\beta_{2,2} = 0$. Due to the symmetricity of the basis elements e_1, e_2 and x_1, x_2, without loss of generality we can assume that $\alpha_{1,1} = 0$, $\beta_{2,2} = -1$. This gives

$$\gamma_{3,1} = 0, \qquad \gamma_{4,2} = 0, \qquad \gamma_{3,2} = -\gamma_{2,2} = 0.$$

  - If $\alpha_{3,3} = -\mu_1$, $\beta_{3,3} = -\mu_2$, then $\gamma_{1,3} = 0$, $\gamma_{4,3} = 0$, $\gamma_{3,3} = -\gamma_{2,3}$. The change $x_1' = x_1 - \gamma_{1,1}e_1 + \gamma_{2,2}e_2$, $x_2' = x_2 - \gamma_{2,1}e_1$, gives the following table of multiplication:

$$\begin{cases} [e_1, x_1] = e_1 & [e_3, x_1] = \mu_1 e_3, \\ [e_2, x_2] = e_2, & [e_3, x_2] = \mu_2 e_3, \\ [x_1, e_3] = -\mu_1 e_3, & \\ [x_2, e_2] = -e_2, & [x_2, e_3] = -\mu_2 e_3, \\ [x_2, x_1] = \gamma_{2,3} e_3, & [x_1, x_2] = -\gamma_{2,3} e_3. \end{cases}$$

    - ∗ If $\mu_1 \neq 0$, then taking the base change $e_1' = e_3$, $e_3' = e_1$, $x_1' = \frac{1}{\mu_1}x_1$, $x_2' = x_2 - \frac{\mu_2}{\mu_1}x_1 - \frac{\gamma_{2,3}}{\mu_1}e_3$ we obtain the algebra $M_2(\mu_1, \mu_2)$.
    - ∗ If $\mu_1 = 0$, then $\mu_2 \neq 0$ and after the base change $x_1' = x_1 + \frac{\gamma_{2,3}}{\mu_2}e_3$ we get $M_3(\mu_2)$.
  - If $\alpha_{3,3} = 0$, $\beta_{3,3} = 0$, then we get $\gamma_{3,3} = \frac{\gamma_{1,3}\mu_2}{\mu_1}$, $\gamma_{4,3} = \frac{\gamma_{2,3}\mu_2}{\mu_1}$ and by the base change $x_1' = x_1 - \gamma_{1,1}e_1 + \gamma_{2,2}e_2 - \frac{\gamma_{1,3}}{\mu_1}e_3$, $x_2' = x_2 - \gamma_{2,1}e_1 - \frac{\gamma_{2,3}}{\mu_1}e_3$, we derive $M_4(\mu_1, \mu_2)$.
- Let $\alpha_{1,1} = 0$, $\beta_{2,2} = 0$, then one has $\gamma_{3,1} = 0$, $\gamma_{2,2} = 0$.
  - If $\alpha_{3,3} = -\mu_1$, $\beta_{3,3} = -\mu_2$, then $\gamma_{1,3} = 0$, $\gamma_{4,3} = 0$, $\gamma_{3,3} = -\gamma_{2,3}$. Applying the base change $x_1' = x_1 - \gamma_{1,1}e_1 - \gamma_{3,2}e_2$, $x_2' = x_2 - \gamma_{2,1}e_1 - \gamma_{4,2}e_2 - \frac{\gamma_{2,3}}{\mu_1}e_3$, we obtain the following table of multiplications

$$\begin{cases} [e_1, x_1] = e_1 & [e_3, x_1] = \mu_1 e_3, \\ [e_2, x_2] = e_2, & [e_3, x_2] = \mu_2 e_3, \\ [x_1, e_3] = -\mu_1 e_3, & [x_2, e_3] = -\mu_2 e_3. \end{cases}$$

It is easy to see that the base change $e_1' = e_2$, $e_2' = e_3$, $e_3' = e_1$, $x_1' = \frac{1}{\mu_1}x_1$, $x_2' = x_2 - \frac{\mu_2}{\mu_1}x_1$ in the table gives $M_4(\mu_1, \mu_2)$.

- If $\alpha_{3,3} = 0$, $\beta_{3,3} = 0$, we get $\gamma_{3,3} = \frac{\gamma_{1,3}\mu_2}{\mu_1}$, $\gamma_{4,3} = \frac{\gamma_{2,3}\mu_2}{\mu_1}$ and taking the change $x_1' = x_1 - \gamma_{1,1}e_1 - \gamma_{3,2}e_2 - \frac{\gamma_{1,3}}{\mu_1}e_3$, $x_2' = x_2 - \gamma_{2,1}e_1 - \gamma_{4,2}e_2 - \frac{\gamma_{2,3}}{\mu_1}e_3$, we obtain $M_5(\mu_1, \mu_2)$.

Case 1.1.2. Let $\mu_1 = \mu_2 = 0$ (i.e., $C(L) = Span\{e_3\}$). Then we get $\alpha_{3,3} = \beta_{3,3} = 0$ and obtain the following table of multiplication

$$\begin{aligned}
&[e_1, x_1] = e_1, \qquad [x_1, e_1] = \alpha_{1,1}e_1, \qquad [x_1, x_1] = \gamma_{1,1}e_1 + \gamma_{1,3}e_3,\\
&[x_2, x_1] = \gamma_{2,1}e_1 + \gamma_{2,2}e_2 + \gamma_{2,3}e_3,\\
&[e_2, x_2] = e_2, \qquad [x_2, e_2] = \beta_{2,2}e_2, \qquad [x_2, x_2] = \gamma_{4,2}e_2 + \gamma_{4,3}e_3,\\
&[x_1, x_2] = \gamma_{3,1}e_1 + \gamma_{3,2}e_2 + \gamma_{3,3}e_3
\end{aligned}$$

with restrictions

$$\begin{gathered}
\alpha_{1,1}(\alpha_{1,1} + 1) = 0, \qquad \beta_{2,2}(\beta_{2,2} + 1) = 0, \qquad \alpha_{1,1}\gamma_{1,1} = 0, \qquad \beta_{2,2}\gamma_{4,2} = 0,\\
\gamma_{3,1}(\alpha_{1,1} + 1) = 0, \qquad \gamma_{2,2}(\beta_{2,2} + 1) = 0, \qquad \gamma_{3,1} = \gamma_{2,1}\alpha_{1,1}, \qquad \gamma_{2,2} = \gamma_{3,2}\beta_{2,2}.
\end{gathered}$$

Considering the basis change $x_1' = x_1 - \gamma_{3,2}e_2$, $x_2' = x_2 - \gamma_{2,1}e_1$, we can assume that

$$\gamma_{2,1} = \gamma_{2,2} = \gamma_{3,1} = \gamma_{3,2} = 0.$$

- If $\alpha_{1,1} = -1$, $\beta_{2,2} = -1$, then we have $\gamma_{1,1} = 0$, $\gamma_{4,2} = 0$, and the algebra $M_6(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ appears.
- If $(\alpha_{1,1}, \beta_{2,2}) = (0, -1)$ or $(-1, 0)$ then without loss of generality we can suppose that $\alpha_{1,1} = 0$, $\beta_{2,2} = -1$. Then we have $\gamma_{4,2} = 0$, and applying the base change $x_1' = x_1 - \gamma_{1,1}e_1$, we obtain $M_7(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.
- But if $\alpha_{1,1} = 0$, $\beta_{2,2} = 0$, then the base change

$$x_1' = x_1 - \gamma_{1,1}e_1, \qquad x_2' = x_2 - \gamma_{4,2}e_2$$

gives $M_8(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

It is easy to see that

$$M_6(0,0,0,0) \cong M_2(0,0), \qquad M_7(0,0,0,0) \cong M_4(0,0), \qquad M_8(0,0,0,0) \cong M_5(0,0),$$

moreover, choosing appropriate base change one of the λ_i which is not zero can be reduced to 1.

Case 1.2. Let $(\mu_1, \mu_2) \in \{(0,1), (1,0)\}$. Without loss of generality we can suppose that $\mu_1 = 1$, $\mu_2 = 0$. Then because of (3.3) we have

$$\alpha_{1,2} = \alpha_{2,1} = \alpha_{2,3} = \alpha_{3,2} = 0,$$
$$\beta_{1,2} = \beta_{2,1} = \beta_{2,3} = \beta_{3,2} = 0.$$

From the Leibniz identity

$$0 = \big[[x_1, x_1], e_2\big] = \big[x_1, [x_1, e_2]\big] + \big[[x_1, e_2], x_1\big] = [x_1, \alpha_{2,2}e_2] + [\alpha_{2,2}e_2, x_1] = \alpha_{2,2}^2 e_2,$$

we get $\alpha_{2,2} = 0$.

Thus the table of multiplication in this case looks like

$$\begin{array}{lll} [e_1, x_1] = e_1, & [x_1, e_1] = \alpha_{1,1}e_1 + \alpha_{1,3}e_3, & [x_1, x_1] = \gamma_{1,1}e_1 + \gamma_{1,2}e_2 + \gamma_{1,3}e_3, \\ [e_3, x_1] = e_3, & [x_1, e_3] = \alpha_{3,1}e_1 + \alpha_{3,3}e_3, & [x_2, x_1] = \gamma_{2,1}e_1 + \gamma_{2,2}e_2 + \gamma_{2,3}e_3, \\ [e_2, x_2] = e_2, & [x_2, e_1] = \beta_{1,1}e_1 + \beta_{1,3}e_3, & [x_1, x_2] = \gamma_{3,1}e_1 + \gamma_{3,2}e_2 + \gamma_{3,3}e_3, \\ [x_2, e_2] = \beta_{2,2}e_2, & [x_2, e_3] = \beta_{3,1}e_1 + \beta_{3,3}e_3, & [x_2, x_2] = \gamma_{4,1}e_1 + \gamma_{4,2}e_2 + \gamma_{4,3}e_3. \end{array}$$

Now we distinguish the two cases depending on the views of the Jordan forms of the matrix $\left(\begin{smallmatrix} \alpha_{1,1} & \alpha_{1,3} \\ \alpha_{3,1} & \alpha_{3,3} \end{smallmatrix}\right)$, i.e. a multiple root case $\left(\begin{smallmatrix} \alpha_{1,1} & 1 \\ 0 & \alpha_{1,1} \end{smallmatrix}\right)$ and simple roots case $\left(\begin{smallmatrix} \alpha_{1,1} & 0 \\ 0 & \alpha_{3,3} \end{smallmatrix}\right)$.

The former case is impossible due to the following observation. Let us consider the Leibniz identity

$$\begin{aligned} 0 &= \big[[x_1, x_1], e_1\big] = \big[x_1, [x_1, e_1]\big] + \big[[x_1, e_1], x_1\big] = [x_1, \alpha_{1,1}e_1 + e_3] + [\alpha_{1,1}e_1, x_1] \\ &= \big(\alpha_{1,1}^2 + \alpha_{1,1}\big)e_1 + (1 + 2\alpha_{1,1})e_3. \end{aligned}$$

From that we get the system of equations

$$\alpha_{1,1}(\alpha_{1,1} + 1) = 0$$
$$1 + 2\alpha_{1,1} = 0$$

which is obviously not consistent.

Therefore we consider the case when $\left(\begin{smallmatrix} \alpha_{1,1} & \alpha_{1,3} \\ \alpha_{3,1} & \alpha_{3,3} \end{smallmatrix}\right)$ is congruent to $\left(\begin{smallmatrix} \alpha_{1,1} & 0 \\ 0 & \alpha_{3,3} \end{smallmatrix}\right)$. There are a few subcases here.

- Let $e_1, e_3 \in Ann_r(L)$. Then we have

  $$\alpha_{1,1} = \alpha_{3,3} = \beta_{1,1} = \beta_{1,3} = \beta_{3,1} = \beta_{3,3} = 0.$$

  Let us consider the Leibniz identity for $[x_1, [x_1, x_2]]$, $[x_2, [x_1, x_2]]$. This yields

  $$\gamma_{1,2} = \gamma_{3,1} = \gamma_{3,3} = \gamma_{4,1} = \gamma_{4,3} = 0, \qquad \gamma_{2,2} = \gamma_{3,2}\beta_{2,2}.$$

  Then as a result of the basis change

  $$x_1' = x_1 - \gamma_{1,1}e_1 - \gamma_{3,2}e_2 - \gamma_{1,3}e_3, \qquad x_2' = x_2 - \gamma_{2,1}e_1 - \gamma_{2,3}e_3,$$

we conclude that $\gamma_{1,1} = \gamma_{1,3} = \gamma_{2,1} = \gamma_{2,3} = \gamma_{3,2} = 0$.
Therefore we derive the following products

$$[e_1, x_1] = e_1, \qquad [e_3, x_1] = e_3, \qquad [e_2, x_2] = e_2,$$
$$[x_2, e_2] = \beta_{2,2} e_2, \qquad [x_2, x_2] = \gamma_{4,2} e_2.$$

- Now if $e_2 \in Ann_r(L)$, then we have $\beta_{2,2} = 0$ and taking the change $x_2' = x_2 - \gamma_{4,2} e_2$, we get $M_5(1,0)$.
- But if $e_2 \notin Ann_r(L)$, then we get $\beta_{2,2} = -1$, $\gamma_{4,2} = 0$. Hence, L is isomorphic to $M_4(1,0)$.

- Let us now consider the case when one of the basis vectors e_1, e_3 is not in $Ann_r(L)$, then without loss of generality we can suppose that $e_1 \notin Ann_r(L)$, $e_3 \in Ann_r(L)$. Therefore

$$\alpha_{1,1} = -1, \qquad \alpha_{3,1} = \beta_{1,1} = \beta_{3,1} = \beta_{3,3} = \gamma_{1,1} = \gamma_{4,1} = 0, \qquad \gamma_{3,1} = -\gamma_{2,1}.$$

Analogously to that of the previous case considering the Leibniz identities $[x_1, [x_1, x_2]]$, $[x_2, [x_1, x_2]]$ we derive

$$\gamma_{1,2} = \gamma_{3,3} = 0, \qquad \gamma_{2,2} = \gamma_{3,2}\beta_{2,2}, \qquad \gamma_{4,3} = \gamma_{2,1}\beta_{1,3}.$$

Then applying the base change

$$x_1' = x_1 - \gamma_{3,2} e_2 - \gamma_{1,3} e_3, \qquad x_2' = x_2 - \gamma_{2,1} e_1 - \gamma_{2,3} e_3,$$

we get $\gamma_{1,3} = \gamma_{2,1} = \gamma_{2,3} = \gamma_{3,2} = 0$. Hence the table of multiplications in this case is given as follows:

$$\begin{array}{llll} [e_1, x_1] = e_1, & [e_3, x_1] = e_3, & [e_2, x_2] = e_2, & \\ [x_1, e_1] = -e_1, & [x_2, e_1] = \beta_{1,3} e_3, & [x_2, e_2] = \beta_{2,2} e_2, & [x_2, x_2] = \gamma_{4,2} e_2. \end{array}$$

- If $e_2 \in Ann_r(L)$, then $\beta_{2,2} = 0$ and the base change $x_2' = x_2 - \gamma_{4,2} e_2$, yields $\gamma_{4,2} = 0$.
  * If $\beta_{1,3} = 0$, then we obtain $M_4(0,1)$.
  * But if $\beta_{1,3} \neq 0$, then the base change $e_3' = \beta_{1,3} e_3$, gives M_9.
- And if $e_2 \notin Ann_r(L)$, then
  $\beta_{2,2} = -1$, $\gamma_{4,2} = 0$. Here if $\beta_{1,3} = 0$, we obtain the algebra $M_2(1,0)$, otherwise considering the base change $e_3' = \beta_{1,3} e_3$, we get M_{10}.

- Let now none of e_1, e_3 is in $Ann_r(L)$. Then

$$\alpha_{1,1} = \alpha_{3,1} = -1, \qquad \beta_{1,1} = \beta_{1,3} = \beta_{3,1} = \beta_{3,3} = 0,$$
$$\gamma_{1,1} = \gamma_{1,3} = \gamma_{4,1} = \gamma_{4,3} = 0, \qquad \gamma_{3,1} = -\gamma_{2,1}, \qquad \gamma_{3,3} = -\gamma_{2,3}.$$

Applying the Leibniz identities $[x_1,[x_1,x_2]]$, $[x_2,[x_1,x_2]]$ we obtain

$$\gamma_{1,2}=0, \qquad \gamma_{2,2}=\gamma_{3,2}\beta_{2,2}.$$

After the base change

$$x_1'=x_1-\gamma_{3,2}e_2, \qquad x_2'=x_2-\gamma_{2,1}e_1-\gamma_{2,3}e_3,$$

we get $\gamma_{2,1}=\gamma_{2,3}=\gamma_{3,2}=0$ and as a result the table of multiplications is written as follows

$$\begin{array}{llll} [e_1,x_1]=e_1, & [e_3,x_1]=e_3, & [e_2,x_2]=e_2, & \\ [x_1,e_1]=-e_1, & [x_1,e_3]=-e_3, & [x_2,e_2]=\beta_{2,2}e_2, & [x_2,x_2]=\gamma_{4,2}e_2. \end{array}$$

- If $e_2 \in Ann_r(L)$, then $\beta_{2,2}=0$ and the base change $x_2'=x_2-\gamma_{4,2}e_2$, gives $M_3(1)$.
- But if $e_2 \notin Ann_r(L)$, then $\beta_{2,2}=-1$, $\gamma_{4,2}=0$ and one obtains $M_1(1,0)$.

Case 2. Let the basis $\{e_1,e_2,e_3,x_1,x_2\}$ be such that R_{x_1} and R_{x_2} have the form B in Proposition 3.5.

Similar to Case 1, applying the Leibniz identity we get the table multiplications:

$$\begin{array}{lll} [e_1,x_1]=e_1, & [x_1,e_1]=\alpha_{1,1}e_1, & [x_1,x_1]=\gamma_{1,1}e_1+\gamma_{1,2}e_2, \\ [e_2,x_1]=e_2, & [x_1,e_2]=\alpha_{1,1}e_2, & [x_2,x_1]=\gamma_{2,1}e_1+\gamma_{2,2}e_2+\gamma_{2,3}e_3, \\ [e_1,x_2]=e_2, & [x_2,e_1]=\beta_{1,2}e_2, & [x_1,x_2]=\gamma_{3,1}e_1+\gamma_{3,2}e_2+\gamma_{3,3}e_3, \\ [e_3,x_2]=e_3, & [x_2,e_3]=\beta_{3,3}e_3, & [x_2,x_2]=\gamma_{4,2}e_2+\gamma_{4,3}e_3 \end{array}$$

with constraints

$$\begin{array}{lll} \alpha_{1,1}(\alpha_{1,1}+1)=0, & \gamma_{3,1}=\gamma_{2,1}\alpha_{1,1}, & \gamma_{3,2}=\gamma_{1,1}+\gamma_{2,2}\alpha_{1,1}, \\ & \gamma_{2,3}=\gamma_{3,3}\beta_{3,3}, & \gamma_{2,1}=\gamma_{4,2}+\gamma_{3,1}\beta_{1,2}. \end{array}$$

Case 2.1. Let $e_1 \in Ann_r(L)$, then

$$\alpha_{1,1}=0, \qquad \beta_{1,2}=0, \qquad \gamma_{3,1}=0, \qquad \gamma_{3,2}=\gamma_{1,1}, \qquad \gamma_{2,1}=\gamma_{4,2}.$$

Then the base change

$$x_1'=x_1-\gamma_{1,1}e_1-\gamma_{1,2}e_2-\gamma_{3,3}e_3, \qquad x_2'=x_2-\gamma_{2,1}e_1-\gamma_{2,2}e_2,$$

yields $\gamma_{1,1}=\gamma_{1,2}=\gamma_{2,1}=\gamma_{2,2}=\gamma_{3,3}=0$.

- If $e_3 \in Ann_r(L)$, then $\beta_{3,3}=0$, and taking the change $x_2=x_2-\gamma_{4,3}e_3$ and we obtain P_1.

- If $e_3 \notin Ann_r(L)$ then $\beta_{3,3} = -1$, $\gamma_{4,3} = 0$ and we get P_2.

Case 2.2. Let $e_1 \notin Ann_r(L)$, then

$$\alpha_{1,1} = -1, \qquad \beta_{1,2} = -1, \qquad \gamma_{1,1} = \gamma_{1,2} = \gamma_{4,2} = 0,$$
$$\gamma_{3,1} = -\gamma_{2,1}, \qquad \gamma_{3,2} = -\gamma_{2,2}.$$

Applying the base change

$$x_1' = x_1 + \gamma_{2,2}e_1 - \gamma_{3,3}e_3, \qquad x_2' = x_2 - \gamma_{2,1}e_1,$$

we can assume that $\gamma_{2,1} = \gamma_{2,2} = \gamma_{3,3} = 0$.

- If $e_3 \notin Ann_r(L)$, then $\beta_{3,3} = 0$, and the base change $x_2 = x_2 - \gamma_{4,3}e_3$ gives P_3.
- If $e_3 \in Ann_r(L)$ then $\beta_{3,3} = -1$, $\gamma_{4,3} = 0$ and we get P_4.

Case 3. Let now the basis $\{e_1, e_2, e_3, x_1, x_2\}$ be such that R_{x_1} and R_{x_2} have the form C in Proposition 3.5.

By using the Leibniz identity, we obtain the table of multiplications as follows

$$\begin{array}{lll}
[e_1, x_1] = e_1 + e_2, & [x_1, e_1] = \alpha_{1,1}e_1 + \alpha_{1,2}e_2, & [x_1, x_1] = \gamma_{1,1}e_1 + \gamma_{1,2}e_2, \\
[e_2, x_1] = e_2, & [x_1, e_2] = \alpha_{1,1}e_2, & [x_2, x_1] = \gamma_{2,1}e_1 + \gamma_{2,2}e_2 + \gamma_{2,3}e_3, \\
[e_1, x_2] = \mu e_2, & [x_2, e_1] = \beta_{1,2}e_2, & [x_1, x_2] = \gamma_{3,1}e_1 + \gamma_{3,2}e_2 + \gamma_{3,3}e_3, \\
[e_3, x_2] = e_3, & [x_2, e_3] = \beta_{3,3}e_3, & [x_2, x_2] = \gamma_{4,2}e_2 + \gamma_{4,3}e_3
\end{array}$$

with constraints

$$\alpha_{1,1}(\alpha_{1,1} + 1) = 0, \qquad \gamma_{3,1} = \gamma_{2,1}\alpha_{1,1},$$
$$\gamma_{3,2} = -\gamma_{3,1} + \gamma_{1,1}\mu + \gamma_{2,1}\alpha_{1,2} + \gamma_{2,2}\alpha_{1,1},$$
$$\alpha_{1,1}(2\alpha_{1,2} + 1) = -\alpha_{1,2}, \qquad \gamma_{2,3} = \gamma_{3,3}\beta_{3,3}, \qquad \gamma_{4,2} = \gamma_{2,1}\mu - \gamma_{3,1}\beta_{1,2}.$$

Case 3.1. Let $e_1 \in Ann_r(L)$. Then

$$\alpha_{1,1} = 0, \qquad \alpha_{1,2} = 0, \qquad \beta_{1,2} = 0, \qquad \gamma_{3,1} = 0,$$
$$\gamma_{3,2} = \gamma_{1,1}\mu, \qquad \gamma_{4,2} = \gamma_{2,1}\mu, \qquad \gamma_{2,3} = \gamma_{3,3}\beta_{3,3}.$$

Taking the base change

$$x_1' = x_1 - \gamma_{1,1}e_1 - (\gamma_{1,2} - \gamma_{1,1})e_2 - \gamma_{3,3}e_3, \qquad x_2' = x_2 - \gamma_{2,1}e_1 - (\gamma_{2,2} - \gamma_{1,2})e_2,$$

we can assume that $\gamma_{1,1} = \gamma_{1,2} = \gamma_{2,1} = \gamma_{2,2} = \gamma_{3,3} = 0$.

- If $e_3 \in Ann_r(L)$, then $\beta_{3,3} = 0$, and taking the base change $x_2 = x_2 - \gamma_{4,3}e_3$ we obtain the algebra $Q_1(\mu)$.
- If $e_3 \notin Ann_r(L)$ then $\beta_{3,3} = -1$, $\gamma_{4,3} = 0$ and we obtain $Q_2(\mu)$.

Case 3.2. Let $e_1 \notin Ann_r(L)$, then

$$\alpha_{1,1} = \alpha_{2,1} = -1, \qquad \beta_{1,2} = -\mu, \qquad \gamma_{1,1} = \gamma_{1,2} = \gamma_{4,2} = 0,$$

$$\gamma_{3,1} = -\gamma_{2,1}, \qquad \gamma_{3,2} = -\gamma_{2,2}, \qquad \gamma_{2,3} = \gamma_{3,3}\beta_{3,3}.$$

The base change

$$x_1' = x_1 - \gamma_{3,3}e_3, \qquad x_2' = x_2 - \gamma_{2,1}e_1 - (\gamma_{2,2} - \gamma_{2,1})e_1,$$

gives

$$\gamma_{2,1} = \gamma_{2,2} = \gamma_{3,3} = 0.$$

- If $e_3 \notin Ann_r(L)$, then $\beta_{3,3} = 0$, and taking the base change $x_2 = x_2 - \gamma_{4,3}e_3$ we obtain $Q_3(\mu)$.
- But if $e_3 \in Ann_r(L)$ then $\beta_{3,3} = -1$, $\gamma_{4,3} = 0$ and we obtain $Q_4(\mu)$.

Remark 3.7. Impossibility of an isomorphism between elements of the classes

- $M_1(\mu_1, \mu_2)$ except for $M_1(\mu_1, \mu_2) \cong M_1(\mu_2, \mu_1) \cong M_1(\frac{1}{\mu_1}, -\frac{\mu_2}{\mu_1})$,
- $M_2(\mu_1, \mu_2)$ except for $M_2(\mu_1, \mu_2) \cong M_2(\mu_2, \mu_1)$,
- $M_5(\mu_1, \mu_2)$ except for $M_5(\mu_1, \mu_2) \cong M_5(\mu_2, \mu_1)$,
- $M_6(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ except for $M_6(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \cong M_6(\lambda_4, \lambda_3, \lambda_2, \lambda_1)$,
- $M_8(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ except for $M_8(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \cong M_8(\lambda_4, \lambda_3, \lambda_2, \lambda_1)$

can be proven by taking general base change in each case. This is a long and rather technical work. We decided not to include these routine examinations in the paper. They are available from the authors. □

Remark 3.8. Due to Proposition 3.5 we conclude that any two algebras from different classes M_i, P_i and Q_i are not isomorphic. Pairwise nonisomorphness of any two algebras from the same classes can be easily seen by comparing the isomorphism invariants which are presented below.

L	$dim\,Ann_r(L)$	$dim\,L^2$	$dim\,L^3$	$dim\,Ann_l(L)$	$dim\,Lie(L)$
$M_1(\mu_1,\mu_2)$	0				
$M_2(0,0)$	1	2			
$M_6(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$	1	3	2		
$M_3(\mu)$	1	3	3	1	
M_{10}	1	3	3	0	3
$M_2(\mu_1,\mu_2),(\mu_1,\mu_2)\neq(0,0)$	1	3	3	0	4
$M_4(0,0)$	2	2			
$M_7(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$	2	3	2		
M_9	2	3	3	0	
$M_4(\mu_1,\mu_2),(\mu_1,\mu_2)\neq(0,0)$	2	3	3	1	
$M_5(0,0)$	3	2			
$M_8(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$	3	3	2		
$M_5(\mu_1,\mu_2),(\mu_1,\mu_2)\neq(0,0)$	3	3	3		

L	P_1	P_2	P_3	P_4
$dim\,Ann_r(L)$	3	2	1	0

L	$Q_1(\mu)$	$Q_2(\mu)$	$Q_3(\mu)$	$Q_4(\mu)$
$dim\,Ann_r(L)$	3	2	1	0

The list of isomorphism classes of 5-dimensional solvable complex Leibniz algebras with 3-dimensional nilradical.

Representative	Table of multiplication
H	$[e_1,e_2]=e_3,\ [e_2,e_1]=-e_3,\ [e_1,x_1]=e_1,\ [e_3,x_1]=e_3,\ [x_1,e_1]=-e_1,$ $[x_1,e_3]=-e_3,\ [e_2,x_2]=e_2,\ [e_3,x_2]=e_3,\ [x_2,e_2]=-e_2,\ [x_2,e_3]=-e_3.$
L_1	$[e_2,e_1]=e_3,\ [e_1,x_1]=e_1,\ [e_2,x_2]=e_2,$ $[x_1,e_1]=-e_1,\ [e_3,x_1]=e_3,\ [e_3,x_2]=e_3.$
L_2	$[e_1,e_1]=e_3,\ [e_1,x_1]=e_1,\ [x_1,e_1]=-e_1,\ [e_3,x_1]=2e_3,\ [e_2,x_2]=e_2.$
L_3	$[e_1,e_1]=e_3,\ [e_1,x_1]=e_1,\ [x_1,e_1]=-e_1,$ $[e_3,x_1]=2e_3,\ [e_2,x_2]=e_2,\ [x_2,e_2]=-e_2.$
$M_1(\mu_1,\mu_2)$ $\mu_1\neq0$	$[e_1,x_1]=e_1,\ [e_3,x_1]=\mu_1e_3,\ [e_2,x_2]=e_2,\ [e_3,x_2]=\mu_2e_3,$ $[x_1,e_1]=-e_1,\ [x_1,e_3]=-\mu_1e_3,\ [x_2,e_2]=-e_2,\ [x_2,e_3]=-\mu_2e_3.$
$M_2(\mu_1,\mu_2)$	$[e_1,x_1]=e_1,\ [e_3,x_1]=\mu_1e_3,\ [e_2,x_2]=e_2,$ $[e_3,x_2]=\mu_2e_3,\ [x_1,e_1]=-e_1,\ [x_2,e_2]=-e_2.$
$M_3(\mu)$ $\mu\neq0$	$[e_1,x_1]=e_1,\ [e_2,x_2]=e_2,\ [e_3,x_2]=\mu e_3,\ [x_2,e_2]=-e_2,\ [x_2,e_3]=-\mu e_3.$
$M_4(\mu_1,\mu_2)$	$[e_1,x_1]=e_1,\ [e_3,x_1]=\mu_1e_3,\ [e_2,x_2]=e_2,\ [e_3,x_2]=\mu_2e_3,\ [x_2,e_2]=-e_2.$
$M_5(\mu_1,\mu_2)$	$[e_1,x_1]=e_1,\ [e_3,x_1]=\mu_1e_3,\ [e_2,x_2]=e_2,\ [e_3,x_2]=\mu_2e_3.$
$M_6(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ $\neq(0,0,0,0)$	$[e_1,x_1]=e_1,\ [e_2,x_2]=e_2,\ [x_1,e_1]=-e_1,\ [x_2,e_2]=-e_2,$ $[x_1,x_1]=\lambda_1e_3,\ [x_2,x_1]=\lambda_2e_3,\ [x_1,x_2]=\lambda_3e_3,\ [x_2,x_2]=\lambda_4e_3.$
$M_7(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ $\neq(0,0,0,0)$	$[e_1,x_1]=e_1,\ [e_2,x_2]=e_2,\ [x_2,e_2]=-e_2,$ $[x_1,x_1]=\lambda_1e_3,\ [x_2,x_1]=\lambda_2e_3,\ [x_1,x_2]=\lambda_3e_3,\ [x_2,x_2]=\lambda_4e_3.$
$M_8(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ $\neq(0,0,0,0)$	$[e_1,x_1]=e_1,\ [e_2,x_2]=e_2,\ [x_1,x_1]=\lambda_1e_3,$ $[x_2,x_1]=\lambda_2e_3,\ [x_1,x_2]=\lambda_3e_3,\ [x_2,x_2]=\lambda_4e_3.$
M_9	$[e_1,x_1]=e_1,\ [e_2,x_2]=e_2,\ [e_3,x_1]=e_3,$ $[x_1,e_1]=-e_1,\ [x_2,e_1]=-e_3.$
M_{10}	$[e_1,x_1]=e_1,\ [e_3,x_1]=e_3,\ [x_1,e_1]=-e_1,$ $[x_2,e_1]=e_3,\ [e_2,x_2]=e_2,\ [x_2,e_2]=-e_2.$

(*continued*)

Representative	Table of multiplication
P_1	$[e_1,x_1]=e_1,\ [e_2,x_1]=e_2,\ [e_1,x_2]=e_2,\ [e_3,x_2]=e_3.$
P_2	$[e_1,x_1]=e_1,\ [e_2,x_1]=e_2,\ [e_1,x_2]=e_2,$ $[e_3,x_2]=e_3,\ [x_2,e_3]=-e_3.$
P_3	$[e_1,x_1]=e_1,\ [e_2,x_1]=e_3,\ [e_1,x_2]=e_2,\ [e_3,x_2]=e_3,$ $[x_1,e_1]=-e_1,\ [x_1,e_2]=-e_2,\ [x_2,e_1]=-e_2.$
P_4	$[e_1,x_1]=e_1,\ [e_2,x_1]=e_2,\ [e_1,x_2]=e_2,\ [e_3,x_2]=e_3,$ $[x_1,e_1]=-e_1,\ [x_1,e_2]=-e_2,\ [x_2,e_1]=-e_2,\ [x_2,e_3]=-e_3.$
$Q_1(\mu)$	$[e_1,x_1]=e_1+e_2,\ [e_2,x_1]=e_2,\ [e_1,x_2]=\mu e_2,\ [e_3,x_2]=e_3.$
$Q_2(\mu)$	$[e_1,x_1]=e_1+e_2,\ [e_2,x_1]=e_2,\ [e_1,x_2]=\mu e_2,$ $[e_3,x_2]=e_3,\ [x_2,e_3]=-e_3.$
$Q_3(\mu)$	$[e_1,x_1]=e_1+e_2,\ [e_2,x_1]=e_2,\ [e_1,x_2]=\mu e_2,\ [e_3,x_2]=e_3,$ $[x_1,e_1]=-e_1-e_2,\ [x_1,e_2]=-e_2,\ [x_2,e_1]=-\mu e_2.$
$Q_4(\mu)$	$[e_1,x_1]=e_1+e_2,\ [e_2,x_1]=e_2,\ [e_1,x_2]=\mu e_2,\ [e_3,x_2]=e_3,$ $[x_1,e_1]=-e_1-e_2,\ [x_1,e_2]=-e_2,\ [x_2,e_1]=-\mu e_2,\ [x_2,e_3]=-e_3.$

4. Conclusion

Combining the results of [12] and the present paper we conclude that there are 12 parametric families and 10 concrete nonisomorphic solvable Leibniz algebra structures with three-dimensional nilradicals on 5-dimensional complex vector space.

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