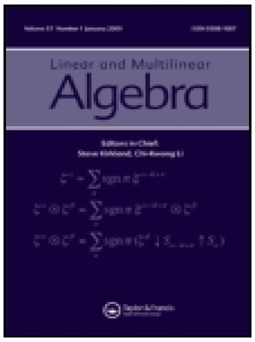

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On solvable Leibniz algebras whose nilradical is a direct sum of null-filiform algebras

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In this paper, we continue the investigation of complex finite-dimensional solvable Leibniz algebras with nilradical $NF_1 \oplus NF_2 \oplus \cdots \oplus NF_s$, where NF_i are ideals of maximal nilindex of the nilradical. The multiplication tables of such solvable algebras with restrictions to structural constants are obtained. In the case when the complemented space to the nilradical is one-dimensional, we present a multiplication table without any restrictions to structural constants. The classification of solvable Leibniz algebras, whose dimension of the complemented vector space to the nilradical is equal to s, is also given. Moreover, the description of solvable Leibniz algebras with the condition of each NF_i is ideal of the algebra is presented.

Keywords: Leibniz algebra; solvability; nilpotency; nilradical; derivation; null-filiform Leibniz algebra

AMS Subject Classifications: 17A32; 17A36; 17A60; 17A65; 17B30

1. Introduction

The notion of Leibniz algebras was introduced by Loday [1] in 1993 as a generalization of the Lie algebras. These algebras preserve a unique property of Lie algebras – the right multiplication operators are derivations. Many (co)homological and structure properties of Lie algebras are extended to the case of Leibniz algebras in [1–8] and many others. In fact, Engel's theorem on nilpotency, the structure theorems on Levi's decomposition and conjugacy of Cartan subalgebras of the theory of Lie algebras [9] are also true for Leibniz algebras as well (see [5,10,11]). Moreover, the semisimple part of a Leibniz algebra is a Lie algebra. It means that the study of finite-dimensional Leibniz algebras (similar to Lie algebras case) is reduced to the investigation of the solvable radical. Based on the work of [12], a new approach for the investigation of solvable Lie algebras by using their nilradicals is developed in the works,[13–19] etc.

Due to results of Mubarakzjanov [12] for Lie algebras and their analogues for Leibniz algebras case,[20] the importance of solvable Leibniz algebras consists of their nilradical. Only papers [21] and [20] are devoted to the study of solvable non-Lie Leibniz algebras with a given nilradical.

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It is known that an n-dimensional Leibniz algebra of maximal nilindex is unique (up to isomorphism) [3] and it has index $n+1$. This algebra is one-generated and, hence, it is not a Lie algebra (which should be at least two generated). In the paper,[20] solvable Leibniz algebras with null-filiform nilradical are described. Moreover, the description of solvable Leibniz algebras, whose nilradical is a direct sum of null-filiform algebras and with additional restriction, is given there. In particular, the case of the nilradical being direct sum of two copies of null-filiform algebras is described. However, in the case when the complemented space is one-dimensional this work has a defect, because in the case of the nilradical being a direct sum of s ($s \geq 3$) copies it is not necessary that each null-filiform is ideal of the algebra. In the present work, we improve this defect and we study the general case.

The aim of the present paper is to investigate the structure of solvable Leibniz algebras whose nilradical is a direct sum of null-filiform Leibniz algebras.

In order to achieve our goal, we organize the paper as follows: in Section 2, we give some necessary notions and preliminary results about Leibniz algebras. Section 3 is devoted to study the structure of complex finite-dimensional solvable Leibniz algebras whose nilradical is a direct sum of null-filiform Leibniz algebras. Firstly, for such algebras we present a family with restrictions to structure constants (Theorem 3.3), after that, we give descriptions for various cases on the nilradical and the complemented space to the nilradical. Namely, for the case when the number of null-filiform algebras is equal to the complemented space to the nilradical we prove that up to isomorphism there is a unique such algebra (Corollary 3.4). We give description, as well, for the case when each null-filiform is ideal of the algebra (Theorem 3.5). Finally, solvable Leibniz algebras with one-dimensional complemented space to the nilradical are described (Theorem 3.6).

Throughout the paper, we consider finite-dimensional vector spaces and algebras over a field of characteristic zero. Moreover, in the multiplication table of an algebra omitted products are assumed to be zero and if it is not noticed we shall consider non-nilpotent solvable algebras.

2. Preliminaries

In this section, we give necessary definitions and results for understanding the main part of the work.

Definition 2.1 ([1]) A vector space L over a field F with a binary operation $[-,-]$ is called a Leibniz algebra, if for any $x, y, z \in L$ the so-called Leibniz identity holds

$$\big[x,[y,z]\big] = \big[[x,y],z\big] - \big[[x,z],y\big].$$

Every Lie algebra is a Leibniz algebra, but the bracket in a Leibniz algebra needs not to be skew-symmetric.

The set $\mathrm{Ann}_r(L) = \{x \in L \mid [L,x] = 0\}$ is called *right annihilator* of the Leibniz algebra L. It is easy to see that $\mathrm{Ann}_r(L)$ is a two-sided ideal of the algebra L and for any $x, y \in L$, elements of the form $[x,x],\ [x,y]+[y,x]$ belong to $\mathrm{Ann}_r(L)$.

For a Leibniz algebra L consider the following central lower and derived series:

$$L^1 = L, \quad L^{k+1} = \left[L^k, L^1\right], \quad L^{[1]} = L, \quad L^{[k+1]} = \left[L^{[k]}, L^{[k]}\right], \quad k \geq 1.$$

Definition 2.2 A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists $p \in \mathbb{N}$ ($q \in \mathbb{N}$) such that $L^p = 0$ (respectively, $L^{[q]} = 0$).

It is known that in Leibniz algebras case in each dimension there exists a unique (up to isomorphism) algebra of maximal index of nilpotency [3] and with multiplication table

$$NF_n : [e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq n-1.$$

The next theorem presents a crucial property of the nilradical of a solvable Leibniz algebra.

THEOREM 2.3 ([20]) *Let R be a solvable Leibniz algebra and N its nilradical. Then the dimension of the complementary vector space to N is not greater than the maximal number of nil-independent derivations of N.*

Let $R = N + Q$ be a solvable Leibniz algebra with nilradical N and complemented vector space Q. Let $N = NF_1 \oplus NF_2 \oplus \cdots \oplus NF_s$ be a direct sum of null-filiform Leibniz algebras.

Further, we shall use the following notations: $\dim N = n,\ \dim(NF_i) = n_i,\ \dim Q = q$ and $\left\{e_1^i, e_2^i, \ldots, e_{n_i}^i\right\}$ be a basis of the algebra NF_i such that the table has the following form:

$$\left[e_t^i, e_1^i\right] = e_{t+1}^i, \quad 1 \leq t \leq n_i - 1.$$

We set

$$A_i = \sum_{j=1}^{n_i} j a_1^i e_{n_1+\cdots+n_{i-1}+j, n_1+\cdots+n_{i-1}+j},$$
$$B_i = \sum_{1 \leq j < k \leq n_i} a_{k+1-j}^i e_{n_1+\cdots+n_{i-1}+j, n_1+\cdots+n_{i-1}+k}, \quad 1 \leq i \leq s,$$

where $e_{k,t}$ is the identity matrix of size $n \times n$ (see below Proposition 3.1 for notation).

For convenience of further reading, instead of $i \neq j,\ i \neq t$, we shall use notations $i_{\neq j,t}$.

3. The main part

In this part, we describe the derivations of the algebra $N = NF_1 \oplus NF_2 \oplus \cdots \oplus NF_s$ and then we shall apply the general theory in order to investigate solvable Leibniz algebras with nilradical N.

PROPOSITION 3.1 *Any derivation of the algebra N has the following matrix form:*

$$D = \sum_{i=1}^{s} A_i + \sum_{i=1}^{s} B_i + \sum_{\substack{j=1 \\ i_{\neq j}}}^{s} \delta_i^j e_{n_1+\cdots+n_{i-1}+1, n_1+\cdots+n_j}.$$

Proof Any derivation of the algebra $N = NF_1 \oplus NF_2 \oplus \cdots \oplus NF_s$ has the following matrix form:

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$$D = \begin{pmatrix} D_1 & \delta_2^1 E_{n_1,n_2} & \dots & \delta_s^1 E_{n_1,n_s} \\ \delta_1^2 E_{n_2,n_1} & D_2 & \dots & \delta_s^2 E_{n_2,n_s} \\ \vdots & \vdots & \vdots & \vdots \\ \delta_1^s E_{n_s,n_1} & \delta_2^s E_{n_s,n_2} & \dots & D_s \end{pmatrix}, \text{ where } D_i = \begin{pmatrix} a_1^i & a_2^i & \dots & a_{n_i}^i \\ 0 & 2a_1^i & \dots & a_{n_i-1}^i \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & n_i a_1^i \end{pmatrix},$$

$$D_i = A_i + B_i,\ A_i = \begin{pmatrix} a_1^i & 0 & \dots & 0 \\ 0 & 2a_1^i & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & n_i a_1^i \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & a_2^i & a_3^i & \dots & a_{n_i}^i \\ 0 & 0 & a_2^i & \dots & a_{n_i-1}^i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_2^i \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

and $E_{i,j}$ is the $i \times j$ matrix $E_{i,j} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$ (cf. [20]). □

PROPOSITION 3.2 *The matrix*

$$\widetilde{D} = \sum_{i=1}^{s} B_i + \sum_{\substack{j=1 \\ i \neq j}}^{s} \delta_i^j e_{n_1+\dots+n_{i-1}+1, n_1+\dots+n_j}$$

is nilpotent.

Proof The proof follows from the equality $\widetilde{D}^t = \sum_{i=1}^{s} B_i^t$ and nilpotency of the matrices B_i. □

Thanks to Propositions 3.1 and 3.2, we conclude that the maximal number of nil-independent derivation of the algebra N is equal to s. From Theorem 2.3, we have that the dimension of the complemented space to the nilradical N in the algebra R is less or equal to s.

Let R be a solvable Leibniz algebra with nilradical $N = NF_1 \oplus NF_2 \oplus \dots \oplus NF_s$ and let $R = N \oplus Q$, with $\dim Q = q \leq s$. It is known that we can choose a basis

$$\left\{x_1, x_2, \dots, x_q, e_1^1, e_2^1, \dots, e_{n_1}^1, e_1^2, e_2^2, \dots, e_{n_2}^2, \dots, e_1^s, e_2^s, \dots, e_{n_s}^s\right\}$$

of R such that the restrictions of the right multiplication operators $\mathcal{R}_{x_1}, \mathcal{R}_{x_2}, \dots \mathcal{R}_{x_q}$ to the nilradical N are nil-independent derivations. Moreover, due to Proposition 3.1, we have the products

$$[e_1^i, x_j] = \sum_{k=1}^{n_i} \alpha_{j,k}^i e_k^i + \sum_{\substack{k=1 \\ k \neq i}}^{s} \delta_{j,k}^i e_{n_k}^k, \quad [e_t^i, x_j] = t\alpha_{j,1}^i e_t^i + \sum_{k=t+1}^{n_i} \alpha_{j,k-t+1}^i e_k^j,$$

where $1 \leq i \leq s,\ 1 \leq j \leq q,\ 2 \leq t \leq n_i$.

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Since rank $\begin{pmatrix} \alpha_{1,1}^1 & \alpha_{1,1}^2 & \dots & \alpha_{1,1}^s \\ \alpha_{2,1}^1 & \alpha_{2,1}^2 & \dots & \alpha_{2,1}^s \\ \vdots & \vdots & \dots & \vdots \\ \alpha_{q,1}^1 & \alpha_{q,1}^2 & \dots & \alpha_{q,1}^s \end{pmatrix} = q$, then without loss of generality we can assume that

$$\alpha_{i,1}^i = 1, \quad \alpha_{i,1}^j = 0, \quad 1 \le i \ne j, \quad j \le q.$$

By taking the change of basis

$$x' = x, \quad e_1^{i'} = e_1^i + \sum_{k=2}^{n_i} A_k^i e_k^i + \sum_{\substack{k=1 \\ k \ne i}}^{q} \delta_{i,k}^i e_{n_k}^k,$$

$$e_t^{i'} = e_t^i + \sum_{k=t+1}^{n_i} A_{k-t+1}^i e_k^i, \quad 1 \le i \le q, \quad 2 \le t \le n_i,$$

with

$$A_2^i = -\alpha_{i,2}^i, \quad A_k^i = -\frac{1}{k-1}\left(\sum_{p=2}^{k-1} A_{k-p+1}^i \alpha_{i,p}^i + \alpha_{i,k}^i\right), \quad 3 \le k \le n_i,$$

we obtain

$$\begin{cases} [e_t^i, e_1^i] = e_{t+1}^i, & 1 \le i \le s, \quad 1 \le t \le n_i - 1, \\ [e_1^i, x_i] = e_1^i + \sum\limits_{k=q+1}^{s} \delta_{i,k}^i e_{n_k}^k, & 1 \le i \le q, \\ [e_t^i, x_i] = t e_t^i, & 1 \le i \le q, \quad 2 \le t \le n_i, \\ [e_1^i, x_j] = \sum\limits_{k=2}^{n_i} \alpha_{j,k}^i e_k^i + \sum\limits_{\substack{k=1 \\ k \ne i}}^{s} \delta_{j,k}^i e_{n_k}^k, & 1 \le i \ne j, j \le q, \\ [e_t^i, x_j] = \sum\limits_{k=t+1}^{n_i} \alpha_{j,k-t+1}^i e_k^j, & 1 \le i \ne j, j \le q, \quad 2 \le t \le n_i, \\ [e_1^i, x_j] = \sum\limits_{k=1}^{n_i} \alpha_{j,k}^i e_k^i + \sum\limits_{\substack{k=1 \\ k \ne i}}^{s} \delta_{j,k}^i e_{n_k}^k, & 1 \le j \le q, \quad q+1 \le i \le s, \\ [e_t^i, x_j] = t\alpha_{j,1}^i e_t^i + \sum\limits_{k=t+1}^{n_i} \alpha_{j,k-t+1}^i e_k^i, & 1 \le j \le q, \quad q+1 \le i \le s, \ 2 \le t \le n_i. \end{cases} \tag{3.1}$$

Let us introduce notations:

$$\begin{aligned} [x_j, e_1^i] &= \sum_{p=1}^{s} \sum_{k=1}^{n_p} \beta_{j,k}^{i,p} e_k^p, \quad 1 \le j \le q, \ 1 \le i \le s, \\ [x_i, x_j] &= \sum_{p=1}^{s} \sum_{k=1}^{n_p} \gamma_{i,j,k}^{p} e_k^p, \quad 1 \le i, j \le q. \end{aligned} \tag{3.2}$$

In the following theorem, we verify the Leibniz identity for the products, which are given in (3.1), (3.2) and simplify them using basis transformations.

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THEOREM 3.3 *Let R be a solvable Leibniz algebra with nilradical $NF_1 \oplus NF_2 \oplus \ldots \oplus NF_s$ and $\dim Q = q \leq s$. Then there exists a basis $\{x_1, x_2, \ldots, x_q, e_1^1, e_2^1, \ldots, e_{n_1}^1, e_1^2, e_2^2, \ldots, e_{n_2}^2, \ldots, e_1^s, e_2^s, \ldots, e_{n_s}^s\}$ of R such that the multiplication in this basis has the form:*

$$
\begin{array}{ll}
[e_t^i, e_1^i] = e_{t+1}^i, & 1 \leq i \leq s,\ 1 \leq t \leq n_i - 1,\\
[x_i, e_1^i] = -e_1^i, \quad [e_t^i, x_i] = te_t^i, & 1 \leq i \leq q,\ 1 \leq t \leq n_i,\\
[e_1^i, x_j] = \sum\limits_{k=1}^{n_i} \alpha_{j,k}^i e_k^i + \sum\limits_{\substack{k=1\\ k\neq i}}^{s} \delta_{j,k}^i e_{n_k}^k, & q+1 \leq i \leq s,\ 1 \leq j \leq q,\\
[e_t^i, x_j] = t\alpha_{j,1}^i e_t^i + \sum\limits_{k=t+1}^{n_i} \alpha_{j,k-t+1}^i e_k^i, & q+1 \leq i \leq s,\ 1 \leq j \leq q,\ 2 \leq t \leq n_i,\\
[x_j, e_1^i] = \sum\limits_{p=q+1}^{s} \beta_{j,n_p}^{i,p} e_{n_p}^p, & 1 \leq i_{\neq j}, j \leq q,\\
[x_j, e_1^i] = -\alpha_{j,1}^i e_1^i + \sum\limits_{\substack{p=1\\ p\neq i}}^{s} \beta_{j,n_p}^{i,p} e_{n_p}^p, & q+1 \leq i \leq s,\ 1 \leq j \leq q,\\
[x_i, x_j] = \sum\limits_{p=q+1}^{s} \sum\limits_{k=2}^{n_p} \gamma_{i,j,k}^p e_k^p, & 1 \leq i, j \leq q.
\end{array}
$$

Moreover, the structural constants should satisfy the conditions

$$
\begin{array}{lll}
\alpha_{t,1}^i \alpha_{j,2}^i = 0, & q+1 \leq i \leq s, & 1 \leq j, t \leq q,\\
\alpha_{t,1}^i \alpha_{j,k}^i - \alpha_{j,1}^i \alpha_{t,k}^i = 0, & q+1 \leq i \leq s, & 1 \leq j_{\neq t}, t \leq q,\\
 & & 3 \leq k \leq n_i,\\
\left(n_p \alpha_{i,1}^p - 1\right) \beta_{j,n_p}^{i,p} = 0, & 1 \leq i_{\neq j}, j \leq q, & q+1 \leq p \leq s,\\
\alpha_{t,1}^p \beta_{j,n_p}^{i,p} = 0, & 1 \leq i_{\neq j,t}, j, t \leq q, & q+1 \leq p \leq s,\\
\gamma_{j,t,k-1}^i = -\alpha_{j,1}^i \alpha_{t,k}^i, & q+1 \leq i \leq s, & 1 \leq j, t \leq q,\\
 & & 3 \leq k \leq n_i,\\
\alpha_{j,1}^i \delta_{t,t}^i = \left(n_t - \alpha_{t,1}^i\right) \beta_{j,n_t}^{i,t}, & q+1 \leq i \leq s, & 1 \leq j, t \leq q,\\
\alpha_{j,1}^i \delta_{t,p}^i = -\alpha_{t,1}^i \beta_{j,n_p}^{i,p}, & q+1 \leq i \leq s, & 1 \leq j, t_{\neq p}, p \leq q,\\
\alpha_{j,1}^i \delta_{t,p}^i = \left(n_p \alpha_{t,1}^p - \alpha_{t,1}^i\right) \beta_{j,n_p}^{i,p}, & q+1 \leq i_{\neq p}, p \leq s, & 1 \leq j, t \leq q,\\
\alpha_{j,1}^i \delta_{t,k}^i - \alpha_{t,1}^i \delta_{j,k}^i = 0, & q+1 \leq i \leq s, & 1 \leq j_{\neq k,t}, k_{\neq t}, t \leq q,\\
\alpha_{j,1}^i \delta_{t,t}^i = \delta_{j,t}^i \left(\alpha_{t,1}^i - n_t\right), & q+1 \leq i \leq s, & 1 \leq j_{\neq t}, t \leq q,\\
\delta_{t,k}^i \left(\alpha_{j,1}^i - n_k \alpha_{j,1}^k\right) = \delta_{j,k}^i \left(\alpha_{t,1}^i - n_k \alpha_{t,1}^k\right), & q+1 \leq i_{\neq k}, k \leq s, & 1 \leq j_{\neq t}, t \leq q,\\
\gamma_{i,j,2}^p \alpha_{t,1}^p - \gamma_{i,t,2}^p \alpha_{j,1}^p = 0, & q+1 \leq p \leq s, & 1 \leq i, j_{\neq t}, t \leq q,
\end{array}
$$

$$
k\left(\gamma_{i,j,k}^p \alpha_{t,1}^p - \gamma_{i,t,k}^p \alpha_{j,1}^p\right) + \sum_{r=2}^{k-1} \left(\gamma_{i,j,k+1-r}^p \alpha_{t,r}^p - \gamma_{i,t,k+1-r}^p \alpha_{j,r}^p\right) = 0,
$$
$$
1 \leq i, j_{\neq t}, t \leq q,\ q+1 \leq p \leq s,\ 3 \leq k \leq n_p.
$$

Proof Starting from the products (3.1) and (3.2), we shall verify the Leibniz identity for the basis elements.

Consider the Leibniz identity

$$0 = \big[x_j, [e_1^i, e_1^t]\big] = \big[[x_j, e_1^i], e_1^t\big] - \big[[x_j, e_1^t], e_1^i\big] = \sum_{k=1}^{n_t-1} \beta_{j,k}^{i,t} e_{k+1}^t - \sum_{k=1}^{n_i-1} \beta_{j,k}^{t,i} e_{k+1}^i$$

with $1 \leq i_{\neq t},\ t \leq s,\ 1 \leq j \leq q$.

Then we have

$$\beta_{j,k}^{i,t} = 0,\ 1 \leq k \leq n_t - 1, \quad \beta_{j,k}^{t,i} = 0, \quad 1 \leq k \leq n_i - 1.$$

Since $[e_1^i, x_j] + [x_j, e_1^i] \in \mathrm{Ann}_r(R)$, then it implies

$$\begin{array}{ll} \beta_{i,1}^{i,i} = -1, & 1 \leq i \leq q, \\ \beta_{j,1}^{i,i} = 0, & 1 \leq i_{\neq j},\ j \leq q, \\ \beta_{j,1}^{i,i} = -\alpha_{j,1}^i, & q+1 \leq i \leq s, \quad 1 \leq j \leq q. \end{array}$$

Thus, for $1 \leq j \leq q$, we have

$$[x_i, e_1^i] = -e_1^i + \sum_{k=2}^{n_i} \beta_{i,k}^{i,i} e_k^i + \sum_{\substack{p=1\\ p\neq i}}^{s} \beta_{i,n_p}^{i,p} e_{n_p}^p, \quad 1 \leq i \leq q,$$

$$[x_j, e_1^i] = \sum_{k=2}^{n_i} \beta_{j,k}^{i,i} e_k^i + \sum_{\substack{p=1\\ p\neq i}}^{s} \beta_{j,n_p}^{i,p} e_{n_p}^p, \quad 1 \leq i_{\neq j} \leq q,$$

$$[x_j, e_1^i] = -\alpha_{j,1}^i e_1^i + \sum_{k=2}^{n_i} \beta_{j,k}^{i,i} e_k^i + \sum_{\substack{p=1\\ p\neq i}}^{s} \beta_{j,n_p}^{i,p} e_{n_p}^p, \quad q+1 \leq i \leq s.$$

By taking the change of basis

$$e_1^{i'} = e_1^i - \sum_{p=q+1}^{s} \beta_{i,n_p}^{i,p} e_{n_p}^p, \quad 1 \leq i \leq q, \quad e_1^{i'} = e_1^i, \quad q+1 \leq i \leq s,$$

$$x_j' = x_j - \sum_{k=3}^{n_j} \beta_{j,k}^{j,j} e_{k-1}^j - \sum_{\substack{p=1\\ p\neq j}}^{s} \sum_{k=2}^{n_p} \beta_{j,k}^{p,p} e_{k-1}^p, \quad 1 \leq j \leq q,$$

we get

$$[x_i, e_1^i] = -e_1^i + \beta_{i,2}^{i,i} e_2^i + \sum_{\substack{p=1\\ p\neq i}}^{q} \beta_{i,n_p}^{i,p} e_{n_p}^p, \quad 1 \leq i \leq q,$$

$$[x_j, e_1^i] = \sum_{\substack{p=1\\ p\neq i}}^{s} \beta_{j,n_p}^{i,p} e_{n_p}^p, \quad 1 \leq i_{\neq j}, \quad j \leq q,$$

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$$[x_j, e_1^i] = -\alpha_{j,1}^i e_1^i + \sum_{\substack{p=1\\p\neq i}}^{s} \beta_{j,n_p}^{i,p} e_{n_p}^p, \quad q+1 \leq i \leq s, \quad 1 \leq j \leq q.$$

Since $[x_i, x_i], [x_i, x_j] + [x_j, x_i] \in \mathrm{Ann}_r(R)$, then $\gamma_{i,i,1}^p = 0,\ \gamma_{i,j,1}^p = -\gamma_{j,i,1}^p$.

Consider the equalities

$$\begin{aligned}\left[x_i, [e_1^i, x_i]\right] &= \left[[x_i, e_1^i], x_i\right] - \left[[x_i, x_i], e_1^i\right]\\ &= \left[-e_1^i + \beta_{i,2}^{i,i} e_2^i + \sum_{\substack{p=1\\p\neq i}}^{q} \beta_{i,n_p}^{i,p} e_{n_p}^p, x_i\right] - \left[\sum_{p=1}^{s}\sum_{k=2}^{n_p} \gamma_{i,i,k}^p e_k^p, e_1^i\right]\\ &= -e_1^i - \sum_{k=q+1}^{s} \delta_{i,k}^i e_{n_k}^k + 2\beta_{i,2}^{i,i} e_2^i - \sum_{k=2}^{n_i-1} \gamma_{i,i,k}^i e_{k+1}^i.\end{aligned}$$

On the other hand,

$$\left[x_i, [e_1^i, x_i]\right] = \left[x_i, e_1^i + \sum_{k=q+1}^{s} \delta_{i,k}^i e_{n_k}^k\right] = -e_1^i + \beta_{i,2}^{i,i} e_2^i + \sum_{\substack{p=1\\p\neq i}}^{q} \beta_{i,n_p}^{i,p} e_{n_p}^p.$$

Comparing the coefficients at the basis elements, we derive

$$\begin{aligned}&\beta_{i,2}^{i,i} = 0,\ 1 \leq i \leq q, \qquad \beta_{i,n_p}^{i,p} = 0, \qquad 1 \leq p_{\neq i} \leq q,\\ &\delta_{i,k}^i = 0,\ q+1 \leq k \leq s, \qquad \gamma_{i,i,t}^i = 0, \qquad 2 \leq t \leq n_i - 1.\end{aligned}$$

Taking into account that for $1 \leq i_{\neq j}, j \leq q$, the products $[x_j, e_1^i]$ belong to $\mathrm{Ann}_r(R)$, and considering the Leibniz identity for the elements $\{x_j, x_j, e_1^i\}$ with $1 \leq i_{\neq j}, j \leq q$, then we deduce

$$\beta_{j,n_j}^{i,j} = \gamma_{j,j,k}^i = \alpha_{j,1}^p \beta_{j,n_p}^{i,p} = 0, \quad 1 \leq i_{\neq j}, j \leq q, \quad 2 \leq k \leq n_i - 1, \quad q+1 \leq p \leq s.$$

Bearing in mind the Leibniz identity $\left[x_j, [x_j, e_1^i]\right]$ for $q+1 \leq i \leq s$, then we have

$$\begin{aligned}\left[x_j, [x_j, e_1^i]\right] &= \left[[x_j, x_j], e_1^i\right] - \left[[x_j, e_1^i], x_j\right]\\ &= \sum_{k=2}^{n_i-1} \gamma_{j,j,k}^i e_{k+1}^i + \alpha_{j,1}^i \left(\sum_{k=1}^{n_i} \alpha_{j,k}^i e_k^i + \sum_{\substack{p=1\\p\neq i}}^{s} \delta_{j,p}^i e_{n_p}^p\right) - n_j \beta_{j,n_j}^{i,j} e_{n_j}^j\\ &\quad - \sum_{\substack{p=q+1\\p\neq i}}^{s} n_p \alpha_{j,1}^p \beta_{j,n_p}^{i,p} e_{n_p}^p.\end{aligned}$$

On the other hand,

$$\left[x_j, [x_j, e_1^i]\right] = \left[x_j, -\alpha_{j,1}^i e_1^i + \sum_{\substack{p=1\\p\neq i}}^{s} \beta_{j,n_p}^{i,p} e_{n_p}^p\right] = \alpha_{j,1}^i \left(\alpha_{j,1}^i e_1^i - \sum_{\substack{p=1\\p\neq i}}^{s} \beta_{j,n_p}^{i,p} e_{n_p}^p\right).$$

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Comparing the coefficients at the basis elements, we get

$$\begin{array}{llll}
\alpha^i_{j,1}\alpha^i_{j,2}=0, & q+1\le i\le s, & 1\le j\le q, & \\
\gamma^i_{j,j,k-1}=-\alpha^i_{j,1}\alpha^i_{j,k}, & q+1\le i\le s, & 1\le j\le q, & 3\le k\le n_i,\\
\alpha^i_{j,1}\delta^i_{j,j}=\left(n_j-\alpha^i_{j,1}\right)\beta^{i,j}_{j,n_j}, & q+1\le i\le s, & 1\le j\le q, & \\
\alpha^i_{j,1}\delta^i_{j,p}=-\alpha^i_{j,1}\beta^{i,p}_{j,n_p}, & q+1\le i\le s, & 1\le j_{\neq p},\, p\le q, & \\
\alpha^i_{j,1}\delta^i_{j,p}=\left(n_p\alpha^p_{j,1}-\alpha^i_{j,1}\right)\beta^{i,p}_{j,n_p}, & q+1\le i_{\neq p},\, p\le s, & 1\le j\le q. &
\end{array}$$

From the chain of equalities

$$\begin{aligned}
0&=\left[x_i,\ \sum_{\substack{p=1\\ p\neq i,j}}^{s}\beta^{i,p}_{j,n_p}e^p_{n_p}\right]=\left[x_i,[x_j,e^i_1]\right]=\left[[x_i,x_j],e^i_1\right]-\left[[x_i,e^i_1],x_j\right]\\
&=\left[\sum_{p=1}^{s}\sum_{k=1}^{n_p}\gamma^p_{i,j,k}e^p_k,\ e^i_1\right]-[-e^i_1,x_j]=\sum_{k=1}^{n_i-1}\gamma^i_{i,j,k}e^i_{k+1}+\sum_{k=2}^{n_i}\alpha^i_{j,k}e^i_k+\sum_{\substack{k=1\\ k\neq i}}^{s}\delta^i_{j,k}e^k_{n_k},
\end{aligned}$$

we obtain

$$\begin{array}{lll}
\delta^i_{j,k}=0, & 1\le i_{\neq j},\, j\le q, & 1\le k\le s,\\
\gamma^i_{i,j,k-1}=-\alpha^i_{j,k}, & 1\le i_{\neq j},\, j\le q, & 2\le k\le n_i.
\end{array}$$

Applying the Leibniz identity for the triples of elements $\{x_j, x_i, e^i_1\}$, $\{e^i_1, x_j, x_i\}$ and $\{x_j, x_t, e^i_1\}$, we derive the following conditions for the structural constants:

$$\begin{array}{llll}
\beta^{i,p}_{j,n_p}=0, & 1\le i_{\neq j,p},\, j_{\neq p},\, p\le q, & & \\
\left(n_p\alpha^p_{i,1}-1\right)\beta^{i,p}_{j,n_p}=0, & 1\le i_{\neq j},\, j\le q, & q+1\le p\le s, & \\
\gamma^i_{j,i,k-1}=0, & 1\le i_{\neq j},\, j\le q, & 2\le k\le n_i, & \\
\alpha^i_{j,k}=0, & 1\le i,\, j\le q, & 2\le k\le n_i, & \\
\gamma^i_{i,j,k-1}=0, & i\neq j, & & \\
\alpha^p_{t,1}\beta^{i,p}_{j,n_p}=\gamma^i_{j,t,k-1}=0, & 1\le i_{\neq j,t},\, j_{\neq t},\, t\le q, & q+1\le p\le s, & 2\le k\le n_i,\\
\gamma^i_{j,t,k-1}=-\alpha^i_{j,1}\alpha^i_{t,k}, & q+1\le i\le s, & 1\le j_{\neq t},\, t\le q, & 2\le k\le n_i,\\
\alpha^i_{j,1}\delta^i_{t,t}=\left(n_t-\alpha^i_{t,1}\right)\beta^{i,t}_{j,n_t}, & q+1\le i\le s, & 1\le j_{\neq t},\, t\le q, & \\
\alpha^i_{j,1}\delta^i_{t,p}=-\alpha^i_{t,1}\beta^{i,p}_{j,n_p}, & q+1\le i\le s, & 1\le j_{\neq t},\, p,\, t_{\neq p}\le q, & \\
\alpha^i_{j,1}\delta^i_{t,p}=\left(n_p\alpha^p_{t,1}-\alpha^i_{t,1}\right)\beta^{i,p}_{j,n_p}, & q+1\le i_{\neq p},\, p\le s, & 1\le j_{\neq t},\, t\le q. &
\end{array}$$

If $1\le i\le q$, then the Leibniz identity for the elements $\{e^i_1, x_j, x_t\}$, with $i_{\neq j,t}$ and $j_{\neq t}$, does not give any additional restrictions. Therefore, we shall consider the case $q+1\le i\le s$. Namely, we have the following chain of equalities:

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$$\begin{aligned}\left[e_1^i,\left[x_j,x_t\right]\right]&=\left[\left[e_1^i,x_j\right],x_t\right]-\left[\left[e_1^i,x_t\right],x_j\right]\\&=\alpha_{j,1}^i\left(\sum_{k=1}^{n_i}\alpha_{t,k}^ie_k^i+\sum_{\substack{k=1\\k\neq i}}^{s}\delta_{t,k}^ie_{n_k}^k\right)+\sum_{k=2}^{n_i}\alpha_{j,k}^i\left(k\alpha_{t,1}^ie_k^i+\sum_{p=k+1}^{n_i}\alpha_{t,p-k+1}^ie_p^i\right)\\&\quad+n_t\delta_{j,t}^ie_{n_t}^t+\sum_{\substack{k=q+1\\k\neq i}}^{s}n_k\delta_{j,k}^i\alpha_{t,1}^ke_{n_k}^k-\alpha_{t,1}^i\left(\sum_{k=1}^{n_i}\alpha_{j,k}^ie_k^i+\sum_{\substack{k=1\\k\neq i}}^{s}\delta_{j,k}^ie_{n_k}^k\right)\\&\quad-\sum_{k=2}^{n_i}\alpha_{t,k}^i\left(k\alpha_{j,1}^ie_k^i+\sum_{p=k+1}^{n_i}\alpha_{j,p-k+1}^ie_p^i\right)-n_j\delta_{t,j}^ie_{n_j}^j-\sum_{\substack{k=q+1\\k\neq i}}^{s}n_k\delta_{t,k}^i\alpha_{j,1}^ke_{n_k}^k\\&=\sum_{k=2}^{n_i}(k-1)\left(\alpha_{t,1}^i\alpha_{j,k}^i-\alpha_{j,1}^i\alpha_{t,k}^i\right)e_k^i+\sum_{k=2}^{n_i}\alpha_{j,k}^i\sum_{p=k+1}^{n_i}\alpha_{t,p-k+1}^ie_p^i\\&\quad-\sum_{k=2}^{n_i}\alpha_{t,k}^i\sum_{p=k+1}^{n_i}\alpha_{j,p-k+1}^ie_p^i+\sum_{\substack{k=1\\k\neq j,t}}^{q}\left(\alpha_{j,1}^i\delta_{t,k}^i-\alpha_{t,1}^i\delta_{j,k}^i\right)e_{n_k}^k\\&\quad+\left(\alpha_{j,1}^i\delta_{t,t}^i-\alpha_{t,1}^i\delta_{j,t}^i+n_t\delta_{j,t}^i\right)e_{n_t}^t+\left(\alpha_{j,1}^i\delta_{t,j}^i-\alpha_{t,1}^i\delta_{j,j}^i-n_j\delta_{t,j}^i\right)e_{n_j}^j\\&\quad+\sum_{k=q+1}^{s}\left(\alpha_{j,1}^i\delta_{t,k}^i-\alpha_{t,1}^i\delta_{j,k}^i+n_k\alpha_{t,1}^k\delta_{j,k}^i-n_k\alpha_{j,1}^k\delta_{t,k}^i\right)e_{n_k}^k\\&=\sum_{k=2}^{n_i}(k-1)\left(\alpha_{t,1}^i\alpha_{j,k}^i-\alpha_{j,1}^i\alpha_{t,k}^i\right)e_k^i+\sum_{\substack{k=1\\k\neq j,t}}^{q}\left(\alpha_{j,1}^i\delta_{t,k}^i-\alpha_{t,1}^i\delta_{j,k}^i\right)e_{n_k}^k\\&\quad+\left(\alpha_{j,1}^i\delta_{t,t}^i-\alpha_{t,1}^i\delta_{j,t}^i+n_t\delta_{j,t}^i\right)e_{n_t}^t+\left(\alpha_{j,1}^i\delta_{t,j}^i-\alpha_{t,1}^i\delta_{j,j}^i-n_j\delta_{t,j}^i\right)e_{n_j}^j\\&\quad+\sum_{\substack{k=q+1\\k\neq i}}^{s}\left(\alpha_{j,1}^i\delta_{t,k}^i-\alpha_{t,1}^i\delta_{j,k}^i+n_k\alpha_{t,1}^k\delta_{j,k}^i-n_k\alpha_{j,1}^k\delta_{t,k}^i\right)e_{n_k}^k.\end{aligned}$$

On the other hand,

$$\left[e_1^i,[x_j,x_t]\right]=\left[e_1^i,\sum_{p=1}^{q}\gamma_{j,t,n_p}^pe_{n_p}^p+\sum_{p=q+1}^{s}\sum_{k=1}^{n_p}\gamma_{j,t,k}^pe_k^p\right]=\gamma_{j,t,1}^ie_2^i.$$

Comparing the coefficients for $q+1\le i\le s$ and for $j_{\neq t}$, we get

$$\begin{aligned}\gamma_{j,t,1}^i&=\alpha_{t,1}^i\alpha_{j,2}^i-\alpha_{j,1}^i\alpha_{t,2}^i,\\\alpha_{t,1}^i\alpha_{j,k}^i&=\alpha_{j,1}^i\alpha_{t,k}^i,\quad 3\le k\le n_i,\\\alpha_{j,1}^i\delta_{t,k}^i&=\alpha_{t,1}^i\delta_{j,k}^i,\quad 1\le k_{\neq j,t}\le q,\end{aligned}$$

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$$\alpha^i_{j,1}\delta^i_{t,t} = \delta^i_{j,t}\left(\alpha^i_{t,1} - n_t\right),$$
$$\alpha^i_{t,1}\delta^i_{j,j} = \delta^i_{t,j}\left(\alpha^i_{j,1} - n_j\right),$$
$$\delta^i_{t,k}\left(\alpha^i_{j,1} - n_k\alpha^k_{j,1}\right) = \delta^i_{j,k}\left(\alpha^i_{t,1} - n_k\alpha^k_{t,1}\right), \quad q+1 \le k_{\neq i} \le s.$$

From which we derive

$$\gamma^i_{j,t,1} = \alpha^i_{t,1}\alpha^i_{j,2} = \alpha^i_{j,1}\alpha^i_{t,2} = 0.$$

Similarly as above, from the chain of equalities

$$\begin{aligned}
0 &= \left[x_i, \sum_{p=1}^{q}\gamma^p_{j,t,n_p}e^p_{n_p} + \sum_{p=q+1}^{s}\sum_{k=2}^{n_p}\gamma^p_{j,t,k}e^p_k\right] \\
&= \left[x_i,[x_j,x_t]\right] = \left[[x_i,x_j],x_t\right] - \left[[x_i,x_t],x_j\right] \\
&= n_t\gamma^t_{i,j,n_t}e^t_{n_t} + \sum_{p=q+1}^{s}\sum_{k=2}^{n_p}\gamma^p_{i,j,k}\Big(k\alpha^p_{t,1}e^p_k + \sum_{r=k+1}^{n_p}\alpha^p_{t,r-k+1}e^t_r\Big) - n_j\gamma^j_{i,t,n_j}e^j_{n_j} \\
&\quad - \sum_{p=q+1}^{s}\sum_{k=2}^{n_p}\gamma^p_{i,t,k}\Big(k\alpha^p_{j,1}e^p_k + \sum_{r=k+1}^{n_p}\alpha^p_{j,r-k+1}e^j_r\Big) = n_t\gamma^t_{i,j,n_t}e^t_{n_t} - n_j\gamma^j_{i,t,n_j}e^j_{n_j} \\
&\quad + \sum_{p=q+1}^{s}\sum_{k=2}^{n_p}k\Big(\gamma^p_{i,j,k}\alpha^p_{t,1} - \gamma^p_{i,t,k}\alpha^p_{j,1}\Big)e^p_k + \sum_{p=q+1}^{s}\sum_{k=2}^{n_p}\sum_{r=k+1}^{n_p}\gamma^p_{i,j,k}\alpha^p_{t,r-k+1}e^t_r \\
&\quad - \sum_{p=q+1}^{s}\sum_{k=2}^{n_p}\sum_{r=k+1}^{n_p}\gamma^p_{i,t,k}\alpha^p_{j,r-k+1}e^j_r = n_t\gamma^t_{i,j,n_t}e^t_{n_t} - n_j\gamma^j_{i,t,n_j}e^j_{n_j} \\
&\quad + \sum_{p=q+1}^{s}\sum_{k=2}^{n_p}\left(k(\gamma^p_{i,j,k}\alpha^p_{t,1} - \gamma^p_{i,t,k}\alpha^p_{j,1}) + \sum_{r=2}^{k-1}(\gamma^p_{i,j,k+1-r}\alpha^p_{t,r} - \gamma^p_{i,t,k+1-r}\alpha^p_{j,r})\right)e^p_k,
\end{aligned}$$

comparing the coefficients at the basis elements, we deduce

$$\gamma^t_{i,j,n_t} = \gamma^j_{i,t,n_j} = 0, \quad \gamma^p_{i,j,2}\alpha^p_{t,1} = \gamma^p_{i,t,2}\alpha^p_{j,1}, \quad q+1 \le p \le s,$$
$$k\left(\gamma^p_{i,j,k}\alpha^p_{t,1} - \gamma^p_{i,t,k}\alpha^p_{j,1}\right) + \sum_{r=2}^{k-1}\left(\gamma^p_{i,j,k+1-r}\alpha^p_{t,r} - \gamma^p_{i,t,k+1-r}\alpha^p_{j,r}\right) = 0,$$
$$q+1 \le p \le s,\ 3 \le k \le n_p.$$

Summarizing all obtained restrictions, we have a family of algebras with the difference with the family of the assertion of the theorem in the product $[x_i, x_j]$. However, it can be improved by taking the following change of basis

$$x'_i = x_i - \sum_{k=1}^{q}\frac{\gamma_{i,k,n_k}}{n_k}e^k_{n_k}, \quad 1 \le i \le q.$$

Thus, we obtain the family of algebras with restrictions to structural constants as in the assertion of the theorem. □

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From Theorem 3.3 in the case when the dimension of the complemented space is equal to s we have the complete classification of such solvable Leibniz algebras.

Corollary 3.4 *Let $R = N \oplus Q$, where $N = NF_1 \oplus NF_2 \oplus \cdots \oplus NF_s$ and* $\dim Q = s$. *Then the algebra R is isomorphic to the algebra:*

$$\begin{cases} [e_t^i, e_1^i] = e_{t+1}^i, & 1 \le i \le s, 1 \le t \le n_i - 1, \\ [e_t^i, x_i] = te_t^i, & 1 \le i \le s, 1 \le t \le n_i, \\ [x_i, e_1^i] = -e_1^i, & 1 \le i \le s, 1 \le t \le n_i. \end{cases}$$

Corollary 3.4 implies that $R \cong (NF_1 + \langle x_1 \rangle) \oplus (NF_2 + \langle x_2 \rangle) \oplus \cdots \oplus (NF_s + \langle x_s \rangle)$.

Below, we present the description of solvable Leibniz algebras with nilradical $N = NF_1 \oplus NF_2 \oplus \cdots \oplus NF_s$ and with the condition that each NF_i is a ideal of the algebra R.

Theorem 3.5 *Let R be a solvable Leibniz algebra with nilradical $N = NF_1 \oplus NF_2 \oplus \cdots \oplus NF_s$ and* $\dim Q = q$. *If NF_i are ideals of R, $1 \le i \le s$, then R is isomorphic to one of the following algebras:*

$$R_{s,0}^q : \begin{cases} [e_t^i, e_1^i] = e_{t+1}^i, & 1 \le i \le s,\ 1 \le t \le n_i - 1, \\ [x_i, e_1^i] = -e_1^i, \quad [e_t^i, x_i] = te_t^i, & 1 \le i \le q,\ 1 \le t \le n_i, \\ [e_t^i, x_j] = \sum\limits_{k=t+1}^{n_i} \alpha_{j,k-t+1}^i e_k^i, & 1 \le j \le q,\ q+1 \le i \le s,\ 1 \le t \le n_i, \\ [x_i, x_j] = \sum\limits_{p=q+1}^{s} \gamma_{i,j,n_p}^p e_{n_p}^p, & 1 \le i, j \le q; \end{cases}$$

$$R_{s,s_0}^q : \begin{cases} [e_t^i, e_1^i] = e_{t+1}^i, & 1 \le i \le s,\ 1 \le t \le n_i - 1, \\ [x_i, e_1^i] = -e_1^i, \quad [e_t^i, x_i] = te_t^i, & 1 \le i \le q,\ 1 \le t \le n_i, \\ [x_j, e_1^i] = -\alpha_{j,1}^i e_1^i, \quad [e_t^i, x_j] = t\alpha_{j,1}^i e_t^i, & 1 \le j \le q,\ q+1 \le i \le q+s_0, \\ & 1 \le t \le n_i, \\ [e_t^i, x_j] = \sum\limits_{k=t+1}^{n_i} \alpha_{j,k-t+1}^i e_k^i, & 1 \le j \le q,\ q+s_0+1 \le i \le s, \\ & 1 \le t \le n_i, \\ [x_i, x_j] = \sum\limits_{p=q+s_0+1}^{s} \gamma_{i,j,n_p}^p e_{n_p}^p, & 1 \le i, j \le q; \end{cases}$$

where for any i $(q+1 \le i \le q+s_0)$ there exists j_i $(1 \le j_i \le q)$ such that $\alpha_{j_i,1}^i \ne 0$.

Proof Taking into account that NF_i, $1 \le i \le s$, are ideals of R, from Theorem 3.3, we have the multiplication table

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$$
\begin{array}{lll}
[e_t^i, e_1^i] = e_{t+1}^i, & 1 \le i \le s, & 1 \le t \le n_i - 1, \\
[x_i, e_1^i] = -e_1^i, \quad [e_t^i, x_i] = te_t^i, & 1 \le i \le q, & 1 \le t \le n_i, \\
[e_1^i, x_j] = \sum\limits_{k=1}^{n_i} \alpha_{j,k}^i e_k^i, & q+1 \le i \le s, & 1 \le j \le q, \\
[e_t^i, x_j] = t\alpha_{j,1}^i e_t^i + \sum\limits_{k=t+1}^{n_i} \alpha_{j,k-t+1}^i e_k^i, & q+1 \le i \le s, & 1 \le j \le q, \quad 2 \le t \le n_i, \\
[x_j, e_1^i] = -\alpha_{j,1}^i e_1^i, & q+1 \le i \le s, & 1 \le j \le q, \\
[x_i, x_j] = \sum\limits_{p=q+1}^{s} \sum\limits_{k=2}^{n_p} \gamma_{i,j,k}^p e_k^p, & 1 \le i, j \le q, &
\end{array}
$$

with the restrictions

$$
\begin{array}{llll}
\alpha_{t,1}^p \alpha_{j,2}^p = 0, & q+1 \le p \le s, & 1 \le j, t \le q, & \\
\alpha_{t,1}^p \alpha_{j,k}^p - \alpha_{j,1}^p \alpha_{t,k}^p = 0, & q+1 \le p \le s, & 1 \le j_{\neq t}, t \le q, & 3 \le k \le n_i, \\
\gamma_{j,t,k-1}^p = -\alpha_{j,1}^p \alpha_{t,k}^p, & q+1 \le p \le s, & 1 \le j, t \le q, & 3 \le k \le n_p, \\
\gamma_{i,j,2}^p \alpha_{t,1}^p - \gamma_{i,t,2}^p \alpha_{j,1}^p = 0, & q+1 \le p \le s, & 1 \le i, j_{\neq t}, t \le q, &
\end{array}
$$

$$
k\left(\gamma_{i,j,k}^p \alpha_{t,1}^p - \gamma_{i,t,k}^p \alpha_{j,1}^p\right) + \sum_{r=2}^{k-1} \left(\gamma_{i,j,k+1-r}^p \alpha_{t,r}^p - \gamma_{i,t,k+1-r}^p \alpha_{j,r}^p\right) = 0,
$$
$$
1 \le i, j_{\neq t}, t \le q, \quad q+1 \le p \le s, \quad 3 \le k \le n_p.
$$

Denote by s_0 the number of non-zero vectors

$$
\left(\alpha_{1,1}^{q+1}, \alpha_{2,1}^{q+1}, \dots, \alpha_{q,1}^{q+1}\right), \quad \left(\alpha_{1,1}^{q+2}, \alpha_{2,1}^{q+2}, \dots, \alpha_{q,1}^{q+2}\right), \dots, \left(\alpha_{1,1}^{s}, \alpha_{2,1}^{s}, \dots, \alpha_{q,1}^{s}\right).
$$

If $s_0 = 0$, i.e. $\alpha_{j,1}^i = 0$ for all $1 \le j \le q$, $q+1 \le i \le s$, then we do not have any restriction and hence, we get the algebra $R_{s,0}^q$.

If $s_0 \ge 1$, then without loss of generality, we can assume

$$
\begin{array}{ll}
(\alpha_{1,1}^p, \alpha_{2,1}^p, \dots, \alpha_{q,1}^p) \ne (0, 0, \dots, 0), & q+1 \le p \le q+s_0, \\
(\alpha_{1,1}^p, \alpha_{2,1}^p, \dots, \alpha_{q,1}^p) = (0, 0, \dots, 0), & q+s_0+1 \le p \le s.
\end{array}
$$

From the above restrictions, we conclude

$$
\alpha_{j,2}^p = 0, \quad q+1 \le p \le q+s_0, \quad 1 \le j \le q.
$$

Let us fix some p $(q+1 \le p \le q+s_0)$. Then there exists j_p $(1 \le j_p \le q)$ such that $\alpha_{j_p,1}^p \ne 0$.

By taking the change

$$
e_t^{p'} = e_t^p + \frac{1}{\alpha_{j_p,1}^p} \sum_{k=t+2}^{n_p} A_{k-t+1}^p e_k^p, \quad 1 \le t \le n_p, \quad x_{j_p}' = x_{j_p} + \frac{1}{\alpha_{j_p,1}^p} \sum_{k=2}^{n_p-1} A_{k+1}^p e_k^p,
$$

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with

$$A_3^p = -\frac{1}{2}\alpha_{j_p,3}^p, \quad A_4^p = -\frac{1}{3}\alpha_{j_p,4}^p,$$
$$A_k^p = -\frac{1}{k-1}\left(\sum_{j=3}^{k-2} A_{k-j+1}^p \alpha_{j_p,j}^p + \alpha_{1,k}^p\right), \quad 5 \le k \le n_p,$$

we deduce

$$\left[x_{j_p}, e_1^p\right] = -\alpha_{j_p,1}^p e_1^p, \quad \left[e_t^p, x_{j_p}\right] = t\alpha_{j_p,1}^p e_t^p, \quad 1 \le t \le n_p.$$

Therefore, we can suppose

$$\alpha_{j_p,k}^p = 0, \quad q+1 \le p \le q+s_0, \quad 2 \le k \le n_p. \tag{3.3}$$

Then the equality

$$\alpha_{j_p,1}^p \alpha_{j,k}^p - \alpha_{j,1}^p \alpha_{j_p,k}^p = 0, \quad 1 \le j_{\neq j_p} \le q, \quad 3 \le k \le n_p,$$

implies

$$\alpha_{j,k}^p = 0, \quad 1 \le j_{\neq j_p} \le q, \quad 2 \le k \le n_p. \tag{3.4}$$

Summarizing (3.3) and (3.4), we obtain $\alpha_{j,k}^p = 0$ for $1 \le j \le q,\ 2 \le k \le n_p$.

Using above argumentation for any p $(q+1 \le p \le q+s_0)$, we get

$$\alpha_{j,k}^p = 0, \quad q+1 \le p \le q+s_0, \quad 1 \le j \le q, \quad 2 \le k \le n_p.$$

Since $\alpha_{j,1}^p = 0$ for $q+s_0+1 \le p \le s$ and $1 \le j \le q$, from the restriction

$$\gamma_{j,t,k-1}^p = -\alpha_{j,1}^p \alpha_{t,k}^p, \quad q+1 \le p \le s, \quad 1 \le j, t \le q, \quad 3 \le k \le n_p,$$

we obtain $\gamma_{j,t,k-1}^p = 0, \quad q+1 \le p \le s,\ 1 \le j,\ t \le q,\ 3 \le k \le n_p$.

Applying the obtained equalities, for $q+1 \le p \le s$, we conclude that the following restrictions

$$\gamma_{i,j,2}^p \alpha_{t,1}^p - \gamma_{i,t,2}^p \alpha_{j,1}^p = 0, \quad 1 \le i, j_{\neq t}, t \le q,$$
$$k\left(\gamma_{i,j,k}^p \alpha_{t,1}^p - \gamma_{i,t,k}^p \alpha_{j,1}^p\right) + \sum_{r=2}^{k-1}\left(\gamma_{i,j,k+1-r}^p \alpha_{t,r}^p - \gamma_{i,t,k+1-r}^p \alpha_{j,r}^p\right) = 0,$$
$$1 \le i, j_{\neq t}, t \le q,\ 3 \le k \le n_p,$$

are reduced to the following one

$$\gamma_{i,j,n_p}^p \alpha_{t,1}^p - \gamma_{i,t,n_p}^p \alpha_{j,1}^p = 0, \quad q+1 \le p \le q+s_0, \quad 1 \le i, j_{\neq t}, t \le q.$$

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Thus, we obtain the following multiplication table:

$$\begin{array}{ll}
[e_t^i, e_1^i] = e_{t+1}^i, & 1 \le i \le s,\ 1 \le t \le n_i - 1, \\
[x_i, e_1^i] = -e_t^i, \quad [e_t^i, x_i] = te_1^i, & 1 \le i \le q,\ 1 \le t \le n_i, \\
[x_j, e_1^i] = -\alpha_{j,1}^i e_1^i, \quad [e_t^i, x_j] = t\alpha_{j,1}^i e_t^i, & 1 \le j \le q,\ q+1 \le i \le q+s_0,\ 1 \le t \le n_i, \\
[e_t^i, x_j] = \sum\limits_{k=t+1}^{n_i} \alpha_{j,k-t+1}^i e_k^i, & 1 \le j \le q,\ q+s_0+1 \le i \le s,\ 1 \le t \le n_i, \\
[x_i, x_j] = \sum\limits_{p=q+1}^{s} \gamma_{i,j,n_p}^p e_{n_p}^p, & 1 \le i, j \le q,
\end{array}$$

with the restriction

$$\gamma_{i,j,n_p}^p \alpha_{t,1}^p - \gamma_{i,t,n_p}^p \alpha_{j,1}^p = 0, \quad q+1 \le p \le q+s_0, \quad 1 \le i,\ j_{\neq t},\ t \le q.$$

Again using the fact about the existence of j_p such that $\alpha_{j_p,1}^p \neq 0$ for any fixed p $(q+1 \le p \le q+s_0)$, we have

$$\gamma_{i,j,n_p}^p = \frac{\gamma_{i,j_p,n_p}^p \alpha_{j,1}^p}{\alpha_{j_p,1}^p}, \quad 1 \le i, j \le q.$$

By taking the change

$$x_i' = x_i - \frac{\gamma_{i,j_p,n_p}^p}{n_p \alpha_{j_p,1}^p} e_{n_p}^p, \quad 1 \le i \le q,$$

we can suppose $\gamma_{i,j,n_p}^p = 0,\ q+1 \le p \le q+s_0,\ 1 \le i, j \le q$ and $[x_i, x_j] = \sum\limits_{p=q+s_0+1}^{s} \gamma_{i,j,n_p}^p e_{n_p}^p,\ 1 \le i, j \le q$. Thus, we obtain the algebra R_{s,s_0}^q. □

The next theorem is an adaptation of Theorem 3.3 for the case $q = 1$.

THEOREM 3.6 *Let R be a solvable Leibniz algebra with nilradical $NF_1 \oplus NF_2 \oplus \cdots \oplus NF_s$ and* $\dim Q = 1$. *Then it is isomorphic to one of the following algebras*

$$R_{s,s_0}^1 : \begin{cases}
[e_t^i, e_1^i] = e_{t+1}^i, & 1 \le i \le s,\ 1 \le t \le n_i - 1, \\
[x_1, e_1^1] = -e_1^1, & \\
[e_t^1, x_1] = te_t^1, & 1 \le t \le n_1, \\
[x_1, e_1^i] = -\alpha_{1,1}^i e_1^i, & 2 \le i \le s_0, \\
[e_t^i, x_1] = t\alpha_{1,1}^i e_t^i, & 2 \le i \le s_0,\ 1 \le t \le n_i, \\
[e_1^i, x_1] = \sum\limits_{k=2}^{n_i} \alpha_{1,k}^i e_k^i + \sum\limits_{\substack{k=s_0+1 \\ k \neq i}}^{s} \delta_{1,k}^i e_{n_k}^k, & s_0+1 \le i \le s, \\
[e_t^i, x_1] = \sum\limits_{k=t+1}^{n_i} \alpha_{1,k-t+1}^i e_k^i, & s_0+1 \le i \le s,\ 2 \le t \le n_i - 1, \\
[x_1, e_1^i] = \sum\limits_{\substack{p=s_0+1 \\ p \neq i}}^{s} \beta_{1,n_p}^{i,p} e_{n_p}^p, & s_0+1 \le i \le s, \\
[x_1, x_1] = \sum\limits_{p=s_0+1}^{s} \gamma_{n_p}^p e_{n_p}^p, &
\end{cases}$$

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where $1 \leq s_0 \leq s$ *and* $\alpha^i_{1,1} \neq 0$.

Proof For $q = 1$ by Theorem 3.3, we have for $2 \leq i \leq s$ the multiplication table

$$\begin{aligned}
&[x_1, e^1_1] = -e^1_1, \\
&[e^1_t, x_1] = te^1_t, && 1 \leq t \leq n_1, \\
&[e^i_1, x_1] = \sum_{k=1}^{n_i} \alpha^i_{1,k} e^i_k + \sum_{\substack{k=1 \\ k \neq i}}^{s} \delta^i_{1,k} e^k_{n_k}, \\
&[e^i_t, x_1] = t\alpha^i_{1,1} e^i_t + \sum_{k=t+1}^{n_i} \alpha^i_{1,k-t+1} e^1_k, && 2 \leq t \leq n_i, \\
&[x_1, e^i_1] = -\alpha^i_{1,1} e^i_1 + \sum_{\substack{p=1 \\ p \neq i}}^{s} \beta^{i,p}_{1,n_p} e^p_{n_p}, \\
&[x_1, x_1] = \sum_{p=2}^{s} \sum_{k=2}^{n_p} \gamma^p_k e^p_k,
\end{aligned}$$

with the following restrictions for $2 \leq i \leq s$:

$$\begin{aligned}
\alpha^i_{1,1}\alpha^i_{1,2} &= 0, \\
\gamma^i_{k-1} &= -\alpha^i_{1,1}\alpha^i_{1,k}, && 3 \leq k \leq n_i, \\
\alpha^i_{1,1}\delta^i_{1,1} &= (n_1 - \alpha^i_{1,1})\beta^{i,1}_{1,n_1}, \\
\alpha^i_{1,1}\delta^i_{1,p} &= (n_p\alpha^p_{1,1} - \alpha^i_{1,1})\beta^{i,p}_{1,n_p}, && 2 \leq p_{\neq i} \leq s.
\end{aligned}$$

Let the number of non-zero constants among $\{\alpha^2_{1,1}, \alpha^3_{1,1}, \dots, \alpha^s_{1,1}\}$ be equal to $s_0 - 1$. Then, without loss of generality, we can assume $\alpha^i_{1,1} \neq 0,\ 2 \leq i \leq s_0,\ \alpha^i_{1,1} = 0,\ s_0 + 1 \leq i \leq s$, where $1 \leq s_0 \leq s$.

Then from the above conditions we have $\alpha^i_{1,2} = 0,\ 2 \leq i \leq s_0$.

By taking the change

$$e^{i'}_1 = e^i_1 - \frac{1}{\alpha^i_{1,1}} \sum_{\substack{p=1 \\ p \neq i}}^{s} \beta^{i,p}_{1,n_p} e^p_{n_p}, \quad 2 \leq i \leq s_0,$$

we can suppose that $[x_1, e^i_1] = -\alpha^i_{1,1} e^i_1,\ 2 \leq i \leq s_0$, i.e. $\beta^{i,p}_{1,n_p} = 0,\ 2 \leq i_{\neq p} \leq s_0,\ 1 \leq p \leq s$.

From the above conditions, we have

$$\begin{aligned}
&\gamma^i_{k-1} = -\alpha^i_{1,1}\alpha^i_{1,k}, && 2 \leq i \leq s_0, && 3 \leq k \leq n_i, \\
&\gamma^i_{k-1} = 0, && s_0 + 1 \leq i \leq s, && 3 \leq k \leq n_i, \\
&\delta^i_{1,p} = 0, && 2 \leq i_{\neq p} \leq s_0 && 1 \leq p \leq s, \\
&\beta^{i,p}_{1,n_p} = 0, && s_0 + 1 \leq i \leq s, && 1 \leq p \leq s_0.
\end{aligned}$$

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By taking the change of basis

$$e_t^{i'} = e_t^i + \frac{1}{\alpha_{1,1}^i} \sum_{k=t+2}^{n_i} A_{k-t+1}^i e_k^i, \quad 2 \le i \le s_0,\ 1 \le t \le n_i,$$
$$x_1' x_1 + \sum_{i=2}^{s_0} \frac{1}{\alpha_{1,1}^i} \sum_{k=2}^{n_i-1} A_{k+1}^i e_k^i,$$

where

$$A_3^i = -\frac{1}{2}\alpha_{1,3}^i, \quad A_4^i = -\frac{1}{3}\alpha_{1,4}^i,$$
$$A_k^i = -\frac{1}{k-1}\left(\sum_{j=3}^{k-2} A_{k-j+1}^i \alpha_{1,j}^i + \alpha_{1,k}^i\right), \quad 2 \le i \le s_0,\ \ 5 \le k \le n_i,$$

we obtain

$$[x_1, e_1^i] = -\alpha_{1,1}^i e_1^i, \quad [e_t^i, x_1] = t\alpha_{1,1}^i e_t^i, \quad 2 \le i \le s_0, \quad 1 \le t \le n_i.$$

That is, we can suppose $\alpha_{1,k}^i = 0,\ 2 \le i \le s_0,\ 3 \le k \le n_i$.

Now taking the change of basis in the following form:

$$e_1^{i'} = e_1^i - \frac{\delta_{1,1}^i}{n_1} e_{n_1}^1 - \sum_{p=2}^{s_0} \frac{\delta_{1,p}^i}{n_p \alpha_{1,1}^p} e_{n_p}^p, \quad s_0 + 1 \le i \le s, \quad x_1' = x_1 - \sum_{p=2}^{s_0} \frac{\gamma_{n_p}^p}{n_p \alpha_{1,1}^p} e_{n_p}^p,$$

we can suppose $\delta_{1,k}^i = 0,\ 1 \le k \le s_0$ and $\gamma_{n_p}^p = 0,\ 2 \le p \le s_0$ and so we obtain the algebra R_{s,s_0}^1. □

Since the multiplication tables of the family R_{s,s_0}^1 are different to the multiplication tables of the family of algebras in the work [20, Theorem 3.9], we can conclude that the condition for the case $s \ge 3$ (which was omitted in [20]) each NF_i is a ideal, is crucial. Thus, the list of algebras presented in the work [20, Theorem 3.9] can be obtained from the description of Theorem 3.6 under the condition that all NF_i are ideals of the algebra R.

Below, we present the description of solvable Leibniz algebras with nilradical $NF_1 \oplus NF_2 \oplus \cdots \oplus NF_s$ and $\dim Q = 1$ for the cases $s = 3, 4$.

COROLLARY 3.7 *Let R be a solvable Leibniz algebra with nilradical $N = NF_1 \oplus NF_2 \oplus NF_3$ and* $\dim Q = 1$. *Then it is isomorphic to one of the following algebras*

$$R_{3,1}^1 : \begin{cases} [e_i^j, e_1^j] = e_{i+1}^j, & 1 \le j \le 3, & 1 \le i \le n_j - 1, \\ [x_1, e_1^1] = -e_1^1, & [e_i^1, x_1] = i e_i^1, & 1 \le i \le n_1, \\ [e_1^2, x_1] = \sum\limits_{k=2}^{n_2} \alpha_{1,k}^2 e_k^2 + \delta_{1,3}^2 e_{n_3}^3, & [e_i^2, x_1] = \sum\limits_{k=i+1}^{n_2} \alpha_{1,k-i+1}^2 e_k^2, & 2 \le i \le n_2 - 1, \\ [e_1^3, x_1] = \sum\limits_{k=2}^{n_3} \alpha_{1,k}^3 e_k^3 + \delta_{1,2}^3 e_{n_2}^2, & [e_i^3, x_1] = \sum\limits_{k=i+1}^{n_3} \alpha_{1,k-i+1}^3 e_k^3, & 2 \le i \le n_3 - 1, \\ [x_1, e_1^2] = \beta_{1,n_3}^{2,3} e_{n_3}^3 & [x_1, e_1^3] = \beta_{1,n_2}^{3,2} e_{n_2}^2, & \\ [x_1, x_1] = \gamma_{n_2}^2 e_{n_2}^2 + \gamma_{n_3}^3 e_{n_3}^3; & & \end{cases}$$

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$$R^1_{3,2}:\begin{cases}[e^j_i,e^j_1]=e^j_{i+1}, & 1\le j\le 3, \quad 1\le i\le n_j-1,\\ [x_1,e^1_1]=-e^1_1, & [e^1_i,x_1]=ie^1_i, \quad 1\le i\le n_1,\\ [x_1,e^2_1]=-\alpha^2_{1,1}e^2_1, & [e^2_i,x_1]=i\alpha^2_{1,1}e^2_i, \quad 1\le i\le n_2,\\ [e^3_i,x_1]=\sum\limits_{k=i+1}^{n_3}\alpha^3_{1,k-i+1}e^3_k, & 1\le i\le n_3-1, \quad [x_1,x_1]=\gamma^3_{n_3}e^3_{n_3};\end{cases}$$

$$R^1_{3,3}:\begin{cases}[e^j_i,e^j_1]=e^j_{i+1}, & 1\le j\le 3, \quad 1\le i\le n_j-1,\\ [x_1,e^1_1]=-e^1_1, & [e^1_i,x_1]=ie^1_i, \quad 1\le i\le n_1,\\ [x_1,e^2_1]=-\alpha^2_{1,1}e^2_1, & [e^2_i,x_1]=i\alpha^2_{1,1}e^2_i, \quad 1\le i\le n_2,\\ [x_1,e^3_1]=-\alpha^3_{1,1}e^3_1, & [e^3_i,x_1]=i\alpha^3_{1,1}e^3_i, \quad 1\le i\le n_3.\end{cases}$$

COROLLARY 3.8 *Let R be a solvable Leibniz algebra with nilradical $N=NF_1\oplus NF_2\oplus NF_3\oplus NF_4$ and* $\dim Q=1$. *Then it is isomorphic to one of the following algebras:*

$$R^1_{4,1}:\begin{cases}[e^j_i,e^j_1]=e^j_{i+1}, \quad 1\le j\le 4, & 1\le i\le n_j-1,\\ [x_1,e^1_1]=-e^1_1, \quad [e^1_i,x_1]=ie^1_i, & 1\le i\le n_1,\\ [e^2_1,x_1]=\sum\limits_{k=2}^{n_2}\alpha^2_{1,k}e^2_k+\delta^2_{1,3}e^3_{n_3}+\delta^2_{1,4}e^4_{n_4}, & 2\le i\le n_2-1,\\ [e^2_i,x_1]=\sum\limits_{k=i+1}^{n_2}\alpha^2_{1,k-i+1}e^2_k, & 2\le i\le n_2-1,\\ [e^3_1,x_1]=\sum\limits_{k=2}^{n_3}\alpha^3_{1,k}e^3_k+\delta^3_{1,2}e^2_{n_2}+\delta^3_{1,4}e^4_{n_4}, & 2\le i\le n_3-1,\\ [e^3_i,x_1]=\sum\limits_{k=i+1}^{n_3}\alpha^3_{1,k-i+1}e^3_k, & 2\le i\le n_3-1,\\ [e^4_1,x_1]=\sum\limits_{k=2}^{n_4}\alpha^4_{1,k}e^4_k+\delta^4_{1,2}e^2_{n_2}+\delta^4_{1,3}e^3_{n_3}, & 2\le i\le n_4-1,\\ [e^4_i,x_1]=\sum\limits_{k=i+1}^{n_4}\alpha^4_{1,k-i+1}e^4_k, & 2\le i\le n_4-1,\\ [x_1,e^2_1]=\beta^{2,3}_{1,n_3}e^3_{n_3}+\beta^{2,4}_{1,n_4}e^4_{n_4} & [x_1,e^3_1]=\beta^{3,2}_{1,n_2}e^2_{n_2}+\beta^{3,4}_{1,n_4}e^4_{n_4},\\ [x_1,e^4_1]=\beta^{4,2}_{1,n_2}e^2_{n_2}+\beta^{4,3}_{1,n_4}e^3_{n_3}, & [x_1,x_1]=\gamma^3_{n_3}e^3_{n_3}+\gamma^4_{n_4}e^4_{n_4};\end{cases}$$

$$R^1_{4,2}:\begin{cases}[e^j_i,e^j_1]=e^j_{i+1}, & 1\le j\le 4, \quad 1\le i\le n_j-1,\\ [x_1,e^1_1]=-e^1_1, & [e^1_i,x_1]=ie^1_i, \quad 1\le i\le n_1,\\ [x_1,e^2_1]=-\alpha^2_{1,1}e^2_1, & [e^2_i,x_1]=i\alpha^2_{1,1}e^2_i, \quad 1\le i\le n_2,\\ [e^3_1,x_1]=\sum\limits_{k=2}^{n_3}\alpha^3_{1,k}e^3_k+\delta^3_{1,4}e^4_{n_4}, & [e^3_i,x_1]=\sum\limits_{k=i+1}^{n_3}\alpha^3_{1,k-i+1}e^3_k, \quad 2\le i\le n_3-1,\\ [e^4_1,x_1]=\sum\limits_{k=2}^{n_4}\alpha^4_{1,k}e^4_k+\delta^4_{1,3}e^3_{n_3}, & [e^4_i,x_1]=\sum\limits_{k=i+1}^{n_4}\alpha^4_{1,k-i+1}e^4_k, \quad 2\le i\le n_4-1,\\ [x_1,e^3_1]=\beta^{3,4}_{1,n_4}e^4_{n_4}, & [x_1,e^4_1]=\beta^{4,3}_{1,n_4}e^3_{n_3},\\ [x_1,x_1]=\gamma^3_{n_3}e^3_{n_3}+\gamma^4_{n_4}e^4_{n_4};\end{cases}$$

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$$R^1_{4,3}:\begin{cases}[e^j_i,e^j_1]=e^j_{i+1}, & 1\le j\le 4, \quad 1\le i\le n_j-1,\\ [x_1,e^1_1]=-e^1_1, & [e^1_i,x_1]=ie^1_i, \quad 1\le i\le n_1,\\ [x_1,e^2_1]=-\alpha^2_{1,1}e^2_1, & [e^2_i,x_1]=i\alpha^2_{1,1}e^2_i, \quad 1\le i\le n_2,\\ [x_1,e^3_1]=-\alpha^3_{1,1}e^3_1, & [e^3_i,x_1]=i\alpha^3_{1,1}e^3_i, \quad 1\le i\le n_3,\\ [e^4_i,x_1]=\sum\limits_{k=i+1}^{n_4}\alpha^4_{1,k-i+1}e^4_k, & 1\le i\le n_4-1, \quad [x_1,x_1]=\gamma^4_{n_4}e^4_{n_4};\end{cases}$$

$$R^1_{4,4}:\begin{cases}[e^j_i,e^j_1]=e^j_{i+1}, & 1\le j\le 4, \quad 1\le i\le n_j-1,\\ [x_1,e^1_1]=-e^1_1, & [e^1_i,x_1]=ie^1_i, \quad 1\le i\le n_1,\\ [x_1,e^2_1]=-\alpha^2_{1,1}e^2_1, & [e^2_i,x_1]=i\alpha^2_{1,1}e^2_i, \quad 1\le i\le n_2,\\ [x_1,e^3_1]=-\alpha^3_{1,1}e^3_1, & [e^3_i,x_1]=i\alpha^3_{1,1}e^3_i, \quad 1\le i\le n_3,\\ [x_1,e^4_1]=-\alpha^4_{1,1}e^4_1, & [e^4_i,x_1]=i\alpha^4_{1,1}e^4_i, \quad 1\le i\le n_4.\end{cases}$$

The next corollary is proved by applying the same methods as we used above.

COROLLARY 3.9 *Let R be a solvable Leibniz algebra with nilradical $N=NF_1\oplus NF_2\oplus NF_3$ and* $\dim Q=2$. *Then it is isomorphic to one of the following algebras:*

$$R^2_{3,1}:\begin{cases}[e^j_i,e^j_1]=e^j_{i+1}, & 1\le j\le 3, \quad 1\le i\le n_j-1,\\ [x_1,e^1_1]=-e^1_1, & [e^1_i,x_1]=ie_i, \quad 1\le i\le n_1,\\ [x_2,e^2_1]=-e^2_1, & [e^2_i,x_2]=ie_i, \quad 1\le i\le n_2,\\ [e^3_1,x_1]=\sum\limits_{i=2}^{n_3}\alpha^3_{1,i}e^3_i, & [e^3_i,x_1]=\sum\limits_{j=i+1}^{n_3}\alpha^3_{1,j-i+1}e^3_j, \quad 2\le i\le n_3,\\ [e^3_1,x_2]=\sum\limits_{i=2}^{n_3}\alpha^3_{2,i}e^3_i, & [e^3_i,x_2]=\sum\limits_{j=i+1}^{n_3}\alpha^3_{2,j-i+1}e^3_j, \quad 2\le i\le n_3,\\ [x_1,x_1]=\gamma^3_{1,1,n_3}e^3_{n_3}, & [x_1,x_2]=\gamma^3_{1,2,n_3}e^3_{n_3},\\ [x_2,x_1]=\gamma^3_{2,1,n_3}e^3_{n_3}, & [x_2,x_2]=\gamma^3_{2,2,n_3}e^3_{n_3};\end{cases}$$

$$R^2_{3,2}:\begin{cases}[e^j_i,e^j_1]=e^j_{i+1}, & 1\le j\le 3, \quad 1\le i\le n_j-1,\\ [x_1,e^1_1]=-e^1_1, & [e^1_i,x_1]=ie^1_i, \quad 1\le i\le n_1,\\ [x_2,e^2_1]=-e^2_1, & [e^2_i,x_2]=ie^2_i, \quad 1\le i\le n_2,\\ [x_1,e^3_1]=-n_1e^3_1, & [e^3_i,x_1]=in_1e^3_i, \quad 1\le i\le n_3,\\ [x_2,e^3_1]=\beta^{3,1}_{2,n_1}e^1_{n_1};\end{cases}$$

$$R^2_{3,3}:\begin{cases}[e^j_i,e^j_1]=e^j_{i+1}, & 1\le j\le 3, \quad 1\le i\le n_j-1,\\ [x_1,e^1_1]=-e^1_1, & [e^1_i,x_1]=ie^1_i, \quad 1\le i\le n_1,\\ [x_2,e^2_1]=-e^2_1, & [e^2_i,x_2]=ie^2_i, \quad 1\le i\le n_2,\\ [x_1,e^3_1]=-\alpha^3_{1,1}e^3_1, & [e^3_i,x_1]=i\alpha^3_{1,1}e^3_i, \quad 1\le i\le n_3,\\ [x_2,e^3_1]=-\alpha^3_{2,1}e^3_1, & [e^3_i,x_2]=i\alpha^3_{2,1}e_i, \quad 1\le i\le n_3.\end{cases} \quad (\alpha^3_{1,1},\alpha^3_{2,1})\ne(0,0),$$

Corollary 3.9 and our further investigations show that in the case when $s=4$ and $q=2$, 10 families of algebras appear with many parameters and without any restrictions on them. It means that the situation for $q\ge 2$ is very complicated.

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