Lecture Notes in Mathematics

1763

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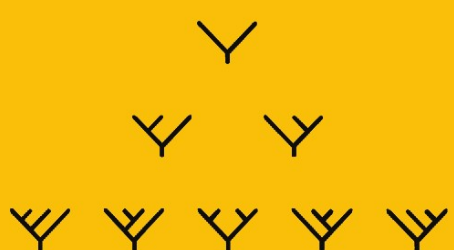
Dialgebras and Related Operads



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Cataloging-in-Publication Data applied for

Mathematics Subject Classification (2000):
05C5, 16A24, 16W30, 17Axx, 17D99, 18D50, 18G60, 55Uxx

ISSN 0075-8434
ISBN 3-540-42194-7 Springer-Verlag Berlin Heidelberg New York

Springer-Verlag Berlin Heidelberg New York
a member of BertelsmannSpringer Science+Business Media GmbH

http://www.springer.de

Typesetting: Camera-ready TEX output by the authors
SPIN: 10841327 41/3142/LK - 543210 - Printed on acid-free paper

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Introduction

The four papers of this volume deal with new notions of algebras whose common feature is to have *two* generating operations. So they are called *dialgebras*. The first motivation to introduce such algebraic structures was a problem in algebraic K-theory. It turned out later that some of them (the dendriform dialgebras) are closely related to Hopf algebras occuring in the theory of renormalization of A. Connes and D. Kreimer. They are also closely related to the notion of homotopy Gerstenhaber algebra.

Let us first describe the motivation from algebraic K-theory. The algebraic K-groups of a ring are not periodic like the topological K-groups, but computation of some of them shows the existence of a periodicity phenomenon. For instance the groups $K_n(\mathbf{Z}) \otimes \mathbf{Q}$ are periodic of period 4 for $n \geq 2$. The algebraic K-groups are constructed on the general linear group GL. If we replace it by its additive counterpart, that is the Lie algebra gl, then the analogue of algebraic K-theory is computable : it is cyclic homology, denoted HC. It turns out that for this theory the periodicity phenomenon is well understood. It takes the form of a long exact sequence :

$$\cdots \to HC_{n+1} \to HC_{n-1} \to HH_n \to HC_n \to HC_{n-2} \to \cdots$$

where HH stands for Hochschild homology. In other words, cyclic homology is not periodic (in general) but the obstruction to periodicity is known, it is Hochschild homology.

It is too naive to expect the existence of a similar sequence in the algebraic K-theory framework, however it is reasonable to conjecture that the so-called (b, B)-bicomplex, giving rise to the above exact sequence (cf. [L1]) has an analogue in the algebraic K-theory setting.

Recall that the (b, B)-bicomplex of the algebra A is of the following form

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \cdots \\
b\downarrow & & b\downarrow & & b\downarrow & & \\
C_2(A) & \xleftarrow{B} & C_1(A) & \xleftarrow{B} & C_0(A) & & \\
b\downarrow & & b\downarrow & & & & \\
C_1(A) & \xleftarrow{B} & C_0(A) & & & & \\
b\downarrow & & & & & & \\
C_0(A) & & & & & &
\end{array}$$

where each vertical complex is a copy of the Hochschild chain complex. The horizontal map is Connes boundary map B. The total homology of this bicomplex is precisely cyclic homology (cf. [L1]).

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In algebraic K-theory I conjecture the existence of a bicomplex

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
C_2^{(1)}(A) & \longleftarrow & C_1^{(2)}(A) & \longleftarrow & C_0^{(3)}(A) & & \\
\downarrow & & \downarrow & & & & \\
C_1^{(1)}(A) & \longleftarrow & C_0^{(2)}(A) & & & & \\
\downarrow & & & & & & \\
C_0^{(1)}(A) & & & & & &
\end{array}$$

made of (different) vertical chain complexes $C_*^{(i)}(A)$, whose total homology would be algebraic K-theory when A is a field.

Taking the homology of the vertical complexes gives rise to horizontal complexes, which are expected to be related to the motivic complexes of Bloch and Goncharov (cf. [G]). The first vertical complex $C_*^{(1)}$, which is our main concern, is expected to be related with the Milnor K-groups, and its homology is a multiplicative analogue of Hochschild homology.

The notion of associative dialgebra came out naturally while trying to construct $C_*^{(1)}$ and compute its homology $H_*^{(1)}$ through the following strategy.

Let us recall the relationship, in characteristic zero, between algebraic K-theory and cyclic homology on one hand and the homology of groups and Lie algebras of matrices on the other hand (see [L1] for details) :

$$H_*^{Lie}(gl(A)) \cong \Lambda^* HC_{*-1}(A), \quad H_*^{Leib}(gl(A)) \cong T^* HH_{*-1}(A).$$

$$H_*(GL(A)) \cong \Lambda^* K_*(A), \qquad (??)_*(GL(A)) \cong T^* H_*^{(1)}(A).$$

In this tableau Λ^* (resp. T^*) stands for the graded symmetric functor (resp. tensor functor), and H_*^{Lie} (resp. H_*^{Leib}) stands for Lie algebra homology (resp. Leibniz algebra homology). So the problem is to find the analogue $(??)_*$ of Leibniz homology for groups. Applied to the group $GL(A)$ it should give the homology $H_*^{(1)}(A)$ of the conjectural complex $C_*^{(1)}(A)$.

Let us recall that a Leibniz algebra is a vector space equipped with a binary operation, denoted $[-,-]$, satisfying the Leibniz relation :

$$[[x,y],z] = [[x,z],y] + [x,[y,z]].$$

So, if the bracket is skew-symmetric, then this is a Lie algebra.

Groups, associative algebras and Lie algebras are related by well-known functors :

$$Gp \underset{(-)^\times}{\overset{\mathbf{Z}[-]}{\rightleftarrows}} As \underset{U}{\overset{-}{\rightleftarrows}} Lie$$

Observe also that the homology of a group can be defined as the Hochschild homology of its group algebra.

Our task is now to find analogous objects in the noncommutative framework, that is when Lie algebras are replaced by Leibniz algebras. The first step is to find the analogue of associative algebras in this new setting. This is where the associative dialgebras (whose category is denoted *Dias*) crop up :

$$? \rightleftarrows Dias \underset{Ud}{\overset{-}{\rightleftarrows}} Leibniz$$

An associative algebra gives rise to a Lie algebra by putting $[x,y] = xy - yx$ (functor denoted $-$ above). The idea, to obtain a Leibniz algebra which is not necessarily Lie, is to start with an algebra with *two* operations, denoted $\dashv$ and $\vdash$, and to put

$$[x,y] := x \dashv y - y \vdash x.$$

In order for this bracket to satisfy the Leibniz relation, we need to impose some relations on $\dashv$ and $\vdash$. There are several possible choices. The choice which is as close as possible to the associative relation is the following :

$$\begin{cases} x \dashv (y \dashv z) = (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (x \dashv y) \vdash z = x \vdash (y \vdash z) = (x \vdash y) \vdash z. \end{cases}$$

This gives the definition of an *associative dialgebra.*

The aim of this volume is to achieve the first step of this program, that is to study the associative dialgebras from the algebraic and the homological point of view.

In particular we need to know up to which extent the properties of Hochschild homology of associative algebras can be generalized to associative dialgebras, including the relationship with Lie algebras (to be replaced by Leibniz algebras, of course).

Here is a brief overview of the content of the four papers.

Dialgebras, by Jean-Louis Loday.

After introducing the notion of associative dimonoid and associative dialgebra, we construct a chain complex which gives rise to homology and cohomology of a dialgebra. This chain complex is modelled on a new type of algebras, whose associated operad is Koszul dual to the operad of associative dialgebras.

They are called *dendriform* algebras (or dendriform dialgebras since they are also generated by two operations) because the free object admits the set of planar binary trees as a basis.

The main technical result is the vanishing of the homology of the free associative dialgebra, which implies the Koszulness of both operads by the Koszul duality theory for operads as devised by Ginzburg and Kapranov in [G-K]. Similarly the (co)homology theory for dendriform algebras is explicitly constructed.

The comparison with Leibniz algebra homology and Zinbiel algebra (the Koszul dual notion) homology is performed. As a by-product we describe the notion of strong homotopy associative dialgebra (analogue of A_∞-algebra).

Dialgebra (co)homology with coefficients, by Alessandra Frabetti.

The first paper deals only with dialgebra (co)homology with trivial coefficients. However both for theoretical and pratical purposes it is necessary to know about dialgebra (co)homology with coefficients. This is the object of study of this second paper. First, the notion of coefficients is made explicit in this context. Second, the theory is constructed. Third, a few important computations are performed. All these results rely on peculiar and intricate combinatorial properties of the set of planar binary trees and operations on them.

Un endofoncteur de la catégorie des opérades, by Frédéric Chapoton.

Among the associative algebras the commutative ones form an important subcategory. Similarly among associative dialgebras, those which satisfy the commutativity property

$$x \dashv y = y \vdash x$$

form a subcategory whose operad, denoted *Perm*, has surprising properties. Indeed, tensoring with *Perm* gives a general device for dichotomization, for instance

$$As \otimes Perm = Dias, \qquad Lie \otimes Perm = Leibniz.$$

The object of this paper is the study of the functor $- \otimes Perm$.

Un théorème de Milnor-Moore pour les algèbres de Leibniz, by François Goichot.

One of the key ingredients in the theorems relating algebraic K-theory and cyclic homology to the homology of matrices (see above) is the Milnor-Moore theorem. This result relates cocommutative Hopf algebras and Lie algebras. In our setting an immediate question arises : what happens when Lie is replaced by Leibniz ? And immediately after another one : what replaces the notion of Hopf algebras when associative algebra is replaced by associative dialgebra ?

This paper gives a solution to these questions and proves an analogue of the Poincaré-Birkhoff-Witt theorem and of the Milnor-Moore theorem in the associative dialgebra framework.

Conclusion. One of the surprises of this work is the relevance of the dendriform algebras to several other topics. A priori it was just an intermediate tool in the study of homology of associative dialgebras. But it turned out that dendriform algebras, and more generally the algebraic and combinatorial operations on trees attached to it, are closely related to

- renormalization theory, cf. [Br], [Br-Fr], [Ch-Li], [C-K],
- strong homotopy Gerstenhaber algebras, cf. [R],
- arithmetic, cf. [L2],
- polytopes, cf. [Ch], [L-R].

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Jean-Louis Loday　　22 March 2001

Dialgebras

Jean-Louis Loday

Contents

There is a notion of "non-commutative Lie algebra" called *Leibniz algebra*, which is characterized by the following property. The bracketing $[-,z]$ is a derivation for the bracket operation, that is, it satisfies the Leibniz identity:

$$[[x,y],z] = [[x,z],y] + [x,[y,z]].$$

cf. [L1]. When it happens that the bracket is skew-symmetric, we get a Lie algebra since the Leibniz identity becomes equivalent to the Jacobi identity.

Any associative algebra gives rise to a Lie algebra by $[x,y] = xy - yx$. The purpose of this article is to introduce and study a new notion of algebra which gives, by a similar procedure, a Leibniz algebra. The idea is to start with two distinct operations for the product xy and the product yx, so that the bracket is not necessarily skew-symmetric any more. Explicitly, we define an *associative dialgebra* (or simply dialgebra for short) as a vector space D equipped with two associative operations $\dashv$ and $\vdash$, called respectively left and right product, satisfying 3 more axioms:

$$\begin{cases} x \dashv (y \dashv z) = x \dashv (y \vdash z), \\ (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (x \dashv y) \vdash z = (x \vdash y) \vdash z. \end{cases}$$

It is immediate to check that $[x,y] := x \dashv y - y \vdash x$ defines a Leibniz bracket. Hence any associative dialgebra gives rise to a Leibniz algebra.

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A typical example of dialgebra is constructed as follows. Let (A, d) be a differential associative algebra, and put

$$x \dashv y := x\ dy \quad \text{and} \quad x \vdash y := dx\ y.$$

One easily checks that $(A, \dashv, \vdash)$ is a dialgebra. For instance there is a natural dialgebra structure on the de Rham complex of a manifold.

Observe that, since the relations defining a dialgebra do not involve sums, there is a well-defined notion of *dimonoid.*

In this article we construct and study a (co)homology theory for dialgebras. Since an associative algebra is a particular case of dialgebra, we get a new (co)homology theory for associative algebras as well. The surprizing fact, in the construction of the chain complex, is the appearance of the combinatorics of *planar binary trees.* The principal result about this homology theory HY is its vanishing on free dialgebras. In order to state some of the properties of the theory HY, we introduce another type of algebras with two operations: the *dendriform algebras* (sometimes called dendriform dialgebras). This notion dichotomizes the notion of associative algebra in the following sense: there are two operations $\prec$ and $\succ$, such that the product $*$ made of the sum of them

$$x * y := x \prec y + x \succ y,$$

is associative. The axioms relating these two products are

$$\begin{aligned}
&\text{(i)} && (a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c), \\
&\text{(ii)} && (a \succ b) \prec c = a \succ (b \prec c), \\
&\text{(iii)} && (a \prec b) \succ c + (a \succ b) \succ c = a \succ (b \succ c).
\end{aligned}$$

The free dendriform algebra can be constructed by means of the planar binary trees, whence the terminology.

The results intertwining associative dialgebras and dendriform algebras are best expressed in the framework of *algebraic operads.* The notion of associative dialgebra defines an algebraic operad *Dias*, which is binary and quadratic. By the theory of Ginzburg and Kapranov (cf. [GK]), there is a well-defined "dual operad" $Dias^!$. We show that this is precisely the operad *Dend* of the "dendriform algebras", in other words a dual associative dialgebra is nothing but a dendriform algebra. The vanishing of HY of a free dialgebra implies that those two operads are of a special kind: they are "Koszul operads". As a consequence the cohomology of a dialgebra is a graded dendriform algebra and, a fortiori, a graded associative algebra. The explicit description of the free dendriform algebra in terms of trees permits us to describe the notion of *strong homotopy associative dialgebra.*

The categories of algebras over these operads assemble into a commutative diagram of functors which reflects the Koszul duality.

$$\begin{array}{ccccccccc} & & \mathbf{Dend} & & & & \mathbf{Dias} & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & \\ \mathbf{Zinb} & & & & \mathbf{As} & & & & \mathbf{Leib} \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & \mathbf{Com} & & & & \mathbf{Lie} & & \end{array}$$

In this diagram **Zinb** denotes the categories of Zinbiel algebras, which are Koszul dual to the Leibniz algebras.

This paper is part of a long-standing project whose ultimate aim is to study periodicity phenomenons in algebraic K-theory. This project is described in [L4] (see also the introduction of this volume). The next step would consist in computing the dialgebra homology of the augmentation ideal of $K[GL(A)]$, for an associative algebra A.

In the first section of this article we introduce the notion of associative dimonoid, or dimonoid for short, and develop the calculus in a dimonoid. In particular we describe the free dimonoid on a given set. In the second section we introduce the notion of dialgebra and give several examples. We explicitly describe the free dialgebra over a vector space. In the third section we construct the chain complex of a dialgebra D, which gives rise to homology and cohomology groups denoted $HY(D)$. The main tool is made of the planar binary trees and operations on them. We prove that HY of a free dialgebra vanishes (hence the operad associated to dialgebras is a Koszul operad). We also introduce a variation of the chain complex by replacing the trees by increasing trees, or, equivalently, by permutations. This variation appears naturally in the computation of the Leibniz homology of dialgebras of matrices (cf. [F1]). (Co)homology of dialgebras with non-trivial coefficients is treated by Alessandra Frabetti in [F4].

Section 4 is devoted to the relationship between Leibniz algebras and dialgebras. The functor which assigns to any dialgebra $(D, \vdash, \dashv)$ the Leibniz algebra $(D, [x,y] := x \dashv y - y \vdash x)$ has a left adjoint which is the *universal enveloping dialgebra* of a Leibniz algebra. Then we compare the diverse types of free algebras and we propose a definition for a *Poisson dialgebra.* The Hopf-type properties of the universal enveloping dialgebra are studied by François Goichot in [Go].

In the fifth section we introduce the notion of *dendriform algebra*, which is closely connected to the notion of associative dialgebra. For instance the tensor product of a dialgebra and of a dendriform algebra is naturally equipped with a structure of Lie algebra. The main result of this section is to make explicit the free dendriform algebra. It turns out that it is best expressed in terms of planar binary trees. The dendriform algebra structure on the vector space generated by the planar binary trees is the core of this section. It uses the grafting operation and the nesting operation on trees and it induces a graded associative algebra structure on the same vector space. In a sense associative algebras are closely connected with the integers (including addition and multiplication). Similarly dendriform algebras are closely connected with planar binary trees and a calculus on them. This arithmetic aspect of the theory will be treated elsewhere.

In section 6 we construct (co)homology groups for dendriform algebras. They vanish on free dendriform algebras.

In section 7 we relate dendriform algebras with Zinbiel algebras (i.e. dual-Leibniz algebras) and associative algebras. It is based on the relationship between binary trees and permutations as described in Appendix A.

The aim of the eighth section is to interpret the preceding results in the context of algebraic operads. The basics on algebraic operads and Koszul duality are recalled in Appendix B. We show that the operads associated to dialgebras and to dendriform algebras are dual in the operad sense. Then we show that the (co)homology groups HY for dialgebras (resp. H^{Dend} for dendriform algebras) constructed in section 3 (resp. 4) are the ones predicted by the operad theory. Hence, by the vanishing of HY of a free dialgebra, both operads $Dias$ and $Dend$ are Koszul. It implies, among several consequences, the vanishing of the homology of a free dendriform algebra. Some of the theorems in sections 2 to 6 can be proved either directly or by appealing to the operad theory. In general we write down the most elementary one.

The last section describes the notion of *strong homotopy associative dialgebras*. For any Koszul operad the notion of algebra up to homotopy is theoretically well-defined from the bar construction over the dual operad. Since, in our case, we know explicitly the structure of a free dendriform algebra, we can make the notion of dialgebra up to homotopy completely explicit.

Part of the results of this article has been announced in a "Note aux Comptes Rendus" [L2]. I thank Ale Frabetti, Benoit Fresse, Victor Gnedbaye, François Goichot, Phil Hanlon, Muriel Livernet, Teimuraz Pirashvili and Maria Ronco for fruitful conversations on this subject.

1. DIMONOIDS

1.1. Definition. An *associative dimonoid*, or dimonoid for short, is a set X equipped with two maps called respectively *left product* and *right product*:

$$\begin{array}{ll} \text{(left)} & \dashv : X \times X \to X, \\ \text{(right)} & \vdash : X \times X \to X, \end{array}$$

satisfying the following axioms

$$\begin{cases} x \dashv (y \dashv z) \overset{1}{=} (x \dashv y) \dashv z \overset{2}{=} x \dashv (y \vdash z), \\ (x \vdash y) \dashv z \overset{3}{=} x \vdash (y \dashv z), \\ (x \dashv y) \vdash z \overset{4}{=} x \vdash (y \vdash z) \overset{5}{=} (x \vdash y) \vdash z, \end{cases}$$

for all x, y and $z \in X$.

In the notation $x \dashv y$, $y \vdash x$, the element x is said to be on the *pointer* side and the element y is said to be on the *bar* side.

The numbers 1 to 5 of the relations are for future reference.

Observe that relations 1 and 5 are the "associativity" of the products $\dashv$ and $\vdash$ respectively. Relation 3 will be referred to as "inside associativity", since the products point inside. Relations 2 and 4 can be replaced by the relations 12 and 45:

$$x \dashv (y \dashv z) \overset{12}{=} x \dashv (y \vdash z) \quad \text{and} \quad (x \dashv y) \vdash z \overset{45}{=} (x \vdash y) \vdash z,$$

which can be summarized as "on the bar side, does not matter which product". All these relations are referred to as "diassociativity".

A *morphism* of dimonoids is a map $f : X \to Y$ between two dimonoids X and Y such that $f(x \dashv x') = f(x) \dashv f(x')$ and $f(x \vdash x') = f(x) \vdash f(x')$ for any $x, x' \in X$.

Observe that one can define a di-object in any monoidal category. One does not need the monoidal category to be symmetric since in each relation the variables stay in the same order.

1.2. Bar-unit. An element $e \in X$ is said to be a *bar-unit* of the dimonoid X if

$$x \dashv e = x = e \vdash x, \quad \text{for any } x \in X.$$

So it is only assumed that e acts trivially from the bar side. There is no reason for a bar-unit to be unique. The set of bar-units is called the *halo*.

A morphism of dimonoids is said to be *unital* if the image of a bar-unit is a bar-unit.

1.3. Examples.

a) Let M be a monoid (without unit), that is a set M with an associative product $(m, m') \mapsto mm'$. Putting $m \dashv m' = mm' = m \vdash m'$ gives a dimonoid structure on M. Indeed each relation 1 to 5 is the associativity property. A unit of the monoid is a bar-unit of the associated dimonoid.

Conversely, if in a dimonoid D there is a unit, that is an element $1 \in D$ which satisfies either $1 \dashv x = x$ or $x = x \vdash 1$ for all $x \in D$, then, by axiom 3 or 5, one has $\dashv = \vdash$ and D is simply the dimonoid associated to a unital monoid.

b) Let X be a set and define

$$x \dashv y = x = y \vdash x, \quad \text{for any} \quad x, y \in X.$$

Then, obviously, X is a (not so interesting) dimonoid and it coincides with its halo.

c) Let M be a monoid. Put $D = M \times M$ and define the products by

$$\begin{cases} (m,n) \dashv (m',n') := (m, nm'n') \\ (m,n) \vdash (m',n') := (mnm', n'). \end{cases}$$

With these definitions $D = (D, \dashv, \vdash)$ is a dimonoid. Let us check relation 3 for instance:

$$\begin{gathered} ((m,n) \vdash (m',n')) \dashv (m'',n'') = (mnm',n') \dashv (m'',n'') = (mnm', n'm''n'') \\ (m,n) \vdash ((m',n') \dashv (m'',n'')) = (m,n) \vdash (m', n'm''n'') = (mnm', n'm''n''). \end{gathered}$$

Let $1 \in M$ be a unit for M. Then $e = (1,1)$ is a bar-unit for D, but one has $e \dashv x \neq x$ and $x \vdash e \neq x$ in D in general. For any invertible element m the element $(m, m^{-1}) \in D$ is a bar-unit.

d) Let G be a group and X a G-set. The following formulas define a dimonoid structure on $X \times G$ (cf. 7.5):

$$\begin{cases} (x,g) \dashv (y,h) := (x, gh), \\ (x,g) \vdash (y,h) := (g \cdot y, gh). \end{cases}$$

1.4. Opposite dimonoid. Let D be a dimonoid. Define new operations $\dashv'$ and $\vdash'$ on D by

$$\begin{aligned} x \dashv' y &:= y \vdash x, \\ x \vdash' y &:= y \dashv x. \end{aligned}$$

It is immediate to check that $(D, \dashv', \vdash')$ is a new dimonoid which we call the *opposite* dimonoid that we denote by D^{op}.

Observe that if we put

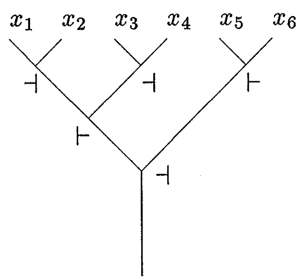
$$\begin{cases} x \dashv'' y := y \dashv x, \\ x \vdash'' y := y \vdash x, \end{cases}$$

then $(D, \vdash'', \dashv'')$ is *not* a dimonoid.

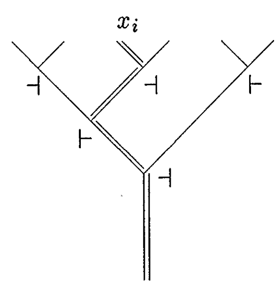
1.5. Monomials in a dimonoid. Let $x_1, \dots, x_n$ be elements in the dimonoid D. A *monomial* in D is a parenthesizing together with product signs, for instance

$$((x_1 \dashv x_2) \vdash (x_3 \dashv x_4)) \dashv (x_5 \vdash x_6),$$

giving rise to an element in D. Such a monomial is completely determined by a binary tree, where each vertex is labelled by $\dashv$ or $\vdash$:



1.6. The middle of a monomial. Given a monomial as above we define the *middle* of the monomial as being the entry x_i determined by the following algorithm. Starting at the root of the tree one goes up by choosing the route indicated by the pointer. The middle of the monomial is the abutment of the path. In this example x_3 is the middle.



1.7. Theorem (Dimonoid calculus). *Let x_i, $i \in \mathbf{Z}$, be elements in a dimonoid D.*

a) Any parenthesizing of

$$x_{-n} \vdash x_{-n+1} \vdash \dots \vdash x_{-1} \vdash x_0 \dashv x_1 \dashv \dots \dashv x_{m-1} \dashv x_m$$

gives the same element in D, which we denote by

$$x_{-n} \dots x_{-1} \check{x}_0 x_1 \dots x_m.$$

b) Let $m = x_1 \dots x_k$ be a monomial in D. Let x_i be its middle entry. Then $m = x_1 \dots \check{x}_i \dots x_k$.

c) One has the following formulas in D:

$$(x_1 \dots \check{x}_i \dots x_k) \dashv (x_{k+1} \dots \check{x}_j \dots x_\ell) = x_1 \dots \check{x}_i \dots x_k x_{k+1} \dots x_j \dots x_\ell$$
$$(x_1 \dots \check{x}_i \dots x_k) \vdash (x_{k+1} \dots \check{x}_j \dots x_\ell) = x_1 \dots x_i \dots x_k x_{k+1} \dots \check{x}_j \dots x_\ell.$$

For instance, in the above example, one has

$$((x_1 \dashv x_2) \vdash (x_3 \dashv x_4)) \dashv (x_5 \vdash x_6) = x_1 x_2 \check{x}_3 x_4 x_5 x_6.$$

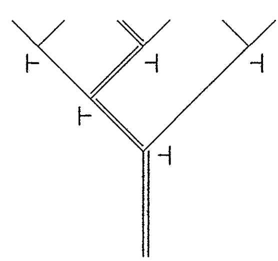
Proof. By axiom 1 (associativity of $\dashv$) any parenthesizing of $x_1 \dashv \dots \dashv x_m$ gives the same element. So, in such a monomial we can ignore the parentheses (and analogously for $\vdash$ thanks to axiom 5).

Consider a generic monomial with first entry x_{-n}, last entry x_m and middle entry x_0 (where $-n \leq 0 \leq m$). By axioms $1-3-5$ it is clear that the element

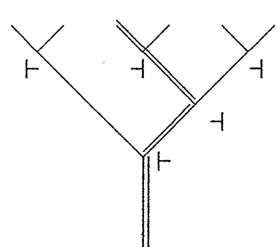
$$(*) \qquad (x_{-n} \vdash \dots \vdash x_{-1}) \vdash x_0 \dashv (x_1 \dashv \dots \dashv x_m)$$

is well-defined. We denote it by $x_{-n} \dots \check{x}_0 \dots x_m$.

Consider the labelled tree of our generic monomial. Let v be a vertex which is on the route from the root to the middle entry x_0. Thanks to axioms 12 and 45 all the vertices on the bar side of v can be labelled with the same label as v. In our example



Then by axiom 3 we can modify the tree so that all labels $\vdash$ come first:



This new tree corresponds to a monomial of the form $(*)$ and therefore we have proved that our starting monomial has value $x_{-n} \dots \check{x}_0 \dots x_m$. So parts a) and b) are proved.

By a) and b) it follows that in order to compute

$$(x_1 \dots \check{x}_i \dots x_k) \dashv (x_{k+1} \dots \check{x}_j \dots x_\ell) \quad \text{and} \quad (x_1 \dots \check{x}_i \dots x_k) \vdash (x_{k+1} \dots \check{x}_j \dots x_\ell)$$

it suffices to determine which entry is the middle of these monomials. By the algorithm described in 1.6, the middle entry is x_i in the first case and x_j in the second case. □

1.8. Corollary. *The free dimonoid on the set X is the disjoint union*

$$\mathcal{D}(X) = \coprod_{n\geq 1} (\underbrace{X^n \cup \dots \cup X^n}_{n \text{ copies}}).$$

Denoting by $x_1 \dots \check{x}_i \dots x_n$ an element in the i-th summand, the products are given by

$$(x_1 \dots \check{x}_i \dots x_k) \dashv (x_{k+1} \dots \check{x}_j \dots x_\ell) = x_1 \dots \check{x}_i \dots x_\ell \,,$$
$$(x_1 \dots \check{x}_i \dots x_k) \vdash (x_{k+1} \dots \check{x}_j \dots x_\ell) = x_1 \dots \check{x}_j \dots x_\ell \,.$$

□

2. ASSOCIATIVE DIALGEBRAS

In the sequel K denotes a field referred to as the *ground field.* Later on it will be supposed to be of characteristic zero. The tensor product over K is denoted by $\otimes_K$ or, more often, by $\otimes$.

After introducing the notion of dialgebra, we give some examples, including free dialgebras, which we describe explicitly, and define modules and representations over a dialgebra.

2.1. Definition. An *associative dialgebra*, or *dialgebra* for short, over K is a K-module D equipped with two K-linear maps

$$\dashv : D \otimes D \longrightarrow D,$$
$$\vdash : D \otimes D \longrightarrow D,$$

satisfying the *di-associativity* axioms

$$\begin{cases} (1) & (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ (2) & (x \dashv y) \dashv z = x \dashv (y \dashv z), \\ (3) & (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (4) & (x \dashv y) \vdash z = x \vdash (y \vdash z), \\ (5) & (x \vdash y) \vdash z = x \vdash (y \vdash z). \end{cases}$$

The maps $\dashv$ and $\vdash$ are called respectively the *left product* and the *right product.*

Here is an equivalent formulation of these axioms: the products $\dashv$ and $\vdash$ are associative and satisfy:

$$\begin{cases} (12) & x \dashv (y \dashv z) = x \dashv (y \vdash z), \\ (3) & (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (45) & (x \dashv y) \vdash z = (x \vdash y) \vdash z. \end{cases}$$

Observe that the analogue of formula (3), but with the product symbols pointing outward, is not valid in general: $(x \dashv y) \vdash z \neq x \dashv (y \vdash z)$.

A *morphism* of dialgebras from D to D' is a K-linear map $f : D \to D'$ such that

$$f(x \dashv y) = f(x) \dashv f(y) \quad \text{and} \quad f(x \vdash y) = f(x) \vdash f(y) \quad \text{for all} \quad x, y \in D.$$

We denote by **Dias** the category of dialgebras.

A *bar-unit* in D is an element $e \in D$ such that

$$x \dashv e = x = e \vdash x \quad \text{for all} \quad x \in D.$$

A bar-unit need not be unique. The subset of bar-units of D is called its *halo*.

A *unital dialgebra* is a dialgebra with a specified bar-unit e. This choice gives rise to a preferred K-linear map $K \hookrightarrow D, \lambda \mapsto \lambda e$.

A morphism of dialgebras is said to be *unital* if the image of any bar-unit is a bar-unit.

Observe that if a dialgebra has a unit ϵ, that is an element which satisfies $\epsilon \dashv x = x$ for any x, then $\dashv = \vdash$ by axiom 12, and D is an associative algebra with unit.

An *ideal* I in a dialgebra D is a submodule of D such that $x \dashv y$ and $x \vdash y$ are in I whenever one of the variables is in I. Clearly the quotient D/I is a dialgebra. Conversely, the kernel of a dialgebra morphism is an ideal.

2.2. Examples.

a) Associative algebra. If A is an associative algebra over K, then the formulas $a \dashv b = ab = a \vdash b$ define a structure of dialgebra on A. If 1 is a unit of the associative algebra, then $e = 1$ is a unit of the dialgebra and the halo is just $\{1\}$.

b) Differential associative algebra. Let (A, d) be a differential associative algebra. So, by hypothesis, $d(ab) = da\, b + a\, db$ (here we work in the non-graded setting) and $d^2 = 0$. Define left and right products on A by the formulas

$$x \dashv y := x\ dy \quad \text{and} \quad x \vdash y := dx\ y.$$

It is immediate to check that A equipped with these two products is a dialgebra. A similar construction holds in the graded (or more accurately super) algebra framework.

c) Dimonoid algebra. Let X be a dimonoid, and denote by $K[X]$ the free K-module on X. Then obviously $K[X]$ is a dialgebra.

d) Bimodule map. Let A be an associative algebra and let M be an A-bimodule. Let $f : M \to A$ be an A-bimodule map. Then one can put a dialgebra structure on M as follows:

$$m \dashv m' := mf(m'),$$
$$m \vdash m' := f(m)m',$$

The verification is left to the reader. One can systematize this procedure by considering the tensor category of linear maps as follows (cf. [LP2], [Ku] for details). The category of linear maps over K is made of the K-linear maps $f : V \to W$ as objects. It can be equipped with a tensor product by

$$(V \xrightarrow{f} W) \otimes (V' \xrightarrow{f'} W') = V \otimes W' \ \oplus \ W \otimes V' \xrightarrow{f\otimes 1 + 1\otimes f'} W \otimes W'.$$

An associative algebra in this tensor category defines a dialgebra structure on the source object.

The particular case of the projection $M \oplus A \to A$ shows that there is a dialgebra structure on $M \oplus A$ (cf. P. Higgins [Hi]).

e) Tensor product, matrices. If D and D' are two dialgebras, then the tensor product $D \otimes D'$ is also a dialgebra by $(a\otimes a')\star(b\otimes b') = (a\star b)\otimes(a'\star b')$ for $\star = \dashv$ and $\vdash$. For instance the module of $n \times n-$matrices $\mathcal{M}_n(D) = \mathcal{M}_n(K)\otimes D$ is a dialgebra. The left and right products are given by

$$(\alpha \dashv \beta)_{ij} = \sum_k \alpha_{ik} \dashv \beta_{kj} \quad \text{and} \quad (\alpha \vdash \beta)_{ij} = \sum_k \alpha_{ik} \vdash \beta_{kj}.$$

f) Opposite dialgebra. As for dimonoids, the opposite dialgebra of D is the dialgebra D^{op} with the same underlying K-module and with products given by

$$x \dashv' y = y \vdash x, \ \ x \vdash' y = y \dashv x.$$

g) Let A be an associative algebra over K. Put $D = A \otimes A$ and define

$$a \otimes b \dashv a' \otimes b' := a \otimes ba'b',$$
$$a \otimes b \vdash a' \otimes b' := aba' \otimes b'.$$

Extending these formulas by linearity on $A \otimes A$ gives well-defined product maps $\dashv$ and $\vdash$ on D which satisfy the diassociativity axioms. If $1 \in A$ is a unit of the associative algebra, then $1 \otimes 1$ is a bar-unit for the dialgebra. More generally, for any invertible element x in A, the element $x \otimes x^{-1}$ is a bar-unit. If I is a left ideal and J is a right ideal, then the same formulas define a diassociative algebra structure on $I \otimes_K J$.

h) Let A be an associative algebra and n be a positive integer. On the module of n-vectors $D = A^n$ one puts:

$$(x \dashv y)_i = x_i \Big(\sum_{j=1}^{n} y_j\Big) \quad \text{for} \quad 1 \leq i \leq n \quad \text{and}$$

$$(x \vdash y)_i = \Big(\sum_{j=1}^{n} x_j\Big) y_i \quad \text{for} \quad 1 \leq i \leq n.$$

One easily checks that D is a dialgebra. For $n = 1$, this is example (a). In fact this construction can be extended to any dialgebra A.

2.3. Module, bimodule, extension. A *left module over a dialgebra* D is a K-module M equipped with two linear maps

(right structure) $\qquad \dashv : D \otimes M \to M,$
(left structure) $\qquad \vdash : D \otimes M \to M,$

satisfying the axioms (1)-(5) whenever they make sense. There is, of course, a similar definition for right modules.

A *bimodule* over a dialgebra D, also called a *representation*, is a K-module M equipped with four linear maps

(right structures) $\qquad \dashv, \vdash : M \otimes D \to M,$
(left structures) $\qquad \dashv, \vdash : D \otimes M \to M,$

satisfying the axioms (1) to (5), whenever one of the entries x, y or z is in M and the two others are in D.

Obviously a bimodule over D is, a fortiori, a left module and also a right module over D ; and D is a bimodule over itself.

Let

$$0 \to M \to \overline{D} \to D \to 0$$

be an abelian extension of dialgebras, that is an exact sequence of dialgebras such that any product of two elements in M is trivial. Then, it is immediate to check that M is a representation of D in the above sense.

2.4. Free associative dialgebra. Let V be a K-module. By definition the *free dialgebra* on V is the dialgebra $Dias(V)$ equipped with a K-linear map $i : V \to Dias(V)$ such that for any K-module map $f : V \to D$, where D is a dialgebra over K, there is a unique factorization

$$f : V \xrightarrow{i} Dias(V) \xrightarrow{\phi} D,$$

where ϕ is a dialgebra morphism.

Equivalently the functor $Dias : (K\text{-}\mathrm{Mod}) \to \mathbf{Dias}$ is left adjoint to the forgetful functor. The following proposition proves the existence of the free dialgebra $Dias(V)$ and gives an explicit description of it in terms of the tensor module
$$T(V) := K \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots .$$

2.5. Theorem. *The free dialgebra on V is the K-module*

$$Dias(V) = T(V) \otimes V \otimes T(V)$$

equipped with the two products induced by:

$$\begin{aligned}
&(v_{-n} \cdots v_{-1} \otimes v_0 \otimes v_1 \cdots v_m) \dashv (w_{-p} \cdots w_{-1} \otimes w_0 \otimes w_1 \cdots w_q)\\
&\qquad\qquad\qquad = v_{-n} \cdots v_{-1} \otimes v_0 \otimes v_1 \cdots v_m w_{-p} \cdots w_q,\\
&(v_{-n} \cdots v_{-1} \otimes v_0 \otimes v_1 \cdots v_m) \vdash (w_{-p} \cdots w_{-1} \otimes w_0 \otimes w_1 \cdots w_q)\\
&\qquad\qquad\qquad = v_{-n} \cdots v_m w_{-p} \cdots w_{-1} \otimes w_0 \otimes w_1 \cdots w_q,
\end{aligned}$$

where v_i, $w_j \in V$.

With our notation (cf. 1.7) any additive generator of $Dias(V)$ can be written
$$v_{-n} \cdots v_{-1} \otimes v_0 \otimes v_1 \cdots v_m = v_{-n} \cdots v_{-1} \check{v}_0 v_1 \cdots v_m .$$

Proof. It is immediate to check that $Dias(V) = (T(V) \otimes V \otimes T(V), \dashv, \vdash)$ is a dialgebra (cf. 1.7). The map $i : V \to Dias(V)$ is the composite $V \simeq 1 \cdot K \otimes V \otimes 1 \cdot K \hookrightarrow T(V) \otimes V \otimes T(V)$. Starting with $f : V \to D$ the map $\phi : Dias(V) \to D$ is given by

$$\phi(v_{-n} \cdots v_{-1} \check{v}_0 v_1 \cdots v_m) = f(v_{-n}) \cdots f(v_{-1}) \check{f(v_0)} f(v_1) \cdots f(v_m).$$

It is obviously a dialgebra morphism. Moreover, by theorem 1.7, it is uniquely determined since it should coincide with f on $V \cong 1 \cdot K \otimes V \otimes 1 \cdot K$ and it should be a morphism of dialgebras. Hence the inclusion $V \to Dias(V)$ is universal.
□

Remark. A free dialgebra is a particular case of example 2.2.d, with $M = T(V) \otimes V \otimes T(V)$, $A = T(V)$ (the associative tensor algebra) and $f : M \to A$ being the concatenation.

Let V be finite dimensional over K generated by $x_1, \cdots, x_n$. Let us describe the degree n part of $T(V) \otimes V \otimes T(V)$ which is generated by all the monomials containing x_i once and only once, $1 \leq i \leq n$. We denote it by $Dias(n)$. These monomials are the elements

$$(\sigma, i)(x_1, \cdots, x_n) := x_{\sigma^{-1}(1)} \cdots \check{x}_{\sigma^{-1}(i)} \cdots x_{\sigma^{-1}(n)}, \quad \sigma \in S_n,\ 1 \leq i \leq n,$$

where S_n is the symmetric group. Therefore, as a left S_n-module, the multilinear part of this space is isomorphic to n copies of the regular representation of S_n:

$$Dias(n) \cong nK[S_n].$$

The element σ in the i-th copy corresponds to the operation (σ, i) described above (cf. Corollary 1.8).

Examples:

- $n = 1$, one generator: $\check{x}_1$.
- $n = 2$, four generators: $\check{x}_1 x_2, \check{x}_2 x_1, x_1 \check{x}_2, x_2 \check{x}_1$.
- $n = 3$, eighteen generators: $\check{x}_i x_j x_k, x_i \check{x}_j x_k, x_i x_j \check{x}_k$

for all permutations i, j, k of 1,2,3.

2.6. Associative algebra associated to a dialgebra. For any dialgebra D let D_{As} be the quotient of D by the ideal generated by the elements $x \dashv y - x \vdash y$, for all $x, y \in D$. It is clear that $\dashv = \vdash$ in D_{As}, hence D_{As} is an associative algebra (non-unital in general). The quotient map $\mu : D \twoheadrightarrow D_{As}$ is universal among the maps from D to associative algebras. In other words the associativization functor $(-)_{As}$: **Dias** $\to$ **As** is left adjoint to inc : **As** $\to$ **Dias**.

Axioms 12 and 45 imply that the element $x \vdash y \dashv z$ in D depends only on the values of x and z in D_{As}. Hence D is a D_{As}-bimodule and the projection map μ is a D_{As}-bimodule map. On the other hand the dialgebra structure of D is completely determined by μ and the D_{As}-bimodule structure on the space D since

$$x \dashv y = x\, \mu(y) \quad \text{and} \quad x \vdash y = \mu(x)\, y\,,$$

cf. example 2.2.d. It is useful to write the element $x \vdash y \dashv z$ as $x\check{y}z$. Under this notation the dialgebra calculus rules are

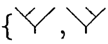
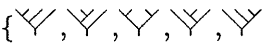
$$\begin{aligned} x\check{y}z \dashv s\check{t}u &= x\check{y}zstu, \\ x\check{y}z \vdash s\check{t}u &= xyzs\check{t}u. \end{aligned}$$

3. (CO)HOMOLOGY OF ASSOCIATIVE DIALGEBRAS

In this section we introduce a chain complex which permits us to define homology groups $HY_*(D)$ and cohomology groups $HY^*(D)$ of a dialgebra D. The main ingredient is the set of *planar binary trees*. The main result of this section is the vanishing of the dialgebra homology of a free dialgebra.

An extension of this theory to a theory with coefficients is to be found in [F4].

3.1. Planar binary trees. A planar tree is *binary* if any vertex is trivalent. We denote by Y_n the set of planar binary trees with $n+1$ leaves. Since we only use planar binary trees in this section we abbreviate it into tree (or n-tree to specify that it has $n+1$ leaves, or, equivalently, n interior vertices).

$Y_0 = \{\,|\,\}$, $Y_1 = \{$  $\}$, $Y_2 = \{$  $\}$, $Y_3 = \{$  $\}$.

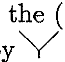
We will use the permutation-like notation of trees (cf. Appendix A):

$$[0] \quad , \quad [1] \quad , \quad [12], [21] \quad , \quad [123], [213], [131], [312], [321]$$

The number of elements in Y_n is the Catalan number $c_n = \frac{(2n)!}{n!(n+1)!}$. For any $y \in Y_n$ we label the $n+1$ leaves by $\{0, 1, \cdots, n\}$ from left to right.

3.2. Face and degeneracy maps. For any i, $0 \le i \le n$, there is a map, called a *face map*, $d_i : Y_n \to Y_{n-1}$ which assigns to the tree y the tree $d_i y$ obtained from y by deleting the i-th leaf. For instance:

$$d_0[213] = [12],\ d_1[213] = [12],\ d_2[213] = [12],\ d_3[213] = [21].$$

For any i, $0 \le i \le n$, there is a map, called a *degeneracy map*, $s_i : Y_n \to Y_{n+1}$ which assigns to the tree y the $(n+1)$-tree $s_i y$ obtained by bifurcating the i-th leaf, that is replace it by  . For instance

$$s_0[0] = [1], s_0[1] = [12]\ , s_1[1] = [21].$$

The face and degeneracy maps satisfy all the classical simplicial relations, *except* for the relation $s_i s_i = s_{i+1} s_i$. Indeed, this relation is not fulfilled on trees, because

$$s_0 s_0([0]) = [12] \text{ and } s_1 s_0([0]) = [21]\ .$$

So $Y_.$ is not a simplicial set, but only an *almost simplicial set*, (cf. [F3]).

For any i, $1 \le i \le n-1$, there is a map

$$\circ_i : Y_n \to \{\dashv, \vdash\}$$

defined as follows. The image of $y \in Y_n$ is $\circ_i^y = \dashv$ (resp. $\vdash$) if the i-th leaf points from the vertex to the left (resp. to the right). For instance:

$$\circ_1^{[131]} = \vdash \quad \text{and} \quad \circ_2^{[131]} = \dashv$$

More generally one has

$$\circ_i^{[j_1,\cdots,j_n]} = \dashv \text{ if } j_i > j_{i+1} \text{ and } \circ_i^{[j_1,\cdots,j_n]} = \vdash \text{ if } j_i < j_{i+1}, \text{ for } 1 \le i \le n-1.$$

Here is the table in low dimension:

y	d_1	$\circ_1$	d_2	$\circ_2$
[12]	[1]	$\vdash$		
[21]	[1]	$\dashv$		
[123]	[12]	$\vdash$	[12]	$\vdash$
[213]	[12]	$\dashv$	[12]	$\vdash$
[131]	[21]	$\vdash$	[12]	$\dashv$
[312]	[21]	$\dashv$	[21]	$\vdash$
[321]	[21]	$\dashv$	[21]	$\dashv$

3.3. The chain complex of a dialgebra. Let D be a dialgebra over K. Define the module of n-chains by

$$CY_n(D) := K[Y_n] \otimes D^{\otimes n},$$

in particular $CY_1(D) \cong D$, $CY_2(D) \cong D^{\otimes 2} \oplus D^{\otimes 2}$, more generally $CY_n(D)$ is isomorphic to the direct sum of c_n copies of $D^{\otimes n}$ (indexed by Y_n).

Define a linear map $d : CY_n(D) \to CY_{n-1}(D)$ by the following formula:

$$d(y; a_1, \cdots, a_n) := -\sum_{i=1}^{n-1} (-1)^i (d_i(y); a_1, \cdots, a_{i-1}, a_i \circ_i^y a_{i+1}, \cdots, a_n),$$

where $y \in Y_n$ and $a_i \in D$. This formula has a meaning since $\circ_i^y = \dashv$ or $\vdash$ and D is a dialgebra. It is convenient to define

$$d_i(y; a_1, \cdots, a_n) := (d_i(y); a_1, \cdots, a_{i-1}, a_i \circ_i^y a_{i+1}, \cdots, a_n)$$

so that $d = -\sum_{i=1}^{n-1} (-1)^i d_i$.

3.4. Lemma. *The face maps $d_i : CY_n(D) \to CY_{n-1}(D)$ satisfy the simplicial relations*

$$d_i d_j = d_{j-1} d_i, \text{ for any } 1 \le i < j \le n-1.$$

Proof. We first prove this identity in the lowest dimension, that is

$$(*) \qquad d_1 d_2 = d_1 d_1 : CY_3(D) \to CY_2(D)$$

The computation of $d_i d_j(y; a, b, c)$ splits into 5 cases corresponding to the five trees with 4 leaves (cf. 3.1).

- *Case* [123] :

$$d_1 d_2([123]; a, b, c) = d_1([12]; a, b \vdash c) = ([1]; a \vdash (b \vdash c)),$$
$$d_1 d_1([123]; a, b, c) = d_1([12]; (a \vdash b, c) = ([1]; (a \vdash b) \vdash c).$$

So relation $(*)$ follows from axiom 5.

- *Case* [213] :

$$d_1d_2([213];a,b,c) = d_1([12];a,b\vdash c) = ([1];a\vdash(b\vdash c)),$$
$$d_1d_1([213];a,b,c) = d_1([12];(a\dashv b,c) = ([1];(a\dashv b)\vdash c).$$

So relation $(*)$ follows from axiom 4.

- *Case* [131] :

$$d_1d_2([131];a,b,c) = d_1([12];a,b\dashv c) = ([1];a\vdash(b\dashv c)),$$
$$d_1d_1([131];a,b,c) = d_1([21];(a\vdash b,c) = ([1];(a\vdash b)\dashv c).$$

So relation $(*)$ follows from axiom 3.

- *Case* [312] :

$$d_1d_2([312];a,b,c) = d_1([21];a,b\vdash c) = ([1];a\dashv(b\vdash c)),$$
$$d_1d_1([312];a,b,c) = d_1([21];(a\dashv b,c) = ([1];(a\dashv b)\dashv c).$$

So relation $(*)$ follows from axiom 2.

- *Case* [321] :

$$d_1d_2([321];a,b,c) = d_1([21];a,b\dashv c) = ([1];a\dashv(b\dashv c)),$$
$$d_1d_1([321];a,b,c) = d_1([21];(a\dashv b,c) = ([1];(a\dashv b)\dashv c).$$

So relation $(*)$ follows from axiom 1.

The proof of the general case $d_id_j = d_{j-1}d_i$ for $i<j$ splits into two different cases.

First, if $j=i+1$, then the proof is exactly as in low dimension and so follows from the axioms of a dialgebra. Second, if $j>i+1$, then both operations d_id_j and $d_{j-1}d_i$ amount to perform the same modification: removing the leaves number j and i of the tree y, and replace $(a_1,\cdots,a_n)$ by

$$(a_1,\cdots,a_i\circ_i^y a_{i+1},\cdots,a_j\circ_j^y a_{j+1},\cdots,a_n).$$

The point is that the leaf number j of y is the leaf number $j-1$ of $d_i(y)$.

So we have proved that $d_id_j = d_{j-1}d_i$ for $i<j$. □

3.5. Proposition. *One has $d\circ d=0$ and so $(CY_*(D),d)$ is a chain-complex.*

Proof. This is an immediate consequence of the previous lemma, like for a pre-simplicial module. □

Observe that in the chain complex

$$CY_*(D):\quad \cdots\to K[Y_n]\otimes D^{\otimes n}\to\cdots\to K[Y_3]\otimes D^{\otimes 3}\to K[Y_2]\otimes D^{\otimes 2}\xrightarrow{(\dashv,\vdash)} D$$

the module of n-chains is the direct sum of c_n copies of $D^{\otimes n}$ (indexed by the set of trees Y_n), the first differential is induced by the two products $\dashv, \vdash$, and the first relation $d^2 = 0$ coincides precisely with the 5 axioms of a dialgebra.

3.6. Homology and cohomology of a dialgebra. By definition the *homology of the dialgebra D* is the homology of the chain-complex $CY_*(D)$:

$$HY_n(D) := H_n(CY_*(D), d), \ n \geq 1.$$

For $n = 1$ it is immediate that $HY_1(D)$ is the quotient of D by the submodule generated by all the elements $x \dashv y$ and $x \vdash y$,

$$HY_1(D) = D/\{x \dashv y, x \vdash y \mid x, y \in D\},$$

which we denote, sometimes, by D/D^2.

By definition the *cohomology of the dialgebra D* is

$$HY^n(D) := H^n(\mathrm{Hom}(CY_*(D), K)), \ n \geq 1.$$

3.7. The chain bicomplex of a dialgebra. The chain complex of a dialgebra is in fact the total chain complex associated to a bicomplex. Indeed, let $Y_{p,q}$ be the subset of Y_n made of the trees which are obtained by grafting a p-tree with a q-tree (cf. Appendix A), where $p + q + 1 = n$. For instance

$$Y_{0,2} = \{[321], [312]\}, \quad Y_{1,1} = \{[131]\}, \quad Y_{2,0} = \{[213], [123]\}.$$

Let $CY_{p,q} := K[Y_{p,q}] \otimes D^{\otimes n}$. Since for any $y \in Y_{p,q}$ the element $d_i(y)$ is either in $Y_{p-1,q}$ or in $Y_{p,q-1}$, the face map d_i takes value either in $CY_{p-1,q}$ or in $CY_{p,q-1}$. So the chain bicomplex $CY_{**}(D)$ is well-defined and its associated total complex is $CY_*(D)$. Remark that, with our choice of notation, one has $CY_n = \bigoplus_{p+q+1=n} CY_{p,q}$. This bicomplex gives rise to two spectral sequences abutting to $HY_*(D)$.

3.8. Theorem. *Let V be a vector space over K and $Dias(V) = TV \otimes V \otimes TV$ be the free dialgebra over V (cf.2.5). Then, one has*

$$\begin{aligned} &HY_1(Dias(V)) \cong V, \\ &HY_n(Dias(V)) = 0, \ for\ n > 1. \end{aligned}$$

Proof. Notation. We denote by P_m the m-dimensional part of the free dimonoid on the generator x. So one has $P_1 = \{\check{x}\}, P_2 = \{\check{x}x, x\check{x}\}, P_3 = \{\check{x}xx, x\check{x}x, xx\check{x}\}$, etc, cf. 1.7.

The complex $CY_*(Dias(V))$ is the Koszul complex associated to the operad $Dias$. Its acyclicity is proved in several steps as follows.

1. We show that it is sufficient to treat the case $V = K$.

2. The chain complex $CY_*(Dias(K))$ is split into the direct sum of chain complexes $C_*(u)$, one for each element u in P_m, $m \geq 1$.
3. The chain complex $C_*(u)$ is shown to be the cell complex of a simplicial set $X(u)$.
4. We show that it suffices to handle the case $u = xx \cdots \check{x}$ by using the join construction of augmented simplicial sets.
5. The space $X(u)$ is shown to be contractible for $u = xx \cdots \check{x}$ by constructing a series of retractions by deformation.

1. *First step.* Recall from 1.8 and 2.5 that $Dias(V) = \oplus_{n\geq 1} K[P_n] \otimes V^{\otimes n}$. Therefore one has

$$\begin{aligned} CY_j(Dias(V)) &= K[Y_j] \otimes \big(\bigoplus_{n\geq 1} K[P_n] \otimes V^{\otimes n}\big)^{\otimes j} \\ &= K[Y_j] \otimes \bigoplus_{m\geq 1} \big(\bigoplus_{n_1+\cdots+n_j=m} K[P_{n_1} \times \cdots \times P_{n_j}]\big) \otimes V^{\otimes m}. \end{aligned}$$

Since d is homogeneous in V, the complex CY_* splits into the direct sum of subcomplexes, one for each $m \geq 1$. This subcomplex is in fact of finite length and, up to tensoring by $V^{\otimes m}$, is of the following form:

$$\begin{aligned} C_*(P_m): \qquad & 0 \to K[Y_m \times P_1 \times \cdots \times P_1] \to \cdots \\ & \to \bigoplus_{n_1+\cdots+n_j=m} K[Y_j \times P_{n_1} \times \cdots \times P_{n_j}] \to \cdots \to K[Y_1 \times P_m]. \end{aligned}$$

Recall that P_1 and Y_1 have only one element. The case $m = 1$ gives the subcomplex of length 0 reduced to V. This shows that $HY_1(Dias(V))$ contains V as expected.

For $m \geq 2$, the differential is simply the differential of $C_*(P_m)$ tensored by the identity of $V^{\otimes m}$, hence it is sufficient to prove the acyclicity of $C_*(P_m)$ in order to prove the theorem.

2. *Second step.* The chain complex $C_*(P_m)$ can still be split into the direct sum of smaller complexes indexed by the elements u of P_m. Indeed, let $\alpha := (t; u_1, \cdots, u_j) \in Y_j \times P_{n_1} \times \cdots \times P_{n_j}$ be a basis element. Under applying $j-1$ face operators to α, we get an element $(\curlyvee\, ; u) \in Y_1 \times P_m$ which does not depend on the choice of the face operators because of the simplicial relations (cf. 3.4). Fixing u, let $C_*(u)$ be the subchain complex linearly generated by the elements α whose image is $(\curlyvee\, ; u) \in Y_1 \times P_m$. It is clear that $C_*(P_m)$ is the direct sum of the chain complexes $C_*(u)$, $u \in P_m$.

Observe that $C_*(u)$ is of simplicial type, that is, its boundary is of the form $d = -\sum_{i=1}^{i=n-1} (-1)^i d_i$.

3. *Third step.* We fix $u = x \cdots x\check{x}x \cdots x \in P_{p+1+q} = P_m$. At this point it is helpful to modify slightly our indexing of the faces and have them to run from 0 to $n-2$ rather than from 1 to $n-1$. For any generator α of $C_*(u)$ the

faces $d_i(\alpha), 0 \leq i \leq n-2$, are still generators of $C_*(u)$. Hence $C_*(u)$ is the normalized augmented complex of an augmented simplicial set that we denote by $X(u)$. The nondegenerate simplices of $X(u)$ are the linear generators α of $C_*(u)$. The top dimensional ones are of the form $(t; x, \cdots, x) \in X(u)_{m-2}$ where $t = t^l \vee t^r, t^l \in Y_p, t^r \in Y_q$. We denote by $Y_{\{u\}}$ this subset of Y_m. At the other end the augmentation set is $X(u)_{-1} = Y_1 \times \{u\}$ (one element). The geometric realization of $X(u)$ is the amalgamation of simplices Δ^{m-2} (one for each $t \in Y_{\{u\}}$) under the following rule:

if $d_{i_k} \cdots d_{i_1}(t) = d_{i_k} \cdots d_{i_1}(t')$ for some $m-2 \geq i_k \geq \cdots \geq i_1 \geq 0$, then we identify the corresponding (oriented) faces of the simplices t and t'. Observe that under this rule a vertex of type i is identified only with a vertex of type i.

4. *Fourth step.* Let $u = x \cdots x\check{x}x \cdots x \in P_m, m = p+1+q$. By direct inspection we see that $X(u)$ is the simplicial join (cf. [EP]) of $X(x \cdots \check{x})$ and of $X(\check{x} \cdots x)$. Hence it is sufficient to show the contractibility of $X(u)$ in the case $u = x \cdots \check{x}$.

5. *Fifth step*: the case $u = x \cdots x\check{x} \in P_m$. We will show that there exists a retraction by deformation of $X(u)$ onto a subspace $X'(u)$ and that $X'(u) = X(v) * \Delta^1$ (simplicial join), where $v = x \cdots x\check{x} \in P_{m-2}$. Hence by induction the contractibility of $X(u)$ follows from the contractibility of $X(x\check{x})$ and of $X(xx\check{x})$.

For each simplex Δ^m the last degeneracy operator is a retraction by deformation on its m-th face. Recall that $X(u)$ is an amalgamated sum of such simplices Δ^{m-2}. In order to show that these retractions assemble to a retraction of $X(u)$ it is sufficient to prove that, in $X(u)$, there is at most one 1-cell from a vertex of type $m-3$ to a vertex of type $m-2$ (these intervals are being squeezed by the retraction). Indeed the vertices of type $m-2$ are of the form $(\curlyvee\,; \check{v}, \check{x})$, where $v\check{x} = u$ and those of type $m-3$ are of the form $(\curlyvee\,; \check{w}, \check{x}, \check{x})$ where $wx\check{x} = u$. There is a 1-simplex relating these two vertices if and only if $\check{v} = w\check{x}$ or $\check{v} = \check{w}x$. In each case there is only one possibility: $(\curlyvee\,; \check{w}, \check{x}, \check{x})$.

We now claim that we can take the following space $X'(u)$ as the image of the retraction. It is the subspace of $X(u)$ which contains all the faces of the form $((t' \vee |) \vee |; \cdots, \check{x}, x\check{x}) \in Y_{m-1} \times P_1 \cdots \times P_1 \times P_2$. Indeed for any tree $t \in Y_{\{u\}}$ one can find a finite sequence ot trees $t = t_0, t_1, \cdots, t_k$ in $Y_{\{u\}}$ such that t_k is as above and

$$d_{m-2}t_0 = d_{m-2}t_1, d_{m-3}t_1 = d_{m-3}t_2, d_{m-2}t_2 = d_{m-2}t_3, \cdots.$$

So the Δ^{m-2} simplex t is mapped onto the Δ^{m-3} simplex $d_{m-2}(t_k)$. The space $X'(u)$ is the join of $X(v), v = x \cdots x\check{x} \in P_{m-2}$ with Δ^1 (same argument as in step 4): $X(u) = X(v) * \Delta^1$. So, by induction, it remains to show that $X(x\check{x})$ and $X(xx\check{x})$ are contractible.

The space $X(x\check{x})$ is simply a point since the only cell is $(\curlyvee\,; x, x)$.

The space $X(xx\check{x})$ has two 1-cells: $(\curlyvee\,; x, x, x)$ and $(\curlyvee\,; x, x, x)$ whose faces are:

$$d_0(\curlyvee\,; x, x, x) = (\curlyvee\,; \check{x}x, x), \quad d_1(\curlyvee\,; x, x, x) = (\curlyvee\,; x, x\check{x})$$
$$d_0(\curlyvee\,; x, x, x) = (\curlyvee\,; x\check{x}, x), \quad d_1(\curlyvee\,; x, x, x) = (\curlyvee\,; x, x\check{x})$$

Since they have the same d_1 the space $X(xx\check{x})$ is the amalgamation of two intervals onto their endpoint:

This space is obviously contractible. □

For the convenience of the reader we illustrate the proof in the next case, that is $u = xxx\check{x}$ $(m = 3)$.

$$\begin{array}{lllll}
 & & d_0 & d_1 & d_2 \\
a = (& ;x,x,x,x) & (\ ;x\check{x},x,x), & (\ ;x,x\check{x},x), & (\ ;x,x,x\check{x}), \\
b = (& ;x,x,x,x) & (\ ;\check{x}x,x,x), & (\ ;x,x\check{x},x), & (\ ;x,x,x\check{x}), \\
c = (& ;x,x,x,x) & (\ ;x\check{x},x,x), & (\ ;x,\check{x}x,x), & (\ ;x,x,x\check{x}), \\
d = (& ;x,x,x,x) & (\ ;\check{x}x,x,x), & (\ ;x,x\check{x},x), & (\ ;x,x,x\check{x}), \\
e = (& ;x,x,x,x) & (\ ;\check{x}x,x,x), & (\ ;x,\check{x}x,x), & (\ ;x,x,x\check{x}),
\end{array}$$

Hence the amalgamation of the simplices a, b, c, d, e of type Δ^2 are under the following rules:

$$d_0(d) = d_0(e), \quad d_1(a) = d_1(b), \quad d_2(a) = d_2(c), \quad d_2(b) = d_2(d) = d_2(e).$$

We let the reader draw the 2-dimensional space $X(xxx\check{x})$.

The retraction by deformation smashes it onto the 1-simplex $d_2(b)$, which is precisely $X(x\check{x}) * \Delta^1$ since $X(x\check{x})$ is a point. For instance the sequence of trees for c is: c, a, b.

3.9. Theorem. *For any dialgebra D the graded module $HY_*(D)$ is a graded dual-codialgebra and the graded module $HY^*(D)$ is a graded dendriform algebra (see section 5). As a consequence $HY^*(D)$ is a graded associative algebra.*

Proof. Though one could prove these statements directly, they are consequences of general facts about Koszul operads (see Appendix B). We will show in section 6 that the operad of dendriform algebras is dual to the operad of associative dialgebras. Moreover, by theorem 3.8 these operads are Koszul, hence the statement follows from general properties of Koszul operads (cf. Appendix B5d). The last statement is a consequence of the preceding one and of Lemma 7.3. □

3.10. Simplicial properties of the chain-modules. We have seen in Lemma 3.4 that the face maps $d_i : CY_n(D) \to CY_{n-1}(D)$ satisfy the standard simplicial relations. Suppose that D is equipped with a bar-unit e and let us define $s_j : CY_n(D) \to CY_{n+1}(D)$ by

$$s_j(y; a_1, \cdots, a_n) := (s_j(y); a_1, \cdots, a_j, e, a_{j+1}, \cdots, a_n), \ 0 \le j \le n,$$

where $s_j(y)$ is described in 3.2. From the properties of the bar-unit, it is immediate to check that

$$d_i s_j = \begin{cases} s_{j-1}d_i & \text{for } i < j, \\ \text{id} & \text{for } i = j,\ i = j+1, \\ s_j d_{i-1} & \text{for } i > j+1, \end{cases}$$
$$s_i s_j = s_{j+1} s_i \quad \text{for } i < j.$$

So the family $(CY_n(D); d_i, s_j)_{n\geq 0}$ is an *almost simplicial module*, that is the face and degeneracy operators satisfy all the standard relations of a simplicial module, *except* for the relation $s_i s_i = s_{i+1} s_i$ (cf. 3.2).

A variation of the Eilenberg-Zilber theorem is still valid for pseudo-simplicial modules (cf. Inassaridze [I]) and a fortiori for almost simplicial modules. It is used in the proof of the following result which is due to Alessandra Frabetti.

3.11. Theorem [F4]. *If D is a dialgebra equipped with a bar-unit, then*

$$HY_n(D) = 0, \textit{ for any } n \geq 0.$$

□

Comment. This result is similar to the vanishing of the bar-homology for a unital associative algebra.

3.12. Generalization of the homology of a dialgebra to the symmetric group. There is a generalization of the complex CY_* consisting in replacing the set of planar binary trees Y_n by the symmetric group S_n, or, equivalently, by the set $\tilde{Y}_n$ of *binary increasing trees* (cf. Appendix A).

The formulas for the maps d_i are the same as in 3.2 and 3.3 once S_n has been identified with $\tilde{Y}_n$ (observe that deleting a leaf in an increasing tree still gives an increasing tree). Hence we get a new complex

$$CS_*(D): \quad \cdots \to K[S_n]\otimes D^{\otimes n} \to \cdots \to K[S_3]\otimes D^{\otimes 3} \to K[S_2]\otimes D^{\otimes 2} \xrightarrow{(\dashv,\vdash)} D$$

for any dialgebra D, and new homology groups $HS_*(D)$.

The boundary map is still the alternate sum of face maps. When the dialgebra is bar-unital, then there also exist degeneracy maps. All these maps satisfy the simplicial relations *except* the relations involving only the degeneracy maps. Such an object is called a *pseudo-simplicial* module (cf. [I]).

Forgetting the levels gives a map $\Psi : S_n = \tilde{Y}_n \to Y_n$ (cf. Appendix A) which induces a chain map $CS_*(D) \to CY_*(D)$ and hence a morphism

$$HS_*(D) \to HY_*(D).$$

This new theory HS_*, or, more accurately, its variant with non trivial coefficients, crops up naturally when one wants to compute the Leibniz homology of the Leibniz algebra of matrices $\mathfrak{gl}(D)$ over a dialgebra D (cf. Frabetti [F2]).

3.13. Remark. We will show in section 6 that $CY_*(D)$ is the chain complex of the dialgebra D predicted by the operad theory. One could wonder if there is another notion of algebra for which $CS_*(D)$ would be the predicted chain complex. If this would be the case, then the Poincaré series of the dual operad would be the inverse of the series

$$\sum_{n\geq 1}(-1)^n \# S_n x^n = \sum_{n\geq 1}(-1)^n n! x^n.$$

However this inverse is not of the form $\sum_{n\geq 1}(-1)^n a_n x^n$ with $a_n \in \mathbf{N}$, hence this operad, even if it existed, could not be a Koszul operad.

4. LEIBNIZ ALGEBRAS, ASSOCIATIVE DIALGEBRAS AND HOMOLOGY

A Leibniz algebra is a non-commutative version of a Lie algebra. In this section we show that, when we replace Lie algebras by Leibniz algebras, then the role of associative algebras is played by the associative dialgebras. In particular we show that any Leibniz algebra has a *universal enveloping associative dialgebra.*

4.1. Leibniz algebras [L1], [LP]. Recall that a *Leibniz algebra* over K is a K-module $\mathbf{g}$ equipped with a binary operation (called a bracket):

$$[-,-] : \mathbf{g}\otimes \mathbf{g} \to \mathbf{g},$$

which satisfies the *Leibniz identity*:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y],$$

for all x, y, z in $\mathbf{g}$. This is in fact a *right* Leibniz algebra. For the opposite structure, that is $[x,y]' = [y,x]$, the left Leibniz identity is

$$[[x,y]',z]' = [x,[y,z]']' - [y,[x,z]']'.$$

If the bracket happens to be anticommutative, then $\mathbf{g}$ is a Lie algebra. Quotienting the Leibniz algebra $\mathbf{g}$ by the ideal generated by the elements $[x,x]$ for all $x\in \mathbf{g}$ gives a Lie algebra that we denote by $\mathbf{g}_{Lie}$.

To any Leibniz algebra $\mathbf{g}$ is associated a chain-complex

$$CL_*(\mathbf{g}) : \qquad \dots \longrightarrow \mathbf{g}^{\otimes n} \xrightarrow{d} \mathbf{g}^{\otimes n-1} \xrightarrow{d} \dots \xrightarrow{d} \mathbf{g}^{\otimes 2} \xrightarrow{d} \mathbf{g}$$

where

$$d(x_1,\dots,x_n) = \sum_{1\leq i<j\leq n} (-1)^j (x_1,\dots,x_{i-1},[x_i,x_j],x_{i+1},\dots,\hat{x}_j,\dots,x_n).$$

The homology groups of this complex are denoted by $HL_n(\mathrm{g})$, for $n \geq 1$.

4.2. Proposition. *Let D be a dialgebra. Then the bracket*

$$[x, y] := x \dashv y - y \vdash x$$

makes D into a Leibniz algebra, denoted by D_{Leib}.

Proof. It is a straightforward checking in which axioms (1), (2), (4), (5) are used once and axiom (3) twice since

$$\begin{aligned} [x,[y,z]] &= x \dashv (y \dashv z) - x \dashv (z \vdash y) - (y \dashv z) \vdash x + (z \vdash y) \vdash x, \\ -[[x,y],z]] &= -(x \dashv y) \dashv z + (y \vdash x) \dashv z + z \vdash (x \dashv y) - z \vdash (y \vdash x), \\ [[x,z],y]] &= (x \dashv z) \dashv y - (z \vdash x) \dashv y - y \vdash (x \dashv z) + y \vdash (z \vdash x). \end{aligned}$$

□

This construction defines a functor

$$\mathbf{Dias} \xrightarrow{-} \mathbf{Leib}$$

from the category **Dias** of dialgebras to the category **Leib** of Leibniz algebras.

4.3. Example. For any dialgebra D over K the $n \times n$-matrices with entries in D form a new dialgebra $\mathcal{M}_n(D)$. Its associated Leibniz algebra is denoted $\mathrm{g}l_n(D)$. It is not a Lie algebra in general. The homology of $\mathrm{g}l(D)$, when D has a bar-unit, has been computed by Frabetti [F2].

4.4. Proposition. *The following diagram of categories and functors is commutative*

$$\begin{array}{ccc} \mathbf{Dias} & \xrightarrow{-} & \mathbf{Leib} \\ \uparrow & & \uparrow \\ \mathbf{As} & \xrightarrow{-} & \mathbf{Lie}. \end{array}$$

□

4.5. Remark. In the definition of a dialgebra, axiom (3) could be relaxed slightly, though proposition 4.2 remains valid. It could be replaced by the weaker axiom:

$$((y \vdash x) \dashv z) + (z \vdash (x \dashv y)) = (y \vdash (x \dashv z)) + ((z \vdash x) \dashv y).$$

Observe that in this formula the variables do not stay in the same order in the monomials. Hence the associated operad would not be a non-Σ-operad anymore.

4.6. Universal enveloping associative dialgebra of a Leibniz algebra. The functor $-$: **As** $\rightarrow$ **Lie** has a left adjoint which is the universal enveloping algebra of a Lie algebra:

$$\overline{U}(\mathrm{g}) = \overline{T}(\mathrm{g})/\{[x,y] - x \otimes y + y \otimes x \mid x, y \in \mathrm{g}\}.$$

($\overline{U}(\mathrm{g})$ is the augmentation ideal of the classical enveloping unital algebra $U(\mathrm{g})$). Similarly, define the *universal enveloping dialgebra* of a Leibniz algebra g as the following quotient of the free dialgebra on g:

$$Ud(\mathrm{g}) := T(\mathrm{g}) \otimes \mathrm{g} \otimes T(\mathrm{g}) / \{[x, y] - x \dashv y + y \vdash x \mid x, y \in \mathrm{g}\}.$$

Under our previous notation, the elements generating the ideal are denoted by $[x, y] - \check{x}y + y\check{x}$.

4.7. Proposition. *The functor* $Ud : \mathbf{Leib} \to \mathbf{Dias}$ *is left adjoint to the functor* $- : \mathbf{Dias} \to \mathbf{Leib}$.

Proof. Let $f : \mathrm{g} \to D_{Leib}$ be a morphism of Leibniz algebras. There is a unique extension of f as a morphism of dialgebras from $T(\mathrm{g}) \otimes \mathrm{g} \otimes T(\mathrm{g})$ to D. Since the image of $[x, y] - x \dashv y + y \vdash x$ under this morphism is 0, it defines a morphism from $Ud(\mathrm{g})$ to D.

On the other hand the restriction of the morphism of dialgebras $g : Ud(\mathrm{g}) \to D$ to $\mathrm{g} = K \otimes \mathrm{g} \otimes K$ yields a morphism of Leibniz algebras $\mathrm{g} \to D_{Leib}$.

It is now immediate to check that these two constructions give rise to isomorphisms

$$\mathrm{Hom}_{\mathbf{Dias}}(Ud(\mathrm{g}), D) \cong \mathrm{Hom}_{\mathbf{Leib}}(\mathrm{g}, D_{Leib}).$$

□

It is well-known that the universal enveloping algebra of a Lie algebra is not only an associative algebra but a Hopf algebra. Similarly the universal enveloping dialgebra of a Leibniz algebra possesses co-operations. They are studied by Goichot in [Go].

4.8. Lemma. *For any Leibniz algebra* g, *one has* $Ud(\mathrm{g})_{As} = U(\mathrm{g}_{Lie})$.

Proof. Since the functor $(-)_{As} : \mathbf{Dias} \to \mathbf{As}$ is left adjoint to $inc : \mathbf{As} \to \mathbf{Dias}$ and since $(-)_{Lie} : \mathbf{Leib} \to \mathbf{Lie}$ is left adjoint to $inc : \mathbf{Lie} \to \mathbf{Leib}$ both composites $U \circ (-)_{Lie}$ and $(-)_{As} \circ Ud$ are left adjoint to the composite $\mathbf{Leib} \to \mathbf{Lie} \to \mathbf{As}$, and so are equal. □

4.9. Proposition. *The universal enveloping dialgebra* $Ud(\mathrm{g})$ *is isomorphic to* $U(\mathrm{g}_{Lie}) \otimes \mathrm{g}$, *equipped with the dialgebra structure issued from a* $U(\mathrm{g}_{Lie})$*-bimodule structure and the bimodule map* $U(\mathrm{g}_{Lie}) \otimes \mathrm{g} \to U(\mathrm{g}_{Lie})$ *(cf. example 2.2.d).*

Proof. Let us define a $U(\mathrm{g}_{Lie})$-bimodule structure on $U(\mathrm{g}_{Lie}) \otimes \mathrm{g}$. The left module structure is given by multiplication in the left factor. The right module structure is induced by

$$(\omega \otimes x) \cdot \bar{y} := \omega \otimes [x, y] + \omega\bar{y} \otimes x,$$

where $\omega \in U(\mathrm{g}_{Lie}), x \in \mathrm{g}, \bar{y} \in \mathrm{g}_{Lie}$ and $y \in \mathrm{g}$ is a lifting of $\bar{y} \in \mathrm{g}_{Lie}$. It is a well-defined element because the bracket $[x, y]$ in the Leibniz algebra g

depends only on the class of y in $\mathbf{g}_{Lie}$. Let us check that this formula provides a representation of $\mathbf{g}_{Lie}$.

Let $\bar{y}$ and $\bar{z}$ be elements in $\mathbf{g}_{Lie}$ and y, z be liftings in $\mathbf{g}$. On one hand one gets

$$((\omega \otimes x \cdot \bar{y}) \cdot \bar{z} = \omega \otimes [[x,y],z] + \omega\bar{z} \otimes [x,y] + \omega\bar{y} \otimes [x,z] + \omega\bar{y}\bar{z} \otimes x,$$

and

$$((\omega \otimes x \cdot \bar{z}) \cdot \bar{y} = \omega \otimes [[x,z],y] + \omega\bar{y} \otimes [x,z] + \omega\bar{z} \otimes [x,y] + \omega\bar{z}\bar{y} \otimes x.$$

Hence one has

$$\begin{aligned}((\omega \otimes x \cdot \bar{z}) \cdot \bar{y} - ((\omega \otimes x \cdot \bar{z}) \cdot \bar{y} &= \omega \otimes ([x,y],z] - [x,z],y]) - \omega(\bar{y}\bar{z} - \bar{z}\bar{y}) \otimes x \\ &= \omega \otimes [x,[y,z]] - \omega\overline{[y,z]} \otimes x \\ &= (\omega \otimes x) \cdot [y,z] .\end{aligned}$$

The right and left module structures are immediately seen to be compatible, hence $U(\mathbf{g}_{Lie}) \otimes \mathbf{g}$ is a $U(\mathbf{g}_{Lie})$-bimodule.

The linear map $U(\mathbf{g}_{Lie}) \otimes \mathbf{g} \to U(\mathbf{g}_{Lie}), \omega \otimes x \mapsto \omega\bar{x}$ is a $U(\mathbf{g}_{Lie})$-bimodule because $\omega\bar{x}\bar{y} = \omega\overline{[x,y]} + \omega\bar{y}\bar{x}$.

So, it follows that $U(\mathbf{g}_{Lie}) \otimes \mathbf{g}$ is equipped with a dialgebra structure (cf. 22.d). The (nonunital) associative algebra associated to this dialgebra is the augmentation ideal of $U(\mathbf{g}_{Lie})$, cf. 2.6.

There is a well-defined dialgebra map

$$Ud(\mathbf{g}) \to U(\mathbf{g}_{Lie}) \otimes \mathbf{g}$$

which sends $\omega \otimes x \otimes 1$ to $\bar{\omega} \otimes x$ (for $\omega \in T(\mathbf{g})$ and $\bar{\omega}$ its image in $U(\mathbf{g}_{Lie})$). Indeed, any element in $Ud(\mathbf{g})$ can be written as a linear combination of elements of the form $\omega \otimes x \otimes 1$ since

$$\omega \otimes x \otimes y = \omega \otimes [x,y] \otimes 1 + \omega y \otimes x \otimes 1.$$

Since by lemma 4.8 one has $Ud(\mathbf{g})_{As} = U(\mathbf{g}_{Lie})$, it follows that the element $\omega\check{x}\omega'$ in $Ud(\mathbf{g})$ depends only on the class of $\omega \in T(\mathbf{g})$ (resp. $\omega' \in T(\mathbf{g})$) in $U(\mathbf{g}_{Lie})$. So, on can define a dialgebra map

$$U(\mathbf{g}_{Lie}) \otimes \mathbf{g} \to Ud(\mathbf{g})$$

by sending $\bar{\omega} \otimes x$ to $\omega\check{x}$, where ω is a lifting of $\bar{\omega}$.

It is immediate to check that both composites are the identity, whence the isomorphism. □

4.10. Free algebras and free dialgebras. Let V be a K-module and let $\overline{T}(V) = V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$ be the tensor module. We denote by

γ the endomorphism of $\overline{T}(V)$ defined inductively by $\gamma(v) = v$ for $v \in V$ and $\gamma(\omega \otimes v) = \omega \otimes v - v \otimes \omega$ for $\omega \in V^{\otimes n}$ and $v \in V$. It is well-known that Im γ is isomorphic to the free Lie algebra $Lie(V)$ over V. Recall that the free associative algebra over V is $\overline{T}(V)$ equipped with the concatenation product, and the free Leibniz algebra over V is $\overline{T}(V)$ equipped with the unique Leibniz bracket which satisfies $[\omega, v] = \omega \otimes v$ for $\omega \in V^{\otimes n}$ and $v \in V$ (cf. [LP]). In the sequence

$$\overline{T}(V) \xrightarrow{\gamma} Lie(V) \hookrightarrow \overline{T}(V),$$

the first map is a map of Leibniz algebras, the second one is a map of Lie algebras (for the Lie structure of $\overline{T}(V)$ coming from its associative algebra structure). From Proposition 4.4 it follows that there is a commutative diagram

$$\begin{array}{ccccc}
Leib(V) = \overline{T}(V) & \xrightarrow{\check{\gamma}_1} & \overline{T}(V) \otimes V \otimes \overline{T}(V) & = Dias(V) \\
\gamma \downarrow & & \downarrow fusion & \\
Lie(V) & \hookrightarrow & \overline{T}(V) & = As(V)
\end{array}$$

where the maps *fusion* and $\check{\gamma}_1$ are described as follows.

The fusion map consists in forgetting the symbol $\check{}$, that is $\omega \check{v} \omega' \mapsto \omega v \omega'$.

The image of $v_1 \cdots v_n$ by $\check{\gamma}_1$ is $\gamma(\check{v}_1 v_2 \cdots v_n)$, which means writing $\gamma(v_1 \cdots v_n)$ and putting the symbol $\check{}$ on the variable v_1 of each monomial. For instance $\check{\gamma}_1(v_1) = \check{v}_1, \check{\gamma}_1(v_1 \otimes v_2) = \check{v}_1 v_2 - v_2 \check{v}_1, \check{\gamma}_1(v_1 \otimes v_2 \otimes v_3) = \check{v}_1 v_2 v_3 - v_2 \check{v}_1 v_3 - v_3 \check{v}_1 v_2 + v_3 v_2 \check{v}_1$.

One observes that this map is very similar to the map α used by Lodder in [Lo] to describe the loop suspension of a wedge product of topological spaces.

4.11. Proposition. *Let V be a K-module and $Leib(V) = \overline{T}(V)$ be the free Leibniz algebra on V. Then one has an isomorphism*

$$Ud(Leib(V)) \cong Dias(V) = T(V) \otimes V \otimes T(V).$$

Proof. The functor *Leib* is left adjoint to the forgetful functor from Leibniz algebras to modules. Similarly the functor *Ud* is left adjoint to the functor from dialgebras to Leibniz algebras, therefore the composite is left adjoint to the forgetful functor from dialgebras to modules, so it is the free diassociative algebra functor. □

4.12. Theorem. *For any dialgebra D there is a natural transformation*

$$HL_*(D_{Leib}) \longrightarrow HS_*(D)$$

induced by the chain complex map

$$\epsilon_n : CL_n(D_{Leib}) = D^{\otimes n} \longrightarrow K[S_n] \otimes D^{\otimes n} = CS_n(D),$$
$$\epsilon_n(x_1, \cdots, x_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \sigma \otimes \sigma^{-1}(x_1, \cdots, x_n).$$

Proof. The boundary map d_L of the Leibniz complex CL_* is described in 4.1. For any element $(\underline{x}, y) := (x_1, \ldots, x_n, y) \in D^{\otimes n+1}$ the map d_L is defined inductively by the formula:

$$d_L(\underline{x}, y) = (d_L(\underline{x}), y) + (-1)^n \operatorname{ad}(y)(\underline{x}), \tag{4.12.1}$$

where $\operatorname{ad}(y)(\underline{x}) = \operatorname{ad}(y)(x_1, \ldots, x_n) := \sum_{i=1}^n (x_1, \ldots, x_i \dashv y - y \vdash x_i, \ldots, x_n)$.

The boundary map d_S of the symmetric complex CS_* is described in 3.12 and 3.2.

One extends the operator $\operatorname{ad}(y)$ to $K[S_n] \otimes D^{\otimes n}$ by putting $\operatorname{ad}(y)(\sigma \otimes \underline{x}) = \sigma \otimes \operatorname{ad}(y)(\underline{x}) \in K[S_n] \otimes D^{\otimes n}$, so that it obviously commutes with ϵ_n:

$$\epsilon_n \operatorname{ad}(y) = \operatorname{ad}(y) \epsilon_n. \tag{4.12.2}$$

It turns out that this new operator is homotopic to 0. The homotopy $h(y) : K[S_n] \otimes D^{\otimes n} \to K[S_{n+1}] \otimes D^{\otimes n+1}$ is given by

$$h(y)(\sigma \otimes \underline{x}) := \sum_{i=0}^{n} (-1)^i s_i(\sigma) \otimes (x_1, \ldots, x_i, y, x_{i+1}, \ldots, x_n),$$

where $s_i(\sigma)$ is the i-th degeneracy of σ (bifurcate the i-th leaf of the corresponding increasing tree, cf. 3.2). The checking of

$$d_S h(y) + h(y) d_S = \operatorname{ad}(y) \tag{4.12.3}$$

is tedious but straightforward.

The comparison of the homotopy operator $h(y)$ with the symmetrization operator ϵ_n gives:

$$\epsilon_{n+1}(\underline{x}, y) = (-1)^n h(y) \epsilon_n(\underline{x}). \tag{4.12.4}$$

The proof of the commutation relation

$$d_S \circ \epsilon_n = \epsilon_{n-1} \circ d_L \tag*{$(*)_n$}$$

is done by induction on n as follows.

For $n = 1$, one has $\epsilon_1 = \mathrm{Id}$.

For $n = 2$, the map $\epsilon_2 : D^{\otimes 2} \to K[S_2] \otimes D^{\otimes 2}$ is given by

$$\epsilon_2(x, y) = [12] \otimes (x, y) - [21] \otimes (y, x).$$

Since $d_L(x, y) = [x, y]$ and $d_S([12] \otimes (x, y)) = x \dashv y$, $d_S([21] \otimes (x, y)) = x \vdash y$, it follows from $[x, y] = x \dashv y - y \vdash x$ that $d_S \circ \epsilon_2 = \epsilon_1 \circ d_L$.

By induction we suppose that $(*)_n$ holds and we will prove $(*)_{n+1}$.

One gets:

$$\begin{array}{lll} d_S\epsilon_{n+1}(\underline{x}, y) & = (-1)^n d_S h(y)\epsilon_n(\underline{x}) & \textit{by } (4.12.4) \\ & = (-1)^n(\mathrm{ad}\,(y) - h(y)d_S)\epsilon_n(\underline{x}) & \textit{by } (4.12.3) \\ & = (-1)^n \,\mathrm{ad}\,(y)\epsilon_n(\underline{x}) + (-1)^{n-1}h(y)\epsilon_{n-1}d_L(\underline{x}) & \textit{by induction} \\ & = (-1)^n \,\mathrm{ad}\,(y)\epsilon_n(\underline{x}) + \epsilon_n(d_L(\underline{x}), y) & \textit{by } (4.12.4) \\ & = (-1)^n \epsilon_n(\underline{x})\,\mathrm{ad}\,(y) + \epsilon_n(d_L(\underline{x}), y) & \textit{by } (4.12.2) \\ & = \epsilon_n d_L(\underline{x}, y) & \textit{by } (4.12.1). \end{array}$$

□

Remark. This proof is mimicked on the proof for the Lie case as done in [L0], Proposition 1.3.5. This Proposition has been extended to homology of dialgebras with coefficients by Alessandra Frabetti in her thesis (unpublished).

4.13. Proposition. *The composite*

$$HL_*(D_{Leib}) \to HS_*(D) \xrightarrow{\psi_*} HY_*(D),$$

where $\psi_n : S_n \twoheadrightarrow Y_n$ is the surjective map described in Appendix A, is the map induced, through the operad theory, by the morphism of operads $Dias^! \to Leib^!$.

Proof. We show in section 8 that the complexes CL_* and CY_* are the complexes predicted by the operad theory. So, by Appendix B5e, it suffices to check that the natural map $K[Y_n] \otimes K[S_n] = Dias^!(n) \to Leib^!(n) = K[S_n]$ on the dual operads (see section 8) is given by $y \otimes \omega \mapsto \sum_{\{\sigma \in S_n \,|\, \psi(\sigma) = y\}} \sigma\omega$, and this is precisely Theorem 7.5. □

4.14. Comparison of $HL_*(\mathrm{g})$ and $HY_*(Ud(\mathrm{g}))$ for a Leibniz algebra g. For a Lie algebra **h** the composite map

$$H_*^{Lie}(\mathbf{h}) \to H_*^{Lie}(\bar{U}(\mathbf{h})_{Lie}) \to H_*^{As}(\bar{U}(\mathbf{h})),$$

is known to be an isomorphism (cf. for instance [L0]).

However, for a Leibniz algebra **g**, the composite map

$$HL_*(\mathbf{g}) \to HL_*(Ud(\mathbf{g})_{Leib}) \to HY_*(Ud(\mathbf{g}))$$

is no longer an isomorphism, contrarily to what was mistakenly announced in [L2]. The main point in the proof of the Lie case, is that the graded space associated to the filtration of $U(\mathbf{g})$ depends only on the vector space **g**, and not on the Lie structure. In the Leibniz case, the graded space associated to $Ud(\mathbf{g}) = U(\mathbf{g}_{Lie}) \otimes \mathbf{g}$ (cf. Proposition 4.9) depends on $\mathbf{g}_{Lie}$, that is, on the Leibniz structure of **g**. Hence the map $HL_*(\mathbf{g}) \to HY_*(Ud(\mathbf{g}))$ is part of a spectral sequence involving the derived functors of "Lie-zation".

4.15. Poisson dialgebra. By definition a *Poisson dialgebra P* is a vector space P equipped with a dialgebra structure $\dashv$ and $\vdash$, and a Leibniz structure $[-,-]$ which are compatible in the sense that they satisfy the following 4 relations:

$$[x, y \dashv z] = y \vdash [x, z] + [x, y] \dashv z = [x, y \vdash z],$$

$$[x \dashv y, z] = x \dashv [y, z] + [x, z] \dashv y,$$
$$[x \vdash y, z] = x \vdash [y, z] + [x, z] \vdash y.$$

This definition generalizes the "non-commutative Poisson algebra" as defined in [Kub], [KS] and [Ak].

5. DENDRIFORM ALGEBRAS

In this chapter we construct a new type of algebra with two binary operations, which dichotomizes the notion of associative algebra. It is closely related to associative dialgebras. In fact, we show in section 8 that its operad is Koszul dual to the operad of associative dialgebras. The terminology is due to the structure of the free dendriform algebra, which is best described in terms of planar binary trees.

5.1. Definition. A *dendriform algebra* E over K is a K-vector space E equipped with two binary operations

$$\prec \; : E \otimes E \to E,$$
$$\succ \; : E \otimes E \to E,$$

which satisfy the following axioms:

$$\begin{aligned} &\text{(i)} && (a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c), \\ &\text{(ii)} && (a \succ b) \prec c = a \succ (b \prec c), \\ &\text{(iii)} && (a \prec b) \succ c + (a \succ b) \succ c = a \succ (b \succ c), \end{aligned}$$

for any elements a, b and c in E.

It is sometimes preferable to call this object *dendriform dialgebra* to insist on the fact that it is defined by two operations, but we do not use this terminology here. It is important to observe that, like for associative algebras and associative dialgebras, the monomials involved in the relations keep the variables in the same order. As a consequence the associated operad is a non-Σ-operad.

Observe that there is no "monoid" version of dendriform algebras, since relations (i) and (iii) involve sums. In other words, the associated operad does not come from a set-operad.

By introducing the operation

$$x * y := x \prec y + x \succ y,$$

these relations take the following more concise form:

$$\begin{aligned} &\text{(i)} && (a \prec b) \prec c = a \prec (b * c), \\ &\text{(ii)} && (a \succ b) \prec c = a \succ (b \prec c), \\ &\text{(iii)} && (a * b) \succ c = a \succ (b \succ c). \end{aligned}$$

5.2. Lemma. *For any dendriform algebra E the product defined by*

$$x * y := x \prec y + x \succ y.$$

is associative.

Proof. Adding up the three equalities (i), (ii) and (iii) we get $(x * y) * z$ on the left hand side and $x * (y * z)$ on the right hand side, whence the statement. □

It follows from this lemma that a dendriform algebra is in fact an associative algebra, whose product has some special property. The category of dendriform algebras is denoted by **Dend**.

5.3. Proposition. *Let D be a dialgebra and E a dendriform algebra. Then, on the tensor product $D \otimes E$, the bracket*

$$\begin{aligned}[x \otimes a, y \otimes b] :=& (x \dashv y) \otimes (a \prec b) - (y \vdash x) \otimes (b \succ a) \\ &- (y \dashv x) \otimes (b \prec a) + (x \vdash y) \otimes (a \succ b),\end{aligned}$$

where $x, y \in D, a, b \in E$, defines a structure of Lie algebra.

Proof. The bracket is antisymmetric by definition. Hence, it suffices to show that the Jacobi identity is fulfilled.

The Jacobi identity for $x \otimes a$, $y \otimes b$, $z \otimes c$ gives a total of 48 terms, in fact $8 \times 3!$ terms. There are 8 terms for which x, y, z (and also a, b, c) stay in the same order. The other sets of 8 terms are permutations of this set which reads:

$$\begin{array}{lcl}
x \dashv (y \dashv z) \otimes a \prec (b \prec c) & - & (x \dashv y) \dashv z \otimes (a \prec b) \prec c, \\
x \vdash (y \dashv z) \otimes a \succ (b \prec c) & - & (x \vdash y) \dashv z \otimes (a \succ b) \prec c, \\
x \dashv (y \vdash z) \otimes a \prec (b \succ c) & - & (x \dashv y) \vdash z \otimes (a \prec b) \succ c, \\
x \vdash (y \vdash z) \otimes a \succ (b \succ c) & - & (x \vdash y) \vdash z \otimes (a \succ b) \succ c.
\end{array}$$

The terms 1 and 3 in column 1 together with the term 1 in column 2 cancel due to axioms (1), (2) and (i). Similarly the terms 41, 32 and 42 cancel due to axioms (4), (5) and (iii). Finally the terms 21 and 22 cancel due to axioms (3) and (ii). □

This result may also be seen as a consequence of Koszul duality (cf. Appendix B).

5.4. Examples of dendriform algebras.

(a) Shuffle algebra. Let V be a vector space and let $T'(V)$ be the reduced tensor module over V equipped with the shuffle product (which is associative and commutative). The shuffle of two generating elements $v_1 \cdots v_p$ and $v_{p+1} \cdots v_{p+q}$ can be split into two parts depending on the fact that the first element is v_1 or v_{p+1}. The first part gives the left product and the second part gives the right product. One can show that the shuffle algebra is then a dendriform algebra (this fact had been previously remarked by Gian-Carlo Rota).

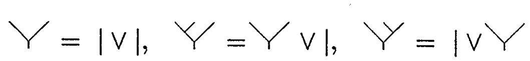
(b) Matrices over dendriform algebras. Since in the axioms of a dendriform algebra the variables a, b, c stay in this order in all the monomials, the tensor product of two dendriform algebras is naturally a dendriform algebra. Similarly, let $\mathcal{M}_n(E)$ be the module of $n \times n$-matrices with entries in the dendriform algebra E. Then the formulas

$$(\alpha \prec \beta)_{ij} = \sum_k \alpha_{ik} \prec \beta_{kj} \quad \text{and} \quad (\alpha \succ \beta)_{ij} = \sum_k \alpha_{ik} \succ \beta_{kj}$$

make $\mathcal{M}_n(E)$ into a dendriform algebra.
(c) Free dendriform algebra. Let V be a K-module and denote by $Dend(V)$ the free dendriform algebra over V. It is a dendriform algebra which satisfies the classical universal property. We will prove its existence and give an explicit description in 5.7. As a first step we describe the free dendriform algebra on one generator by using the sets of planar binary trees Y_n, and some operations on them.

5.5. Grafting operation on trees. By definition the *grafting* of the trees $y \in Y_p$ and $z \in Y_q$ is the tree $y \vee z \in Y_{p+q+1}$ obtained from y and z by joining their roots together and adding a new root. Observe that the number of internal vertices of $y \vee z$ is the sum of the numbers of internal vertices of y and of z plus 1.

Given a tree y (different from $|$) there is a unique decomposition $y = y_1 \vee y_2$. For instance one has:



which, with our notation, reads

$$[1] = [0] \vee [0], \quad [12] = [1] \vee [0], \quad [21] = [0] \vee [1].$$

The grafting operation is easy to write down in terms of the permutation-like notation. Indeed, for $y \in Y_p$ and $z \in Y_q$, one has

$$[y] \vee [z] = [y\ p + q + 1\ z].$$

For instance one has:

$$[1] \vee [1] = [1\ 3\ 1], \qquad [1\ 3\ 1] \vee [2\ 1] = [1\ 3\ 1\ 6\ 2\ 1].$$

We put $K[Y_\infty] := \oplus_{n \geq 0} K[Y_n]$ and $\overline{K[Y_\infty]} := \oplus_{n \geq 1} K[Y_n]$. We introduce recursively the following operations on $K[Y_\infty]$:

(5.5.1) $\qquad y \prec z := y_1 \vee (y_2 * z),$

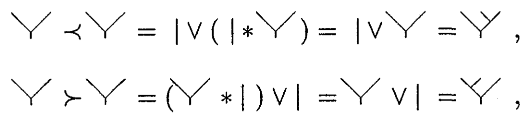
(5.5.2) $\qquad y \succ z := (y * z_1) \vee z_2,$

(5.5.3) $\qquad y * z := y \prec z + y \succ z,$

(5.5.4) $\qquad x \prec | := x =: | \succ x$ and $x \succ | := 0 =: | \prec x$, for $x \neq |$.

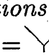
for $y \in Y_p$ and $z \in Y_q$. Observe that $|$ is a unit for $*$.

Since the decomposition $y = y_1 \vee y_2$ is unique, it is clear that these formulas are well-defined by recursion. For instance



or, equivalently, $[1] \prec [1] = [21]$, $\quad [1] \succ [1] = [12]$.

These operations are extended to $K[Y_\infty]$ by linearity. Observe that $[0] * [0] = [0]$, but $[0] \prec [0]$ and $[0] \succ [0]$ are not defined.

5.6. Lemma. *The vector space $\oplus_{n\geq 1} K[Y_n]$ equipped with the two operations $\prec$ and $\succ$ described above is a dendriform algebra, which is generated by $[1] =$*  .

Proof. We prove this assertion by induction on the (total) degree of the trees.

Let $y = y_1 \vee y_2$, $y' = y_1' \vee y_2'$ and $y'' = y_1'' \vee y_2''$ be planar binary trees. The following equalities follow by induction from the definitions of the operations and the associativity of $*$:

$$
\begin{aligned}
(i) \quad & (y \prec y') \prec y'' = (y_1 \vee (y_2 * y')) \prec y'' \\
& = y_1 \vee (y_2 * y' * y'') = y \prec (y' \prec y'') \; + \; y \prec (y' \succ y'').
\end{aligned}
$$

$$
\begin{aligned}
(ii) \quad & y \succ (y' \prec y'') = y \succ (y_1' \vee (y_2' * y'')) = (y * y_1') \vee (y_2' * y'') \\
& = ((y * y_1') \vee y_2') \prec y'' = (y \succ y') \prec y''.
\end{aligned}
$$

$$
\begin{aligned}
(iii) \quad & y \succ (y' \succ y'') = y \succ ((y' * y_1'') \vee y_2'') \\
& = (y * y' * y_1'') \vee y_2'' = (y \prec y') \succ y'' + (y \succ y') \succ y''.
\end{aligned}
$$

Let us show that $\overline{K[Y_\infty]}$ is generated by $[1]$ under the operations $\prec$ and $\succ$. Let $y = y_1 \vee y_2$ be a tree. From the definitions of the operations we have

$$
\begin{aligned}
y_1 \vee y_2 &:= [1] && \text{if } y_1 = [0] = y_2, \\
&:= [1] \prec y_2 && \text{if } y_1 = [0] \neq y_2, \\
&:= y_1 \succ [1] && \text{if } y_1 \neq [0] = y_2, \\
&:= y_1 \succ [1] \prec y_2 && \text{if } y_1 \neq [0] \neq y_2.
\end{aligned}
$$

Therefore, by induction, it is clear that $\overline{K[Y_\infty]}$ is generated by $[1]$. $\square$

5.7. Proposition. *The unique dendriform algebra map $Dend(K) \to \oplus_{n\geq 1} K[Y_n]$ which sends the generator x of $Dend(K)$ to $[1]$ is an isomorphism.*

Proof. Let us show that the dendriform algebra $(\overline{K[Y_\infty]}, \prec, \succ)$ defined in 5.5 satisfies the universal condition to be the free dendriform algebra on one generator.

Let D be a dendriform algebra and a an element in D. Define a linear map $\alpha : \overline{K[Y_\infty]} \to D$ by its value on the trees $y = y_1 \vee y_2$ as follows :

$$\begin{aligned}\alpha(y_1 \vee y_2) &:= a & \text{if } y_1 = [0] = y_2,\\ &:= a \prec \alpha(y_2) & \text{if } y_1 = [0] \neq y_2,\\ &:= \alpha(y_1) \succ a & \text{if } y_1 \neq [0] = y_2,\\ &:= \alpha(y_1) \succ a \prec \alpha(y_2) & \text{if } y_1 \neq [0] \neq y_2.\end{aligned}$$

We claim that α is a morphism of dendriform algebras. The proof is by induction on the degree of the tree. Indeed, on one hand

$$\begin{aligned}\alpha(y \prec z) &= \alpha(y_1 \vee (y_2 * z))\\ &= \alpha(y_1) \succ a \prec \alpha(y_2 * z)\\ &= \alpha(y_1) \succ a \prec (\alpha(y_2) * \alpha(z)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\alpha(y) \prec \alpha(z) &= (\alpha(y_1) \succ a \prec \alpha(y_2)) \prec \alpha(z)\\ &= ((\alpha(y_1) \succ a) \prec \alpha(y_2)) \prec \alpha(z)\\ &= (\alpha(y_1) \succ a) \prec (\alpha(y_2) * \alpha(z))\\ &= \alpha(y_1) \succ a \prec (\alpha(y_2) * \alpha(z)).\end{aligned}$$

Here we supposed that $y_1 \neq [0] \neq y_2$, but the proof is similar for the other cases.

Since by lemma 5.6 $\overline{K[Y_\infty]}$ is generated by $[1]$, the morphism α such that $\alpha([1]) = a$ is unique.

It follows that $(\overline{K[Y_\infty]}, \prec, \succ)$ is the free dendriform algebra on one generator. $\square$

5.8. Theorem (Free dendriform algebra). *The unique dendriform algebra map*

$$Dend(V) \to \oplus_{n\geq 1} K[Y_n] \otimes V^{\otimes n}$$

which sends the generator $v \in V$ to $[1] \otimes v$ is an isomorphism.

Proof. Define the dendriform algebra structure on $\oplus_{n\geq 1} K[Y_n] \otimes V^{\otimes n}$ by

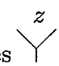
$$\begin{aligned}y \otimes \omega \prec y' \otimes \omega' &:= (y \prec y') \otimes \omega\omega',\\ y \otimes \omega \succ y' \otimes \omega' &:= (y \succ y') \otimes \omega\omega'.\end{aligned}$$

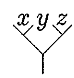
Since in the relations defining a dendriform algebra the variables stay in the same order, the free dendriform algebra over V is completely determined by the free dendriform algebra on one generator:

$$Dend(V) = \bigoplus_{n\geq 1} Dend(K)_n \otimes V^{\otimes n},$$

where $Dend(K)_n$ is the subspace of $Dend(K)$ generated by all the possible products of n copies of the generator. Hence, by proposition 5.7, one gets $Dend(V) \cong \oplus_n K[Y_n] \otimes V^{\otimes n}$. $\square$

5.9. Remark. The inverse isomorphism is obtained as follows. From $y \in Y_n$ we construct a monomial in the variables $x_1, \cdots, x_n$ by first putting the variable x_i in between the leaves $i-1$ and i. Then, for each vertex of depth one, we replace the local patterns with two vertices by local patterns with one vertex:

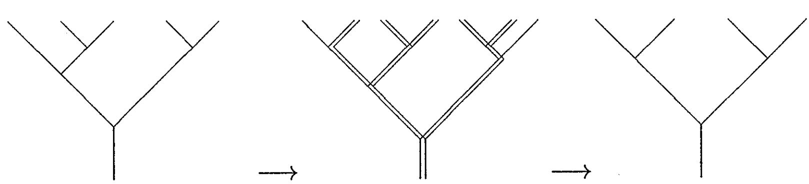
Continue the process until one reaches  . The element $z \in Dend(V)$ is the image of $(y; x_1 \otimes \cdots \otimes x_n) \in K[Y_n] \otimes V^{\otimes n}$. Observe that sometimes one needs to choose an order to perform the process. But, thanks to the second axiom of dendriform algebras, the result does not depend on this choice:

 $\mapsto (x \succ y) \prec z = x \succ (y \prec z)$.

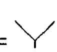
Observe that the right S_n-module $Dend(n)$ which is such that $Dend(V) = \oplus_n Dend(n) \otimes_{S_n} V^{\otimes n}$, is the regular representation $K[Y_n] \otimes K[S_n]$.

5.10. Nested sub-trees and quotients. Given a planar binary tree with $n+1$ leaves and a consecutive sequence of $k+1$ leaves $\{i, \ldots, i+k\}$, the sub-tree of y which contains these leaves is isomorphic to a unique planar binary tree with $k+1$ leaves, which we denote by y'. We say that the sub-tree y' is *nested* in y at i. By definition the quotient $y'' = y/y'$ is the planar binary tree with $n-k+3$ leaves obtained from y by removing the leaves $\{i+1, \ldots, i+k-1\}$.

Example with $n = 6, i = 1, k = 4$ and $y = [131612]$:



y' $\longrightarrow$ y $\longrightarrow$ y''

Observe that y is a sub-tree of itself with quotient $[1] =$  , and that there are n nested sub-trees of y of the form $[1]$ (with quotient y).

In terms of the permutation-like notation the names of y' and of y'' are obtained as follows. Let $y = [a_1 \ldots a_n]$. Start with the sequence of integers $[a_{i+1} \ldots a_{i+k}]$ and make it into a name of tree by first replacing the largest integer by k. Then proceed the same way with the two remaining intervals, and so forth (like in A3). Ultimately one gets the name of y'. In our example we get $[2141]$. The quotient tree y'' of y by y' is obtained by making the sequence of numbers $[a_1 \ldots a_i\ a_j\ a_{i+k+1} \ldots a_n]$ into a name of tree as before (a_j is the largest integer in $[a_{i+1} \ldots a_{i+k}]$). In our example we get $[131]$.

5.11. Proposition. *Under the isomorphism of Proposition 5.7 the composition operation in Dend(K) induces the following composition operation in* $\oplus_{n\geq 1}K[Y_n]$*:*

$$y'' \circ_i y' = \sum y,$$

where the sum is extended over all the trees y *which contain* y' *as a nested sub-tree at* i *and for which* $y/y' = y''$*.*

Proof. Since the operad is quadratic it suffices to check this assertion in low dimension. The isomorphism gives $[21] = x \prec x$ and $[12] = x \succ x$. The eight distinct cases of composition are:

$$\begin{aligned}
&[21] \circ_1 [21] = (x \prec x) \prec x = x \prec (x \prec x) + x \prec (x \succ x) = [213] + [312]\\
&[21] \circ_1 [12] = (x \succ x) \prec x = [131]\\
&[21] \circ_2 [21] = x \prec (x \prec x) = [321]\\
&[21] \circ_2 [12] = x \prec (x \succ x) = [312]\\
&[12] \circ_1 [21] = (x \prec x) \succ x = [213]\\
&[12] \circ_1 [12] = (x \succ x) \succ x = [123]\\
&[12] \circ_2 [21] = x \succ (x \prec x) = [131]\\
&[12] \circ_2 [12] = x \succ (x \succ x) = x \prec (x \succ x) + x \succ (x \succ x) = [213] + [312]
\end{aligned}$$

In each case we verify that the trees of the right hand side are precisely such that y' is nested at i with quotient y''. For instance both [213] and [312] have [21] nested at 1. □.

5.12. Associative algebra structure on $K[Y_\infty]$ and $K[S_\infty]$. By lemma 5.2 the vector space $K[Y_\infty]$ is an associative algebra for the product

$$x * y = x \prec y + x \succ y,$$

hence $K[Y_\infty]$ is a graded associative and unital algebra whose structure is completely determined by the following two conditions: $|$ is a unit, the recursive formula

$$y * z = y_1 \vee (y_2 * z) + (y * z_1) \vee z_2,$$

holds.

Observe that this algebra has an obvious involution: $[i_1, \ldots, i_n] \mapsto [i_n, \ldots, i_1]$ on trees. In theorem 3.8 of [LR] we prove that it is isomorphic to the tensor algebra over $K[Y_{0,\infty}]$, where $Y_{0,n-1} := \{\, y = | \vee y' \in Y_n, \text{ for } y' \in Y_{n-1}\}$. It is also isomorphic to $T(K[Y_{\infty,0}])$, where $Y_{n-1,0} := \{\, y = y' \vee |\}$.

The map $\psi_n : S_n \to Y_n$ (cf. Appendix A) induces a linear map $\psi : K[S_\infty] \to K[Y_\infty]$. There is an associative and unital algebra structure on $K[S_\infty]$ given by

$$x * y := sh_{n,m} \cdot (x \times y) \in K[S_{n+m}],$$

where $x \in S_n$, $y \in S_m$ and $sh_{n,m} \in K[S_{n+m}]$ is the sum of all the (n,m)-shuffles. It is proved in [LR] that ψ is an associative algebra homomorphism.

5.13. Hopf structure on $Dend(V)$. It is well-known that the free associative algebra $T(V)$ is in fact a cocommutative Hopf algebra, where the coproduct is given by the shuffle. Similarly there exists a structure of Hopf algebra on the associative unital algebra $K \oplus Dend(V) = \oplus_{n\geq 0} K[Y_n] \otimes V^{\otimes n}$. The coproduct is completely determined by the shuffles and the coproduct on $\oplus_{n\geq 0} K[Y_n]$ was constructed in [LR]. Observe that this coproduct is not cocommutative. It is related to the "brace algebras" and has been studied in details in [R].

6. (CO)HOMOLOGY OF DENDRIFORM ALGEBRAS

In this section we show that there exists a chain complex (of Hochschild type), for any dendriform algebra. It enables us to construct a homology and a cohomology theory for dendriform algebras. It will be proved in section 8 that these theories are the ones predicted by the operad theory in characteristic zero.

6.1. The chain complex of a dendriform algebra. Let E be a dendriform algebra and let C_n be the set $\{1, \cdots, n\}$. We define the module of n-chains of E as

$$C_n^{Dend}(E) := K[C_n] \otimes E^{\otimes n},$$

and the differential $d = -\sum_{i=1}^{n-1} (-1)^i d_i : C_n^{Dend}(E) \to C_{n-1}^{Dend}(E)$ as follows. First, we define the face operators $d_i, 1 \leq i \leq n-1$, on $r \in C_n$ by

$$d_i(r) = \begin{cases} r-1 & \text{if } i \leq r-1, \\ r & \text{if } i \geq r. \end{cases}$$

These maps are extended linearly to maps

$$d_i : K[C_n] \to K[C_{n-1}],\ 1 \leq i \leq n-1.$$

Second, we define the symbol $\circ_i^r$ as follows:

$$\circ_i^r = \begin{cases} * & \text{if } i < r-1, \\ \succ & \text{if } i = r-1, \\ \prec & \text{if } i = r, \\ * & \text{if } i > r. \end{cases}$$

Recall that $x * y = x \prec y + x \succ y$.

Finally the map $d_i : C_n^{Dend}(E) \to C_{n-1}^{Dend}(E)$ is given by

$$d_i(r; x_1 \otimes \cdots x_n) := (d_i(r); x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i \circ_i^r x_{i+1} \otimes \cdots \otimes x_n)$$

for $1 \leq i \leq n-1$.

6.2. Lemma. *The maps $d_i : C_n^{Dend}(E) \to C_{n-1}^{Dend}(E)$ satisfy the simplicial relations $d_i d_j = d_{j-1} d_i$, for $i < j$, and so $(C_*^{Dend}(E), d)$ is a chain complex.*

Proof. Let us first prove the lowest dimensional case, that is $d_1 d_1 = d_1 d_2$ on $C_3^{Dend}(E) = 3\ E^{\otimes 3}$.

On the first component ($r = 1$) one gets:

$$d_1 d_1(1; a \otimes b \otimes c) = d_1(1; (a \prec b) \otimes c) = (1; (a \prec b) \prec c),$$
$$d_1 d_2(1; a \otimes b \otimes c) = d_1(1; a \otimes (b * c)) = (1; a \prec (b * c)),$$

hence $d_1 d_1 = d_1 d_2$ by axiom (i).

On the second component ($r = 2$) one gets:

$$d_1 d_1(2; a \otimes b \otimes c) = d_1(1; (a \succ b) \otimes c) = (1; (a \succ b) \prec c),$$
$$d_1 d_2(2; a \otimes b \otimes c) = d_1(2; a \otimes (b \prec c)) = (1; a \succ (b \prec c)),$$

hence $d_1 d_1 = d_1 d_2$ by axiom (ii).

On the third component ($r = 3$) one gets:

$$d_1 d_1(3; a \otimes b \otimes c) = d_1(2; (a * b) \otimes c) = (1; (a * b) \succ c),$$
$$d_1 d_2(3; a \otimes b \otimes c) = d_1(2; a \otimes (b \succ c)) = (1; a \succ (b \succ c)),$$

hence $d_1 d_1 = d_1 d_2$ by axiom (iii).

Higher up, the verification of $d_i d_j = d_{j-1} d_i$ splits up into two different cases. First, if $j = i+1$, then it is the same kind of computation as above, so it is a consequence of the axioms of a dendriform algebra. Second, if $j > i+1$, then both operations agree on C_n (direct checking) and the image of $(a_1 \otimes \cdots \otimes a_n)$ is, in both cases,

$$(a_1 \otimes \cdots \otimes a_i \circ_i^r a_{i+1} \otimes \cdots \otimes a_j \circ_j^r a_{j+1} \otimes \cdots \otimes a_n).$$

□

6.3. The chain bicomplex of a dendriform algebra. Observe that $C_*^{Dend}(E)$ is in fact the total complex of a bicomplex. Indeed, let

$$C_{p,q}^{Dend}(E) = \{p\} E^{\otimes p+q} \quad \text{for} \quad p \geq 1, q \geq 0.$$

The map

$$d^h : C_{p,q}^{Dend}(E) \to C_{p-1,q}^{Dend}(E) \quad \text{by} \quad d^h := \sum_{i=1}^{p-1} (-1)^i d_i,$$

is well-defined because $d_i(p) = p - 1$, when $i \leq p - 1$, and

$$d^v : C_{p,q}^{Dend}(E) \to C_{p,q-1}^{Dend}(E) \quad \text{by} \quad d^v := \sum_{i=p}^{p+q} (-1)^i d_i.$$

is well-defined because $d_i(p) = p$, when $i \geq p$. In other words, the p-th component of $C_n^{Dend}(E)$ is put in bidegree $(p, n-p)$.

6.4. (Co)homology of dendriform algebras. By definition the homology (with trivial coefficients) of a dendriform algebra E is

$$H_*^{Dend}(E) := H_*(C_*^{Dend}(E), d) \ ,$$

and the cohomology of a dendriform algebra E (with trivial coefficients) is

$$H_{Dend}^*(E) := H^*(\mathrm{Hom}(C_*^{Dend}(E), K)).$$

Let us use freely the interpretation of the preceding results in terms of operads as devised in the next section. From Koszul duality and Appendix B5d, the graded module $H_{Dend}^*(E)$ is naturally equipped with a structure of graded dialgebra (and hence a structure of graded Leibniz algebra).

6.5. Theorem. *The dendriform algebra homology of a free dendriform algebra is trivial. More precisely*

$$H_n^{Dend}(Dend(V)) = 0 \ for \ n > 0,$$
$$H_1^{Dend}(Dend(V)) = V.$$

Proof. Since we know already that the analogous theorem for associative dialgebras is true (Theorem 3.8), it is a consequence (by the operad theory, cf. Appendix B), of the operad duality between *Dias* and *Dend* proved in Proposition 8.3. □

7. ZINBIEL ALGEBRAS, DENDRIFORM ALGEBRAS AND HOMOLOGY

In this section we introduce Zinbiel (i.e. dual-Leibniz) algebras and we compare them with dendriform algebras. In particular we compute the natural map from a free dendriform algebra to the free Zinbiel algebra, considered as a dendriform algebra. Finally we compare the homology theories.

7.1. Zinbiel algebras [L3]. A *Zinbiel algebra* R (*) (also called *dual-Leibniz algebra*) is a module over K equipped with a binary operation $(x, y) \mapsto x \cdot y$, which satisfies the identity

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) + x \cdot (z \cdot y) \ , \quad for \ all \ x, y, z \in R. \tag{7.1.1}$$

The category of Zinbiel algebras is denoted **Zinb**. The free Zinbiel algebra over the vector space V is $\overline{T}(V) = \oplus_{n\geq 1} V^{\otimes n}$ equipped with the following product

$$(x_0 \ldots x_p) \cdot (x_{p+1} \ldots x_{p+q}) = x_0 sh_{p,q}(x_1 \ldots x_{p+q})$$

(*) Terminology proposed by J.-M. Lemaire where $sh_{p,q}$ is the sum over all (p,q)-shuffles. We denote it by $Zinb(V)$. Observe that $Zinb(V) = \bigoplus_n Zinb(n) \otimes_{S_n} V^{\otimes n}$ for $Zinb(n) = K[S_n]$. It is immediate to check that the symmetrized product

$$xy := x \cdot y + y \cdot x \tag{7.1.2}$$

is associative (cf. [R], [L3]), so, under the symmetrized product, R becomes an associative and commutative algebra. This construction gives a functor

$$\mathbf{Zinb} \xrightarrow{+} \mathbf{Com}.$$

Let us construct functors to and from the category of dendriform algebras **Dend**.

7.2. Lemma. *Let R be a Zinbiel algebra and put*

$$x \prec y := x \cdot y \ , \ x \succ y := y \cdot x \ , \ \forall\, x, y \in R.$$

Then $(R, \prec, \succ)$ is a dendriform algebra denoted R_{Dend}. Conversely, a commutative dendriform algebra (i.e. a dendriform algebra for which $x \succ y = y \prec x$) is a Zinbiel algebra.

Proof. Indeed, relation (i) is exactly relation (3.3.1) and so is relation (iii). Relation (ii) also follows from (3.3.1) since

$$x \succ (y \prec z) = (yz)x, \quad (x \succ y) \prec z = (yx)z,$$

and the relation $(y \cdot z) \cdot x = (y \cdot x) \cdot z$ follows from (3.3.1) for y, z, x and y, x, z. $\square$

So we have constructed a functor

$$\mathbf{Zinb} \to \mathbf{Dend}.$$

From Lemma 5.2 we get a functor

$$\mathbf{Dend} \xrightarrow{+} \mathbf{As}.$$

Summarizing we get the following

7.4. Proposition. *The following diagram of functors between categories of algebras is commutative*

$$\begin{array}{ccc} \mathbf{Zinb} & \hookrightarrow & \mathbf{Dend} \\ \downarrow + & & \downarrow + \\ \mathbf{Com} & \hookrightarrow & \mathbf{As} \end{array}$$

Proof. The commutativity of the diagram is immediate since for a Zinbiel algebra the associated associative products are equal :

$$x * y = x \prec y + x \succ y = x \cdot y + y \cdot x = xy$$

□

7.5. Theorem. *For any vector space V, the natural map of dendriform algebras $Dend(V) \to Zinb(V)_{Dend}$ is induced, in degree n, by the map*

$$K[Y_n] \otimes K[S_n] \to K[S_n], \quad y \otimes \omega \mapsto \sum_{\{\sigma \in S_n \ | \ \psi'(\sigma) = y\}} \sigma\omega \ ,$$

where $\psi' : S_n \twoheadrightarrow Y_n$ is the surjective map described in Appendix A6.

Proof. First, observe that the map $V \to Zinb(V)$ gives a map (the same on the underlying vector spaces) $V \to Zinb(V)_{Dend}$. Since this latter object is a dendriform algebra, the map factors through the free dendriform algebra on V, whence a natural dialgebra map $\phi : Dend(V) \to Zinb(V)_{Dend}$.

The proof will be done by induction on n.

Restricting ϕ to the degree n part gives a commutative square (cf. 5.8 and 5.9):

$$\begin{array}{ccc} K[Y_n] \otimes V^{\otimes n} & \xrightarrow{\phi_n} & V^{\otimes n} \\ \cap & & \cap \\ Dend(V) & \xrightarrow{\phi} & Zinb(V)_{Dend} \ . \end{array}$$

For $n = 1$, ϕ_1 is clearly the identity of V.

For $n = 2$, $\phi_2 : K[Y_2] \otimes V^{\otimes 2} \to V^{\otimes 2}$ is given by

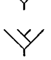
 $\otimes (x, y) = x \prec y \mapsto x \cdot y = [12].(x, y),$

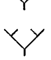
 $\otimes (x, y) = x \succ y \mapsto y \cdot x = [21].(x, y),$

so $\mapsto [12]$ and $\mapsto [21]$, which is the map ψ' of Appendix A6.

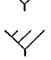
For n=3, one has

 $\otimes (x, y, z) = x \prec (y \prec z) \mapsto x \cdot (y \cdot z) = [123].(x, y, z),$

 $\otimes (x, y, z) = x \prec (y \succ z) \mapsto x \cdot (z \cdot y) = [132].(x, y, z),$

 $\otimes (x, y, z) = (x \succ y) \prec z \mapsto (y \cdot x) \cdot z = y \cdot (x \cdot z + z \cdot x)$
$= ([213] + [312]).(x, y, z),$

 $\otimes (x, y, z) = (z \prec x) \succ y \mapsto x \cdot (y \cdot z) = [231].(x, y, z),$

 $\otimes (x, y, z) = (z \succ y) \succ x \mapsto x \cdot (y \cdot z) = [321].(x, y, z),$

so the map is precisely ψ'.

Let us now prove it for any n. We suppose that the theorem has been proved for any $p < n$. We are going to prove it for n. We use the *grafting* operation on trees (cf. Appendix A).

The element $(y_1 \vee y_2; x_1 \dots x_{p+q+1})$ can be written as

$$(y_1; x_1 \dots x_p) \succ x_{p+1} \prec (y_2; x_{p+2} \dots x_{p+q+1})$$

in $Dend(V)$ when $p \geq 1, q \geq 1$. Here x_{p+1} stands for $([1]; x_{p+1})$. We do not need to put any parenthesis because of the second relation of dendriform algebras. The image of this element under ϕ is

$$\begin{aligned}(x_{p+1} \cdot \phi_q(y_2; x_{p+2} \dots x_{p+q+1})) \cdot \phi_p(y_1; x_1 \dots x_{p+q+1}) \\ = x_{p+1} \cdot (\phi_q(y_2; x_{p+2} \dots x_{p+q+1}) \cdot \phi_p(y_1; x_1 \dots x_{p+q+1}) \\ + \phi_p(y_1; x_1 \dots x_{p+q+1}) \cdot \phi_q(y_2; x_{p+2} \dots x_{p+q+1})).\end{aligned}$$

By induction we know that $\phi_p(y_1; x_1 \dots x_p) = \sum_{\sigma_1 \in \psi'^{-1}(y_1)} \sigma_1(x_1 \dots x_p)$ and that $\phi_q(y_2; x_{p+2} \dots x_{p+q+1}) = \sum_{\sigma_2 \in \psi'^{-1}(y_2)} \sigma_2(x_{p+2} \dots x_{p+q+1})$. For $z_i \in V$ one has, in the free Zinbiel algebra, the equality

$$z_0 \cdot (z_1 \dots z_p \cdot z_{p+1} \dots z_{p+q} + z_{p+1} \dots z_{p+q} \cdot z_1 \dots z_p) = \sum_{\tau} \tau(z_0 \dots z_{p+q}),$$

where the sum is extended over all the (p,q)-shuffles τ (acting on the set $\{1, \dots, p+q\}$. Hence we get

$$\phi_n(y_1 \vee y_2; x_1 \dots x_n) = \sum_{\pi} \pi(x_1 \dots x_n),$$

where the permutation π is of the following form :

$$\begin{aligned}&\pi(i) = \tau\sigma_1(i) \text{ for } 1 \leq i \leq p, \\ &\pi(p+1) = 1, \\ &\pi(i) = \tau(p + \sigma_2(i)) \text{ for } p+2 \leq i \leq p+q+1,\end{aligned}$$

where $\sigma_1 \in \psi'^{-1}(y_1)$, $\sigma_2 \in \psi'^{-1}(y_2)$ and τ is a (p,q)-shuffle.

On the other hand one easily checks that all the permutations σ which belong to $\psi'^{-1}(y_1 \vee y_2)$ are precisely obtained by choosing such a σ_1 and such a σ_2, and then shuffle the associated levels. So we have proved the formula for n.

In the preceding proof we assumed that $p \geq 1, q \geq 1$. We let to the reader the task of modifying the proof when either $p = 0$ or $q = 0$. □

7.6. Comparison of homology theories. As mentioned in Appendix B, for any algebra over a Koszul operad, there is a small chain complex, modelled on the dual operad, whose homology is the homology of the algebra. For Zinbiel algebras it takes the following form (cf. [Li2]):

$$C_*^{Zinb}(R): \qquad \cdots \longrightarrow R^{\otimes n} \xrightarrow{d} R^{\otimes n-1} \xrightarrow{d} \cdots \xrightarrow{d} R^{\otimes 2} \xrightarrow{\cdot} R$$

where

$$d(x_1,\ldots,x_n) = (x_1 \cdot x_2, x_3, \ldots, x_n) + \sum_{i=2}^{n-1}(-1)^{i-1}(x_1,\ldots,x_i * x_{i+1},\ldots,x_n).$$

The homology groups of this complex are denoted $H_n^{Zinb}(R)$, for $n \geq 1$. By Appendix B5e, there is a natural map of complexes

$$C_*^{Dend}(R_{Dend}) \longrightarrow C_*^{Zinb}(R)$$

inducing

$$H_*^{Dend}(R_{Dend}) \longrightarrow H_*^{Zinb}(R).$$

Let us describe explicitly the chain complex map.

7.7. Proposition. *Let $\theta_n^r \in K[S_n]$ be the elements defined recursively by the formulas :*

$$\theta_1^1(1) = (1), \qquad \theta_n^r(1,\ldots,n) = (r, \tilde{\theta}_n^r(1,\ldots,\widehat{r}\ldots,n),$$
$$\tilde{\theta}_n^r(1,\ldots,n-1) = (r-1, \tilde{\theta}_{n-1}^{r-1}(1,\ldots,\widehat{r-1}\ldots,n) + (r, \tilde{\theta}_{n-1}^r(1,\ldots,\widehat{r}\ldots,n).$$

In particular, $\theta_n^1(1,\ldots,n) = (1,\ldots,n)$ and $\theta_n^n(1,\ldots,n) = (n,\ldots,1)$. The map $\Theta_n : K[C_n] \otimes R^{\otimes n} \to R^{\otimes n}$ defined by

$$\Theta_n([r] \otimes (x_1, \cdots x_n)) := \theta_n^r(x_1, \cdots x_n)$$

is the chain complex map $\Theta_* : C_*^{Dend}(R_{Dend}) \longrightarrow C_*^{Zinb}(R)$ induced by the operad morphism $Dend \to Zinb$.

Proof. This is a consequence of the explicit description of the morphism of Leibniz algebras $Leib(V) \to Dias(V)_{Leib}$, cf. Appendix B5e.

Observe that θ_n^r is the sum of the signed action of $\binom{n}{r}$ permutations. Since $\sum\binom{n}{r} = 2^n$, we recover (up to permutation) the 2^n monomials of $[x_1,[x_2,[\ldots,x_n]]]$ (cf. 4.10). $\square$

8. KOSZUL DUALITY FOR THE DIALGEBRA OPERAD

In this section we show that the operad associated to dendriform algebras is dual, in the operadic sense, to the operad associated to dialgebras. Moreover we use the results of section 4 to show that the operad of associative dialgebras is a Koszul operad (and so is the operad of dendriform algebras). The reader not familiar with the notions of operad and Koszul duality may have a look at Appendix B, from which we take the notation.

8.1. The associative dialgebra operad. A dialgebra is determined by two operations (left and right product) on two variables and by relations which make use of the composition of two such operations. Hence the operad *Dias* associated to the notion of dialgebra is a binary (operations on two variables) quadratic (relations involving two operations) operad. Moreover, there is no symmetry property for these operations, and, in the relations, the variables stay in the same order. Hence the operad is a *non-Σ-operad*, that is, as a representation of S_n, the space $Dias(n)$ is a sum of copies of the regular representation. It was proved in 2.5 that the free dialgebra over the vector space V is $Dias(V) = T(V) \otimes V \otimes T(V)$. The degree n part of it is $\bigoplus_{i+1+j=n} V^{\otimes i} \otimes V \otimes V^{\otimes j}$. Hence the operad *Dias* is such that

$$Dias(n) = nK[S_n] \qquad (n \text{ copies of the regular representation}).$$

In particular $Dias(1) = K$ (the only unary operation on a dialgebra is the identity), $E := Dias(2) = E' \otimes K[S_2]$, where E' is 2-dimensional generated by $\dashv$ and $\vdash$. The space $\mathrm{Ind}_{S_2}^{S_3}(E \otimes E)$ is the sum of 8 copies of the regular representation of S_3. Each copy corresponds to a choice of parenthesizing: $(- \circ_1 (- \circ_2 -))$ or $((- \circ_1 -) \circ_2 -)$, and a choice for the two operations $\circ_1$ and $\circ_2$. The space of relations $R \subset \mathrm{Ind}_{S_2}^{S_3}(E \otimes E)$ is of the form $R' \otimes K[S_3]$, where R' is the subspace of $E'^{\otimes 2} \oplus E'^{\otimes 2}$ determined by the relations 1 to 5 of a dialgebra (cf. 2.1).

8.2. The dendriform algebra operad. Analogously the operad *Dend* associated to the notion of dendriform algebra is binary and quadratic, and is a *non-Σ-operad* since there is no symmetry for the operations and, in the relations, the variables stay in the same order. It was proved in 5.7 that the free dendriform algebra over the vector space V is $Dend(V) = \bigoplus_{n\geq 1} K[Y_n] \otimes V^{\otimes n}$. Hence the operad *Dend* is such that

$$Dend(n) = K[Y_n] \otimes K[S_n].$$

In particular $Dend(1) = K, Dend(2) = F' \otimes K[S_2]$, where F' is 2-dimensional generated by $\prec$ and $\succ$. The space

$$\mathrm{Ind}_{S_2}^{S_3}(Dend(2) \otimes Dend(2)) \cong (F'^{\otimes 2} \oplus F'^{\otimes 2}) \otimes K[S_3]$$

is the sum of 8 copies of the regular representation of S_3. The space of relations is of the form $S' \otimes K[S_3]$, where $S' \subset (F'^{\otimes 2} \oplus F'^{\otimes 2})$ is the subspace determined by the 3 relations of a dendriform algebra (cf. 5.1).

8.3. Proposition. *The operad Dend of dendriform algebras is dual, in the operad sense, to the operad Dias of dialgebras :* $Dias^! = Dend$.

Proof. Let us identify $F' = E'^{\vee}$ with E' by identifying the basis $(\prec, \succ)$ with the basis $(\dashv, \vdash)$. Since $R = R' \otimes K[S_3]$, the space of relations for the dual operad is of the form $R'^{ann} \otimes K[S_3]$, where, according to Proposition B3, R'^{ann} is the annihilator of R'. Recall from Proposition B3 that the scalar product on $E'^{\otimes 2} \oplus E'^{\otimes 2}$ is given by the matrix $\begin{bmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{bmatrix}$.

With obvious notation, the subspace R' is determined by the relations

$$\begin{cases} (\dashv, \dashv)_2 - (\dashv, \dashv)_1 = 0, \\ (\dashv, \dashv)_1 - (\dashv, \vdash)_2 = 0, \\ (\vdash, \dashv)_1 - (\vdash, \dashv)_2 = 0, \\ (\dashv, \vdash)_1 - (\vdash, \vdash)_2 = 0, \\ (\vdash, \vdash)_2 - (\vdash, \vdash)_1 = 0. \end{cases}$$

It is immediate to verify that its annihilator $R'^{\perp}$ with respect to the given scalar product is the subspace determined by the relations

$$\begin{cases} (\dashv, \dashv)_1 - (\dashv, \dashv)_2 - (\dashv, \vdash)_2 = 0, \\ (\vdash, \dashv)_1 - (\vdash, \dashv)_2 = 0, \\ (\vdash, \vdash)_2 - (\dashv, \vdash)_1 - (\vdash, \vdash)_1 = 0. \end{cases}$$

This is precisely the relations of dendriform algebras (once we have changed $\dashv$ into $\prec$ and $\vdash$ into $\succ$). □

8.4. Proposition. *The chain complex $CY_*(D)$ associated to a dialgebra D is the chain complex of D in the operad sense (cf. Appendix B4). Hence HY is the (co)homology theory for dialgebras predicted by the operad theory.*

Proof. From the theory of operads recalled in Appendix B we have

$$C_n^{Dias}(D) = Dend(n)^{\vee} \otimes_{S_n} D^{\otimes n}.$$

By Proposition 5.7 we get

$$C_n^{Dias}(D) = K[Y_n] \otimes K[S_n] \otimes_{S_n} D^{\otimes n} = K[Y_n] \otimes D^{\otimes n} = CY_n(D).$$

So, it suffices to prove that the boundary operator d of $CY_*(D)$ agrees with the dialgebra structure of D on $CY_2(D)$ and that it is a coalgebra derivation.

The boundary map on $CY_2(D) = D^{\otimes 2} \oplus D^{\otimes 2}$ is given by $x \otimes y \mapsto x \dashv y$ on the first component and by $x \otimes y \mapsto x \vdash y$ on the second one. So the first condition is fulfilled. Checking the coderivation property is analogous to the associative case (cf. B4). □

8.5. Theorem. *The operad Dias of dialgebras is a Koszul operad, and so is the operad Dend of dendriform algebras.*

Proof. From the definition of a Koszul operad given in B4, this theorem follows from the vanishing of $H_*^{Dias} = HY_*$ of a free dialgebra, as proved in Theorem

3.8. Since $\mathcal{P}$ being Koszul implies $\mathcal{P}^!$ is Koszul (cf. Appendix B5b), the operad *Dend* is Koszul. □

Observe that this theorem, together with the general property of Koszul operads recalled in B4a, implies proposition 5.2.

8.6. Poincaré series. From the description of the operad *Dias* it follows that its Poincaré series is

$$g_{Dias}(x) = \sum_{n\geq 1}(-1)^n n n! \frac{x^n}{n!} = \sum_{n\geq 1}(-1)^n n x^n = \frac{-x}{(1+x)^2}.$$

On the other hand, the Poincaré series of *Dend* is

$$g_{Dend}(x) = \sum_{n\geq 1}(-1)^n c_n n! \frac{x^n}{n!} = \sum_{n\geq 1}(-1)^n c_n x^n = \frac{-1-2x+\sqrt{1+4x}}{2x}.$$

As expected (cf. Appendix B5c) we verify that $g_{Dend}(g_{Dias}(x)) = x$.

8.7. Operad morphisms. Any morphism of operads induces a functor between the associated categories of algebras. Taking the dual gives also a morphism, but in the other direction. For instance the dual of the inclusion of the category of commutative algebras into the category of associative algebras is the "-" functor which transforms an associative algebra into a Lie algebra. All the functors between categories of algebras that we met in the previous sections come from morphisms of operads. They assemble into the following commutative diagram of functors (cf. Proposition 4.4 and Proposition 7.4):

$$\begin{tikzcd}
 & \mathbf{Dend} \arrow[dr, "+"] & & \mathbf{Dias} \arrow[dr, "-"] & \\
\mathbf{Zinb} \arrow[ur, "inc"] \arrow[dr, "+"'] & & \mathbf{As} \arrow[ur, "inc"] \arrow[dr, "-"'] & & \mathbf{Leib} \\
 & \mathbf{Com} \arrow[ur, "inc"'] & & \mathbf{Lie} \arrow[ur, "inc"'] &
\end{tikzcd}$$

The symmetry (around a vertical axis passing through $\mathbf{As}^! = \mathbf{As}$) reflects the Koszul duality of quadratic operads : $Lie^! = Com, Dias^! = Dend, Leib^! = Zinb$.

9. STRONG HOMOTOPY ASSOCIATIVE DIALGEBRAS

Strong homotopy $\mathcal{P}$-algebras are governed by the Koszul dual operad $\mathcal{P}^!$ (cf. Appendix B6). Since for *Dias* we know of an explicit description of its dual $Dias^!$, we are able to describe explicitly the notion of *strong homotopy dialgebra.* In order to do it we use the notion of nested sub-tree of a planar binary tree.

9.1. Nested sub-trees. Recall that in section 5 we defined the notion of nested sub-tree y' of a tree y with quotient $y'' = y/y'$ and we described composition in the free dendriform algebra in terms of nested sub-trees, cf. 5.10 and Proposition 5.11.

9.2. Theorem. *A strong homotopy dialgebra is a graded vector space $A = \oplus_{i \in \mathbf{Z}} A_i$ equipped with operations*

$$m_y : A^{\otimes n} \to A, \textit{ for any } y \in Y_n,\ n \geq 1,$$

which are homogeneous of degree $n-2$ and which satisfy the following relations for any y:

$$(*)_y \qquad \sum_{y' \subset y,\ y''=y/y'} \pm m_{y''}(a_1, \ldots, a_i, m_{y'}(a_{i+1}, \ldots, a_{i+k}), \ldots, a_n) = 0,$$

where the sign $\pm$ is $+$ or $-$ according to the parity of $(k+1)(i+1) + k(n + \sum_{j=1}^{k} |a_j|)$.

In this relation the tree y is fixed and the sum runs over all the nested sub-trees y' of y with $y'' = y/y'$ (as described in 5.10).

Compare with the definition of a A_∞-algebra [St, p. 294].

Proof. Since the operad *Dias* is Koszul (cf. Theorem 8.5), we may apply Theorem B.7. By Proposition 5.8 the dual operad *Dend* is generated in dimension n (as a free S_n-module) by the set of n-trees Y_n. Hence the operations on a $\mathcal{B}(\mathcal{P}^!)^*$-algebra A are generated by operations m_y, for $y \in Y_n$,

$$m_y : A^{\otimes |y|} \to A \quad \text{ of degree } |y| - 2.$$

The relations satisfied by these operations are obtained as follows. First one extends them in order to get a coderivation

$$m : \oplus_n K[Y_n] \otimes A^{\otimes n} \longrightarrow \oplus_n K[Y_n] \otimes A^{\otimes n},$$

and then one writes $m \circ m = 0$. The component in $K[Y_1] \otimes A = A$ of the image under m of an element $K\{y\} \otimes A^{\otimes n} \subset K[Y_n] \otimes A^{\otimes n}$ is given by m_y (up to sign). More precisely we put

$$m_y(a_1, \cdots, a_n) := (-1)^{(k-1)|a_1| + (k-2)|a_2| + \cdots a_{k-1}} m(y; a_1, \cdots, a_n)_1.$$

In order to obtain the component in $K\{y'\} \otimes A^{\otimes k}$ we need to look at the composition in the cofree co-dendriform algebra, or dually, in the free dendriform algebra:

$$Dias^!(k) \otimes Dias^!(i_1) \otimes \cdots \otimes Dias^!(i_k) \longrightarrow Dias^!(i_1 + \cdots + i_k)$$

or equivalently,

$$K[Y_k] \otimes K[Y_{i_1}] \otimes \cdots \otimes K[Y_{i_k}] \longrightarrow K[Y_{i_1+\cdots+i_k}].$$

It is sufficient to write the relation $m \circ m = 0$ for the component in $K[Y_1] \otimes A = A$ of the image, since the vanishing of the other components is a consequence of that one. Hence it is sufficient to compute the composition product for $i_u = 1$ for all i_u except one of them, let us say $i_j = m$. In this case it is precisely the result of Proposition 5.11. □

9.3. Strong homotopy associative dialgebra in low dimensions. Let us write $m_y = m_{ij\dots k}$ in place of $m_{[ij\dots k]}$, when $y = [ij\cdots k]$.

For $n = 1$ the operation $\delta := m_1 : A \to A$ is of degree -1. Since the only nested sub-tree of [1] is [1] itself, the relation $(*)_1$ is

$$\delta \circ \delta = 0.$$

Hence (A, δ) is a chain complex.

For $n = 2$, there are two maps m_{12} and $m_{21} : A^{\otimes 2} \to A$, which are of degree 0. The tree [12] (resp. [21]) has three nested sub-trees, itself with quotient [1], and two (different) copies of [1], both with quotient [12] (resp. [21]). Hence the relation $(*)_{12}$ takes the form (where 1 stands for Id):

$$\delta \circ m_{12} - m_{12} \circ (\delta \otimes 1) - m_{12} \circ (1 \otimes \delta) = 0,$$

and similarly for [21]. In other words δ is a derivation for m_{12} and for m_{21}.

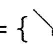
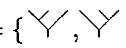
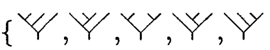
For $n = 3$, there are five maps $A^{\otimes 3} \to A$, denoted $m_{123}, m_{213}, m_{131}, m_{312}, m_{321}$, corresponding to the five trees of Y_3. The five relations $(*)_y$ for y a tree of degree 3 are:

$$\begin{aligned}
&\delta \circ m_{123} + m_{123} \circ (\delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta) = \\
&\qquad\qquad m_{12} \circ (m_{12} \otimes 1) - m_{12} \circ (1 \otimes m_{12}), \\
&\delta \circ m_{213} + m_{213} \circ (\delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta) = \\
&\qquad\qquad m_{12} \circ (m_{21} \otimes 1) - m_{12} \circ (1 \otimes m_{12}), \\
&\delta \circ m_{131} + m_{131} \circ (\delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta) = \\
&\qquad\qquad m_{21} \circ (m_{12} \otimes 1) - m_{12} \circ (1 \otimes m_{21}), \\
&\delta \circ m_{312} + m_{312} \circ (\delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta) = \\
&\qquad\qquad m_{21} \circ (m_{21} \otimes 1) - m_{21} \circ (1 \otimes m_{12}), \\
&\delta \circ m_{321} + m_{321} \circ (\delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta) = \\
&\qquad\qquad m_{21} \circ (m_{21} \otimes 1) - m_{21} \circ (1 \otimes m_{21}).
\end{aligned}$$

One observes that, as expected, if all the maps m_{ijk} are trivial (and also higher up), then $m_{12} = \vdash$, $m_{21} = \dashv$, and the algebra is a graded dialgebra.

Appendix A. PLANAR BINARY TREES AND PERMUTATIONS.

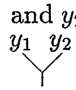
A.1. Planar binary trees. A planar tree is *binary* if any vertex is trivalent. We denote by Y_n the set of planar binary trees with n vertices, that is with $n+1$ leaves (and one root). Since we only use planar binary trees in this section we abbreviate it into tree (or n-tree). The integer n is called the degree of the tree. For any $y \in Y_n$ we label the $n+1$ leaves by $\{0, 1, \ldots, n\}$ from left to right. We label the vertices by $\{1, \ldots, n\}$ so that the i-th vertex is in between the leaves $i-1$ and i. In low dimension these sets are:

$Y_0 = \{|\}$, $Y_1 = \{$  $\}$, $Y_2 = \{$  $\}$, $Y_3 = \{$  $\}$.

The number of elements in Y_n is $c_n = \frac{(2n)!}{n!(n+1)!}$, so $\mathbf{c} = (1, 2, 5, 14, 42, 132, \ldots)$ is the sequence of the Catalan numbers.

We first introduce a notation for these trees as follows. The only element $|$ of Y_0 is denoted by $[0]$. The only element of Y_1 is denoted by $[1]$.

The *grafting* of a p-tree y_1 and a q-tree y_2 is a $(p+q+1)$-tree denoted by $y_1 \vee y_2$ obtained by joining the roots of y_1 and y_2 and creating a new root from that vertex.

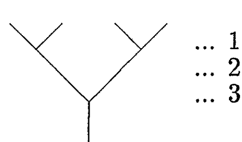
$y_1 \vee y_2 =$ 

Its name is written $[\ y_1\ p+q+1\ y_2\ ]$ with the convention that all 0's are deleted (except for the element in Y_0). For instance one has: $[0] \vee [0] = [1]$, $[1] \vee [0] = [12]$, $[0] \vee [1] = [21]$, $[1] \vee [1] = [131]$, and so on. So the names of the trees pictured above are, from left to right:

$$[0]\ , [1]\ , [12]\ , [21]\ , [123]\ , [213]\ , [131]\ , [312]\ , [321]\ .$$

This labelling has several advantages. For instance if we draw the tree metrically, with the leaves regulary spaced, and the lines at 45 degree angle, then the integers in the sequence are precisely the depth of the successive vertices.

Example



[131]

The orientation of the leaves can be read from the name as follows. Let $y = [a_1 \ldots a_n]$. The i-th leaf is oriented SW-NE (resp. SE-NW) when $a_i < a_{i+1}$ (resp. $a_i > a_{i+1}$).

The following is an inductive criterion to check whether a sequence of integers is the name of a tree.

A.2. Proposition. *A sequence of positive integers $[a_1 \ldots a_n]$ is the name of a tree if and only if it satisfies the following conditions:*

– there is a unique integer p such that $a_p = n$,

– the two sequences $a_1 \ldots a_{p-1}$ and $a_{p+1} \ldots a_n$ are either empty or name of trees.

Proof. It suffices to remark that any tree y is of the form $y_1 \vee y_2$ for two uniquely determined trees y_1 and y_2, whose degree is strictly smaller than the degree of y. □

A.3. From permutations to trees. There is defined a surjective map

$$\psi : S_n \twoheadrightarrow Y_n$$

as follows. The image of $\{1, \ldots, n\}$ under the permutation σ is a sequence of positive integers $[\sigma_1 \ldots \sigma_n]$. We convert it into the name of a tree by the following inductive rule. Replace the largest integer in the interval $\sigma_1 \ldots \sigma_n$, say σ_p, by the length of the interval (which is n here). Then repeat the modification for the intervals $\sigma_1 \ldots \sigma_{p-1}$ and $\sigma_{p+1} \ldots \sigma_n$, and so on, until each integer has been modified. This gives the name of a tree (hence a tree), since it obviously satisfies the above criterion. Let us perform this construction on an example, where the successive modifications are underlined:

$$\sigma = [341652] \mapsto [341\underline{6}52] \mapsto [3\underline{3}1\underline{6}\underline{22}] \mapsto [\underline{131621}] = \psi(\sigma).$$

A.4. Planar binary increasing trees. Let us introduce a variation of trees: the *planar binary trees with levels* also called *increasing trees* in the literature. A tree with levels is an n-tree together with a given level for each vertex. This level takes value in $\{1, \ldots, n\}$, and we suppose that each vertex has a different level, and that the levels are increasing, that is they respect the partial order structure of the tree (the level is the depth of the vertex).

Example: the following are two distinct increasing trees

We denote by $\widetilde{Y}_n$ the set of increasing n-trees.

The following is a well-known result which was brought to my attention by Phil Hanlon [Ha].

A.5. Proposition. *The map which assigns a level to each vertex determines a permutation. This gives a bijection $\widetilde{Y}_n \cong S_n$ between increasing trees and permutations.*

Proof. Label the vertices by their right leaf (i.e. vertex i is in between the leaf $i-1$ and the leaf i). Since each vertex has a different level it is clear that we get a bijection. So any increasing tree gives rise to a permutation.

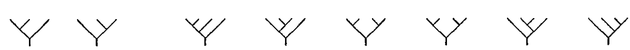
On the other hand a permutation gives rise to a tree under ψ, cf. A3. Labelling the level of the i-th vertex by $\sigma(i)$ gives an increasing tree.

It is immediately seen that the two constructions are inverse to each other. □

Hence the map $\psi : S_n \to Y_n$ is the composite of the bijection from the set of permutations to the set of increasing trees with the forgetful map (forgetting about the levels).

In low dimension ψ is given by:

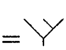
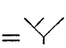
σ	[12]	[21]	[123]	[213]	[132]	[231]	[312]	[321]
$\psi(\sigma)$	[12]	[21]	[123]	[213]	[131]	[131]	[312]	[321]

tree 

A.6. The other choice ψ' to code trees. It is sometimes useful to code the vertices of a planar binary trees by *height* rather than by depth. The resulting map $\psi' : S_n \to Y_n$ is related by the formula

$$\psi'(\sigma) = \psi(\omega\sigma\omega^{-1}), \quad for \quad \omega = [n \cdots 2\ 1].$$

For instance

$\psi'([12]) =$  *and* $\psi'([21]) =$  .

A.7. Geometric interpretation of ψ. The map ψ can be considered as the restriction to the vertices of a cellular map between polytopes. Indeed, starting from the set $< n >= \{1, ..., n\}$ let us define a poset as follows. An element of the poset is an ordered partition of $< n >$. The element X is less than the element Y if Y can be obtained from X by reuniting successive subsets. For instance $\{(35)(2)(14)(7)(6)\} < \{(235)(14)(67)\}$. Note that the minimal elements are precisely the permutations. It can be shown that if we exclude the trivial partition $\{(12...n)\}$ from the poset, then the geometric realization is homeomorphic to the sphere S^{n-2} (it is called the *permutohedron*).

Similarly, starting with planar trees with n interior vertices one defines a poset as follows: a tree x is less than a tree y if y can be obtained from x by scratching some interior edges. Note that the minimal elements are precisely the planar binary trees. It can be shown that if we exclude the trivial tree with only one vertex from the poset, then the geometric realization is a polytope homeomorphic to the sphere S^{n-2} (it is called the Stasheff polytope, or the *associahedron*). The map ψ described above can be obviously extended to a map of posets, and its geometric realization is a homotopy equivalence of spaces, both homeomorphic to the sphere S^{n-2}. One can even compare the associahedron with the hypercubes by looking at the orientation of the leaves, cf. [LR].

Appendix B. ALGEBRAIC OPERADS

In this short appendix we briefly survey Koszul duality for algebraic operads as studied by Ginzburg and Kapranov [GK] and Getzler-Jones [GJ].

The ground field K is supposed to be of characteristic zero. The category of finite dimensional vector spaces over K is denoted by **Vect**. When V is graded its suspension sV is such that $(sV)_n = V_{n-1}$.

B.1. Algebraic operads. For a given "type of algebra" $\mathcal{P}$ (for instance associative algebras, or Lie algebras, etc ...), let $\mathcal{P}(V)$ be the free algebra over the vector space V. One can view $\mathcal{P}$ as a functor from the category **Vect** to itself, which preserves filtered limits. The map $V \to \mathcal{P}(V)$ gives a natural transformation $\mathrm{Id} \to \mathcal{P}$. The functor $\mathcal{P}$ is *analytical.* In characteristic zero it is equivalent (by Schur's lemma) to: $\mathcal{P}$ is of the form

$$\mathcal{P}(V) = \bigoplus_{n \geq 0} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}, \tag{B.1.1}$$

for some right S_n-module $\mathcal{P}(n)$.

From the universal property of the free algebra applied to $\mathrm{Id} : \mathcal{P}(V) \to \mathcal{P}(V)$ one gets a natural map $\mathcal{P}(\mathcal{P}(V)) \to \mathcal{P}(V)$, that is a transformation of functors $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$. By universality property of the free algebra functor, one sees that the operation γ is associative and has a unit.

In order to make precise the notion of "type of algebras", one axiomatizes the above properties and puts up the following definition.

By definition an *algebraic operad* over a characteristic zero field is an analytical functor $\mathcal{P} : \mathbf{Vect} \to \mathbf{Vect}$, such that $\mathcal{P}(0) = 0$, equipped with a natural transformation of functor $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ which is associative and has a unit $1 : \mathrm{Id} \to \mathcal{P}$. In other words $(\mathcal{P}, \gamma, 1)$ is a *monad* in the tensor category $(\mathbf{Funct}, \circ)$, where **Funct** is the category of analytical functors from **Vect** to itself, and $\circ$ stands for composition of functors.

By definition a $\mathcal{P}$-*algebra* (that is an algebra over the operad $\mathcal{P}$) is a vector space A equipped with a map $\gamma_A : \mathcal{P}(A) \to A$ compatible with the composition γ in the following sense: the diagram

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(A)) & \xrightarrow{\mathcal{P}(\gamma_A)} & \mathcal{P}(A) \\
\downarrow \gamma(A) & & \downarrow \gamma_A \\
\mathcal{P}(A) & \xrightarrow{\gamma_A} & A
\end{array}$$

is commutative.

By writing $\mathcal{P}(V)$ and $\mathcal{P}\circ\mathcal{P}(V)$ in terms of the vector spaves $\mathcal{P}(n)$'s we see that the operation γ is determined by linear maps

$$\mathcal{P}(n)\otimes\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_n)\longrightarrow\mathcal{P}(i_1+\cdots+i_n)$$

which satisfy some axioms deduced from the S_n-module structure of $\mathcal{P}(n)$ and from the associativity of γ (cf. for instance [M]).

From this point of view, a $\mathcal{P}$-algebra is determined by linear maps

$$\mathcal{P}(n)\otimes_{S_n} A^{\otimes n}\to A$$

satisfying compatibility properties (cf. [M]). Observe that the space $\mathcal{P}(n)$ is the space of all operations that can be performed on n variables.

The family of S_n-modules $\{\mathcal{P}(n)\}_{n\geq 1}$ is called an **S**-module. There is an obvious forgetful functor from operads to **S**-modules. The left adjoint functor exists and gives rise to the *free operad* over an **S**-module (cf. for instance [BJT]).

We are only dealing here with *binary* operads, that is operads generated by operations on two variables. More explicitly, let E be an S_2-module (module of generating operations) and let $\mathcal{T}(E)$ be the free operad on the **S**-module

$$(0,E,0,\dots\ ).$$

The first components of $\mathcal{T}(E)$ are

$$\begin{aligned}&\mathcal{T}(E)(1)=K,\\&\mathcal{T}(E)(2)=E,\\&\mathcal{T}(E)(3)=\mathrm{Ind}_{S_2}^{S_3}(E\otimes E)=3E\otimes E,\end{aligned}$$

where the action of S_2 on $E\otimes E$ is on the second factor only. Explicitly, $\mathcal{T}(E)(3)$ is the space of all the operations on 3 variables that can be performed out of the operations on 2 variables in E.

A binary operad is *quadratic* if it is the quotient of $\mathcal{T}(E)$ (for some S_2-module E) by the ideal generated by some S_3-submodule R of $\mathcal{T}(E)(3)$. The associated operad is denoted $\mathcal{P}(E,R)$.

For instance, for $\mathcal{P}=As$, one has $E=K[S_2]$, the regular representation, and so $\mathrm{Ind}_{S_2}^{S_3}(E\otimes E)=K[S_3]\oplus K[S_3]$. Labelling the generators of the first summand by $x_i(x_jx_k)$ and the generators of the second summand by $(x_ix_j)x_k$, the space R is the subspace generated by all the elements $x_i(x_jx_k)-(x_ix_j)x_k$.

All the operads $Com, Lie, As, Pois, Leib, Zinb, Dias, Dend$ are binary and quadratic. For all these cases the operadic notion of algebra coincides with the one we started with.

B.2. Dual operad. For any right S_n-module V we denote by $V^\vee$ the right S_n-module $V^* \otimes (sgn)$, where (sgn) is the one-dimensional signature representation. Explicitly, if ${}^\sigma f$ is given by ${}^\sigma f(x) = f(x^\sigma)$, then $f^\sigma = sgn(\sigma)({}^{\sigma^{-1}}f)$. The pairing between $V^\vee$ and V given by:

$$< -,- >: V^\vee \otimes V \longrightarrow K, \quad < f,x >= f(x),$$

is sign-invariant, that is $< f^\sigma, x^\sigma >= sgn(\sigma) < f,x >$.

To any quadratic binary operad $\mathcal{P} = \mathcal{P}(E,R)$ is associated its *dual operad* $\mathcal{P}^!$ defined as follows. Since E is an S_2-module, so is $E^\vee$. There is a canonical isomorphism

$$\mathcal{T}(E^\vee)(3) = \text{Ind}\,^{S_3}_{S_2}(E^\vee \otimes E^\vee) \cong (\text{Ind}\,^{S_3}_{S_2}(E \otimes E))^\vee = \mathcal{T}(E)(3)^\vee,$$

and one defines the orthogonal space of R as

$$R^\perp := \text{Ker}\ (\mathcal{T}(E^\vee)(3) \to R^\vee).$$

By definition one puts

$$\mathcal{P}^! := \mathcal{P}(E^\vee, R^\perp).$$

It is shown in [GK] that $As^! = As$, $Com^! = Lie$, $Lie^! = Com$. The notation *Zinb* indicates that the operad of Zinbiel algebras is precisely the dual, in the above sense, of the operad of Leibniz algebras (cf. [L3]). It is proved in section 6 that the operad associated to dendriform algebras, denoted *Dend* is the dual of the operad associated to dialgebras, as expected. It is a consequence of the following proposition, which makes explicit the dual operad when the operations bear no symmetry. This hypothesis says that the space of operations is of the form $E = E' \otimes K[S_2]$ for some vector space E'. Then, the space of operations on 3 variables is $\mathcal{T}(E)(3) = (E'^{\otimes 2} \oplus E'^{\otimes 2}) \otimes K[S_3]$. For any $\xi \in E'$ we denote by $(\xi)_1$ (resp. $(\xi)_2$ the element corresponding to an operation of the type $(-(--))$ (resp. $((--)-)$). The first (resp. the second) component of $E'^{\otimes 2} \oplus E'^{\otimes 2}$ is made of the elements $(\xi)_1$ (resp. $(\xi)_2$).

B.3. Proposition. *Let $\mathcal{P}$ be an operad whose generating operations have no symmetry, in other words $\mathcal{P} = \mathcal{P}(E' \otimes K[S_2], R)$ for some vector space E', and some S_3-sub-module R of $(E'^{\otimes 2} \oplus E'^{\otimes 2}) \otimes K[S_3]$.*

The dual operad of $\mathcal{P}$ is

$$\mathcal{P}^! = \mathcal{P}(E' \otimes K[S_2], R^{ann}),$$

where R^{ann} is the annihilator of R for the scalar product on $(E'^{\otimes 2} \oplus E'^{\otimes 2}) \otimes K[S_3]$ given by

$$< \xi_1 \otimes \sigma, \xi_1 \otimes \sigma >= \text{sgn}(\sigma),$$
$$< \xi_2 \otimes \sigma, \xi_2 \otimes \sigma >= -\text{sgn}(\sigma),$$

all other scalar products are 0, *where* ξ_1 *(resp* ξ_2*) is a basis vector of the first (resp. second) summand of* $E'^{\otimes 2} \oplus E'^{\otimes 2}$ *and* $\sigma \in S_3$.

Proof. Let $\phi : E \to E^\vee$ be the isomorphism of S_2-modules deduced from the preferred basis of E. It induces an isomorphism of S_3-modules

$$\mathcal{T}(\phi) : \mathcal{T}(E^\vee)(3) \cong \mathcal{T}(E)(3).$$

Hence, the natural evaluation map $\mathcal{T}(E)(3)^\vee \otimes \mathcal{T}(E)(3) \to K$ gives a scalar product $\mathcal{T}(E)(3) \otimes \mathcal{T}(E)(3) \to K$.

Suppose that $E = E' \otimes K[S_2]$. Then $\mathcal{T}(E)(3) = (E'^{\otimes 2} \oplus E'^{\otimes 2}) \otimes K[S_3]$ as mentioned above. In order to prove the Proposition, it suffices to prove it when E' is one-dimensional, generated by the unique product (x_1x_2). The basis of E is $\{(x_1x_2), (x_2x_1)\}$. The isomorphism ϕ is given by $\phi((x_1x_2)) = (x_1x_2)^*, \phi((x_2x_1)) = -(x_2x_1)^*$. The map $\mathcal{T}(\phi)$ sends a basis element to itself (up to sign), so the scalar product is diagonal. Since it is sign-invariant, it suffices to compute $< (x_1(x_2x_3)), (x_1(x_2x_3)) >$ and $< ((x_1x_2)x_3), ((x_1x_2)x_3) >$. With the choice at hand we find $+1$ in the first case and -1 in the second. □

It is clear from this Proposition that $As^! = As$. Indeed, E' is one dimensional generated by $(\cdot\ \cdot)$, and $E'^{\otimes 2} \oplus E'^{\otimes 2}$ is 2-dimensional generated by $(\cdot(\cdot\ \cdot))$ and $((\cdot\ \cdot)\cdot)$. The space R is of the form $R' \otimes K[S_3]$ because, in the associative relation, the variables stay in order. Since R' is determined by the equation $(\cdot(\cdot\ \cdot)) - ((\cdot\ \cdot)\cdot) = 0$, it is immediate to check that it is its own annihilator.

B.4. Homology and Koszul duality. Let $\mathcal{P}$ be a quadratic binary operad and $\mathcal{P}^!$ its dual operad. Let A be a $\mathcal{P}$-algebra. There is defined a chain-complex $C_*^{\mathcal{P}}(A)$:

$$\cdots \to \mathcal{P}^!(n)^\vee \otimes_{S_n} A^{\otimes n} \xrightarrow{d} \mathcal{P}^!(n-1)^\vee \otimes_{S_{n-1}} A^{\otimes n-1} \to \cdots \to \mathcal{P}^!(1)^\vee \otimes A$$

where the differential d agrees, in low dimension, with the $\mathcal{P}$-algebra structure of A

$$\gamma_A(2) : \mathcal{P}(2) \otimes A^{\otimes 2} \to A.$$

In fact d is characterized by this condition plus the fact that on the cofree coalgebra $\mathcal{P}^{!*}(sA)$ it is a graded coderivation. The associated homology groups are denoted by $H_n^{\mathcal{P}}(A)$, for $n \geq 1$. Taking the linear dual of $C_*^{\mathcal{P}}(A)$ over K gives a cochain complex, which permits us to define the cohomology groups $H_{\mathcal{P}}^n(A)$.

If, for any vector space V, the groups $H_n^{\mathcal{P}}(\mathcal{P}(V))$ are trivial for $n > 1$, then the operad $\mathcal{P}$ is called a *Koszul* operad.

One can check that the chain-complex $C_*^{\mathcal{P}}$ is
– the Hochschild complex (of nonunital algebras) for $\mathcal{P} = As$,
– the Harrison complex (of nonunital commutative algebras) for $\mathcal{P} = Com$,
– the Chevalley-Eilenberg complex for $\mathcal{P} = Lie$,
– the chain-complex constructed in [L1] for $\mathcal{P} = Leib$,

– the chain-complex constructed in [Li2] for $\mathcal{P} = Zinb$.

Let us give some details about the check in the case of nonunital associative algebras. Since $As^! = As$, one has

$$C_n^{As}(A) = As^!(n)^\vee \otimes_{S_n} A^{\otimes n} = K[S_n] \otimes_{S_n} A^{\otimes n} = A^{\otimes n}.$$

Since the lowest differential coincides with the product on A, the map $d_2 : A^{\otimes 2} \to A$ is given by $d_2(x,y) = xy$. The coalgebra structure of $C_*^{As}(A)$ is given by "deconcatenation", that is

$$\Delta(a_1, \dots, a_n) = \sum_{i=0}^{n} (a_1, \dots, a_i) \otimes (a_{i+1}, \dots, a_n).$$

Since d is a graded coderivation, we have on $A^{\otimes 3}$

$$\begin{aligned}\Delta d_3(a_1,a_2,a_3) &= (d_2 \otimes 1 + 1 \otimes d_2)\Delta(a_1,a_2,a_3)\\ &= (d_2 \otimes 1 + 1 \otimes d_2)((a_1,a_2) \otimes a_3 + a_1 \otimes (a_2,a_3))\\ &= (a_1a_2) \otimes a_3 - a_1 \otimes (a_2a_3).\end{aligned}$$

Hence we obtain $d_3(a_1,a_2,a_3) = (a_1a_2,a_3) - (a_1,a_2a_3)$.

More generally, the same kind of computation shows that

$$d_n(a_1, \dots, a_n) = \sum_{i=1}^{n-1} (-1)^{i-1}(a_1, \dots, a_ia_{i+1}, \dots, a_n).$$

So $(C_*^{\mathcal{P}}(A), d)$ is precisely the Hochschild complex for non-unital algebras (also called the b'-complex in the literature, cf. [L0] for instance).

B.5. Properties of the operad dualization. Here are some properties of the dualization of operads.

(a) *Lie algebra property.* Let A be a $\mathcal{P}$-algebra and B be a $\mathcal{P}^!$-algebra. The following bracket makes $A \otimes B$ into a Lie algebra :

$$[a \otimes b, a' \otimes b'] := \sum \left(\mu(a,a') \otimes \mu^\vee(b,b') - \mu(a',a) \otimes \mu^\vee(b',b)\right),$$

where the sum is over a basis $\{\mu\}$ of the binary operations of $\mathcal{P}$, $\{\mu^\vee\}$ being the dual basis in $\mathcal{P}^!$.

(b) *Koszulness.* $\mathcal{P}^{!!} = \mathcal{P}$. If $\mathcal{P}$ is Koszul, then so is $\mathcal{P}^!$.

(c) *Poincaré series.* Define the Poincaré series of $\mathcal{P}$ as

$$g_{\mathcal{P}}(x) := \sum_{n \geq 1} (-1)^n \dim\ \mathcal{P}(n)\frac{x^n}{n!}.$$

If $\mathcal{P}$ is Koszul, then the following formula holds:

$$g_{\mathcal{P}}(g_{\mathcal{P}^!}(x)) = x.$$

For instance $g_{As}(x) = \frac{-x}{1+x}$, $g_{Comm}(x) = \exp(-x) - 1$, $g_{Lie}(x) = -\log(1+x)$.

(d) *Multiplicative structure.* For any $\mathcal{P}$-algebra A the homology groups $H_*^{\mathcal{P}}(A)$ form a graded $\mathcal{P}^!$-coalgebra. Equivalently, $H^*_{\mathcal{P}}(A)$ is a graded $\mathcal{P}^!$-algebra.

(e) *Functoriality.* Let $\phi : \mathcal{P} \to \mathcal{Q}$ be a morphism of operads, inducing a functor ($\mathcal{Q}$-algebras) $\to$ ($\mathcal{P}$-algebras), $A \mapsto A_{\mathcal{P}}$ between the categories of algebras. For any $\mathcal{Q}$-algebra A there is a well-defined graded morphism

$$\phi_* : H_*^{\mathcal{P}}(A_{\mathcal{P}}) \to H_*^{\mathcal{Q}}(A),$$

which is induced, at the complex level, by the unique morphism of $\mathcal{Q}^!$-algebra

$$\mathcal{Q}^!(V) \to \mathcal{P}^!(V)_{\mathcal{Q}^!},$$

which commutes with the embeddings of the vector space V.

(f) *Quillen homology.* If $\mathcal{P}$ is Koszul, then the homology theory $H_*^{\mathcal{P}}$ for $\mathcal{P}$-algebras, as defined above, coincides with the Quillen homology with trivial coefficients (cf. [Q], [Li3], [FM]). This is a consequence of the existence of a quasi-isomorphism $\mathcal{B}(\mathcal{P}^!)^* \to \mathcal{P}$ (see below).

B.6. Homotopy algebras over an operad. A *quasi-free resolution* $\mathcal{Q} \to \mathcal{P}$ of the operad $\mathcal{P}$ is a differential graded operad $\mathcal{Q}$ which is a free operad over some graded **S**-module V, and such that, for all n, $\mathcal{Q}(n)$ is a chain-complex whose homology is trivial, except H_0 which is equal to $\mathcal{P}(n)$. The isomorphism $H_0(\mathcal{Q}) \cong \mathcal{P}$ is supposed to be an isomorphism of operads. Observe that one does not require $\mathcal{Q}$ to be free in the category of differential graded operads. By definition a *homotopy $\mathcal{P}$-algebra* is a $\mathcal{Q}$-algebra for some quasi-free resolution $\mathcal{Q}$ of $\mathcal{P}$.

A quasi-free resolution $\mathcal{Q}$ over V, with augmentation ideal $\bar{\mathcal{Q}}$ is called *minimal* when the differential d is *quadratic*, that is, satisfies $d(V) \subset \bar{\mathcal{Q}} \cdot \bar{\mathcal{Q}}$. For a given $\mathcal{P}$ such a minimal model always exists in characteristic zero and is unique up to homotopy.

By definition a *strong homotopy $\mathcal{P}$-algebra* is a $\mathcal{Q}$-algebra for the minimal model $\mathcal{Q}$ of $\mathcal{P}$.

If $\mathcal{P}$ is Koszul, then there is a way of constructing the minimal model as follows. Let $\mathcal{P}^!$ be the Koszul dual of $\mathcal{P}$ and let $\mathcal{B}(\mathcal{P}^!)$ be the bar-construction over $\mathcal{P}^!$. It is the cofree cooperad $\mathcal{T}'(s\bar{\mathcal{P}}^!)$ equipped with the unique coderivation d which is 0 on $\mathcal{P}^!$ and coincides with the composition on $\mathcal{P}^! \circ \mathcal{P}^!$. It is immediate that $d^2 = 0$, and hence $\mathcal{B}(\mathcal{P}^!) := (\mathcal{T}'(s\bar{\mathcal{P}}^!), d)$ is a complex called the *bar-complex* of $\mathcal{P}^!$. It is shown in [GK] (cf. also [GJ]), that, if the operad $\mathcal{P}$ is Koszul, then the differential graded operad $\mathcal{B}(\mathcal{P}^!)^*$ is a resolution of $\mathcal{P}$.

Since it is quasi-free and since the differential m of $\mathcal{B}(\mathcal{P}^!)^*$ is quadratic, $\mathcal{B}(\mathcal{P}^!)^*$ is the minimal model of $\mathcal{P}$.

B.7. Theorem. *For a Koszul operad $\mathcal{P}$, strong homotopy $\mathcal{P}$-algebras are equivalent to $\mathcal{B}(\mathcal{P}^!)^*$-algebras.*

B.8. Strong homotopy associative algebras. The associative operad $\mathcal{P} = \mathbf{As}$ is Koszul and self-dual. Since $\mathbf{As}(n)$ is one-dimensional as a free S_n-module, it gives rise to a generating operation $m_n : A^{\otimes n} \to A$ of the strong homotopy associative algebra A. The condition $m \circ m = 0$ gives, for each $n \geq 1$

$$\sum_{k+l=n-1} m_l \circ m_k = 0.$$

Hence a strong homotopy associative algebra is exactly what is called an A_∞-algebra in the literature (cf. [St]).

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Dialgebra (co)homology with coefficients

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Abstract

Dialgebras are a generalization of associative algebras which gives rise to Leibniz algebras instead of Lie algebras. In this paper we define the dialgebra (co)homology with coefficients, recovering, for constant coefficients, the natural homology of dialgebras introduced by J.-L. Loday in [L6] and denoted by HY_*. We show that the homology HY_* has the main expected properties: it is a derived functor, HY^2 classifies the abelian extensions of dialgebras and Morita invariance of matrices holds for unital dialgebras. For associative algebras, we compare Hochschild and dialgebra homology, and extend the isomorphism proved in [F2] for unital algebras to the case of H-unital algebras.

A feature of the theory HY is that the categories of coefficients for homology and cohomology are different. This leads us to introduce the universal enveloping algebra of dialgebras and the corresponding cotangent complex, analogue to that defined by D. Quillen in [Q] for commutative algebras. Our results follow from a property of Poincaré-Birkhoff-Witt type and from some combinatorial and simplicial properties of the sets of planar binary trees proved in [F4]. Finally, since the faces and degeneracies for unital dialgebras satisfy all the simplicial relations except one, we are lead to study the general properties of the so-called *almost simplicial modules*.

Contents

J.-L. Loday et al: LNM 1763, pp. 67 - 103, 2001

Introduction

It is well known that any associative algebra gives rise to a Lie algebra, with bracket $[a, b] := ab - ba$. In [L2], J.-L. Loday introduces a non-antisymmetric version of Lie algebras, whose bracket satisfies the Leibniz relation

$$(L) \quad [x, [y, z]] = [[x, y], z] - [[x, z], y],$$

and therefore called *Leibniz algebras*. The Leibniz relation, combined with antisymmetry, is a variation of the Jacobi identity, hence Lie algebras are antisymmetric Leibniz algebras. The natural homology theory of Leibniz algebras is denoted by HL_*.

In [L5], Loday introduces an 'associative' version of Leibniz algebras, called *dialgebras*, equipped with two binary operations, $\dashv$ and $\vdash$, which satisfy the five relations

$$(D) \quad \begin{cases} a \dashv (b \dashv c) = (a \dashv b) \dashv c = a \dashv (b \vdash c), \\ (a \vdash b) \dashv c = a \vdash (b \dashv c), \\ (a \dashv b) \vdash c = a \vdash (b \vdash c) = (a \vdash b) \vdash c. \end{cases}$$

These identities are all variations of the associative law, so associative algebras are dialgebras for which the two products coincide. The peculiar point is that the bracket $[a, b] := a \dashv b - b \vdash a$ defines a Leibniz algebra which is not antisymmetric, unless the left and right products coincide. Hence dialgebras yield a commutative diagram of categories and functors

$$\begin{array}{ccc} \mathbf{As}_k & \xrightarrow{-} & \mathbf{Lie}_k \\ \downarrow & & \downarrow \\ \mathbf{Dias}_k & \xrightarrow{-} & \mathbf{Leib}_k \end{array} .$$

A peculiar example of a dialgebra is the *derived dialgebra* of an associative differential algebra (A, d), with products

$$a \dashv b := a \cdot d(b) \qquad \text{and} \qquad a \vdash b := d(a) \cdot b, \qquad a, b \in A.$$

Its associated Leibniz algebra coincides with the derived Leibniz algebra of the differential Lie algebra $(A_{Lie}, [\ ,\ ], d)$, with bracket $[x, y]_d := [x, dy]$. In [KS], Y. Kosmann-Schwarzbach exhibits some examples of derived Leibniz algebras from classical differential geometry and physics.

The natural bar homology theory for dialgebras, found by J.-L. Loday in [L5], is denoted by HY_*, because the module of n-chains of a dialgebra D is given by $k[Y_n] \otimes D^{\otimes n}$, where Y_n is the set of planar binary trees with n internal vertices.

In this paper we define the dialgebra homology and cohomology *with coefficients* and perform some computations. For constant coefficients we recover the bar homology defined by Loday.

In order to determine the coefficients, we consider an extension of a dialgebra D by an abelian dialgebra M. Then M inherits some actions of D which make it into a *representation* of D. Moreover, there is a natural choise of the first terms of a cochain complex $CY^1(D,M) \xrightarrow{\delta} CY^2(D,M) \xrightarrow{\delta} CY^3(D,M)$ such that the second cohomology group classifies the abelian extensions of the dialgebra D by M,

$$HY^2(D,M) \cong \mathrm{Ext}(D,M).$$

We extend the definition of the coboundary operator to higher terms, obtaining the dialgebra cohomolgy groups with coefficients, $HY^n(D,M)$, for any $n \geq 0$.

In order to determine the coefficients for the homology, we interpretate the representations of D as left modules over a unital associative algebra $E(D)$ called *universal enveloping algebra.* (For an associative algebra A, the analogue algebra is $A^e = A \otimes A^{op}$.) Then the coefficients for the homology of D are taken to be the right $E(D)$-modules and they are called *corepresentations* of D. The cochain complex of D with any coefficients can be dualized into a chain complex with coefficients in the universal enveloping algebra $E(D)$. In analogy with a complex defined by D. Quillen for commutative algebras, we call it *cotangent complex* of D. The chain complex of D with coefficients in a corepresentation N is then obtained by considering the tensor product over $E(D)$ of N and the cotangent chain complex. The resulting homology groups are denoted by $HY_n(D,N)$.

We prove that dialgebra homology and cohomology are derived functors,

$$HY_*(D,N) \cong \mathrm{Tor}_*^{E(D)}(N, M(D)), \qquad HY^*(D,M) \cong \mathrm{Ext}^*_{E(D)}(M(D), M).$$

In fact, using a property of Poincaré-Birkhoff-Witt type on the universal enveloping algebra $E(D)$ and some properties of the set of planar binary trees proved in [F4], we show that the cotangent complex is acyclic, with $HY_0(D, E(D)) \cong M(D)$.

Comparing Hochschild and dialgebra homology of associative algebras, using similar arguments, we generalize the isomorphism proved in [F2] for unital associative algebras to the case of H-unital algebras,

$$HY_*(A,M) \cong H_*(A,M).$$

We also introduce the notion of HY-unital dialgebras and show that for associative algebras this property is equivalent to the H-unitality in the sense of Wodzicki, cf. [W].

Both Hochschild and dialgebra chain complexes have boundary operator given by means of some face maps. For associative algebras, a unit permits us to define some degeneracy maps, and the Hochschild chain complex is in fact a simplicial module. The best analogue of a unit, for dialgebras, is an element which is a unit only on the bar-side of the products (bar-unit), since dialgebras with unit are in fact associative algebras. The bar-unit enables us to define some degeneracy maps which satisfy all the simplicial relations except one, namely $s_i s_i = s_{i+1} s_i$, and therefore they are called *almost-simplicial*. Using some simplicial techniques described in [F2], we show that bar-unital dialgebras have homological dimension ≤ 0, that is,

$$HY_n(D,N) = 0, \qquad n \geq 1,$$

that they allow Morita invariance of matrices for dialgebra homology,

$$HY_*(\mathrm{M}_r(D), \mathrm{M}_r(N)) \cong HY_*(D, N),$$

and also a weak version of the Künneth formula for dialgebra homology.

The paper is organized as follows. In section 1 we review the basic definitions of dialgebras and give a characterization of any dialgebra as a bimodule over its canonical associative algebra. In section 2 we define the dialgebra cohomology and prove that the second group classifies the abelian extensions of dialgebras. In section 3 we define the universal enveloping algebra and prove the Poincaré-Birkhoff-Witt isomorphism on its associated graded algebra. In section 4 we define the cotangent chain complex and the dialgebra homology. In section 5 we use some properties of the set of planar binary trees to define a spectral sequence abutting to the dialgebra homology. We then prove the Tor interpretation. With the same techniques, in section 7 we show that for associative algebras the notions of HY-unitality and that of H-unitality are equivalent. In section 6 we study the homology of bar-unital dialgebras, using simplicial tools.

1 Dialgebras

In this section we review the main definition and give a characterization of any dialgebra as a bimodule over its canonical associative algebra. More details and examples can be found in [L5] and [L6].

1.1 - Dialgebras. Let k be a field. An *associative dialgebra over k*, called *dialgebra* in the sequel, is a k-vector space D with two operations, $\dashv, \vdash$: $D \otimes D \longrightarrow D$, called left and right products, satisfying the following five axioms:

$$(D) \quad \begin{cases} a \dashv (b \dashv c) \overset{1}{=} (a \dashv b) \dashv c \overset{2}{=} a \dashv (b \vdash c), \\ (a \vdash b) \dashv c \overset{3}{=} a \vdash (b \dashv c), \\ (a \dashv b) \vdash c \overset{4}{=} a \vdash (b \vdash c) \overset{5}{=} (a \vdash b) \vdash c, \end{cases} \qquad a, b, c \in D.$$

A dialgebra is called *bar-unital* if it is given a specified *bar-unit*: an element $1 \in D$(not necessarily unique) which is a unit for the left and right products only on the bar-side, that is $1 \vdash a = a = a \dashv 1$, for any $a \in D$. A *morphism of dialgebras* is a k-linear map $f : D \longrightarrow D'$ which preserves the products, i.e. such that $f(a \star b) = f(a) \star f(b)$, where $\star$ denotes either the product $\dashv$ or the product $\vdash$. It is called *unital* if the image of any bar-unit is a bar-unit. Denote by $\mathbf{Dias}_k$ the category of dialgebras over k, and by $\mathbf{UDias}_k$ that of bar-unital dialgebras.

Any associative (unital) algebra A is clearly a (bar-unital) dialgebra, that we denote by A_{di}, for which the left and right products coincide:

$$a \dashv b := a \cdot b =: a \vdash b, \qquad a, b \in A.$$

The category $\mathbf{As}_k$ of associative algebras over k is a full subcategory of $\mathbf{Dias}_k$, and similarly that of unital associative algebras $\mathbf{UAs}_k$ is a full subcategory of $\mathbf{UDias}_k$.

1.2 - Canonical Leibniz algebra of a dialgebra. Recall, from [L2], that a *Leibniz algebra* is a k-module $\mathfrak{g}$ equipped with a bracket $[\ ,\ ] : \mathfrak{g}\otimes\mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying the Leibniz relation

$$(L) \quad [x,[y,z]] = [[x,y],z] - [[x,z],y], \qquad x,y,z \in \mathfrak{g}.$$

Then Lie algebras are antisymmetric Leibniz algebras. In [L5], Loday observed that, given a dialgebra D, the bracket

$$[a,b] := a \dashv b - b \vdash a, \qquad a,b \in D,$$

defines a Leibniz algebra D_L which is not antisymmetric, unless the left and right products coincide. The assignement $D \mapsto D_L$ yields a commutative diagram of categories and functors

$$\begin{tikzcd} \mathbf{As}_k \arrow[r, "-"] \arrow[d] & \mathbf{Lie}_k \arrow[d] \\ \mathbf{Dias}_k \arrow[r, "-"] & \mathbf{Leib}_k \end{tikzcd} .$$

1.3 - Example. An interesting example of a dialgebra, which is not an associative algebra, is constructed on an associative differential algebra (A,d), by introducing the *derived products*

$$\begin{aligned} a \dashv b &:= a \cdot d(b), \\ a \vdash b &:= d(a) \cdot b, \end{aligned} \qquad a,b \in A,$$

(with an appropriate sign in the case of graded algebras). The assumptions on d imply the relation $d(a\ db) = da\ db = d(da\ b)$, from which follow immediately the dialgebra axioms (D) for $\dashv$ and $\vdash$. We call *derived dialgebra* of (A,d) the triple $A_d := (A, \dashv, \vdash)$.

If the algebra A is endowed with a unit 1, and the field k has characteristic different from 2, then necessarily $d1 = 0$. Hence, in the derived dialgebra A_d, we have $a \dashv 1 = a \cdot d1 = 0$ and similarly $1 \vdash a = d1 \cdot a = 0$. This means that the element 1 is not a bar-unit in the dialgebra A_d. A bar-unit, in A_d, is an element $x \in A$ such that $dx = 1$, and it does not necessarily exist.

The canonical Leibniz algebra of the derived dialgebra A_{di} is an example of a Leibniz algebra, which is not Lie, whose role in classical differential geometry has recently been pointed out by Y. Kosmann-Schwarzbach. In [KS], she observes that a differential Lie algebra $(\mathfrak{g}, [\ ,\ ], d)$, always gives rise to the *derived Leibniz algebra* $\mathfrak{g}_d$, by introducing the *derived bracket*

$$[x,y]_d := [x, dy], \qquad x,y \in \mathfrak{g},$$

which satisfies (L) but it is not antisymmetric. In fact, if $(\mathfrak{g}, d)$ is the differential Lie algebra of (A,d), with $\mathfrak{g} = A_L$, its derived Leibniz algebra $\mathfrak{g}_d$ coincides with the Leibniz algebra of A_d, i.e. $(A_d)_L \cong (A_L)_d$.

1.4 - Canonical associative algebra of a dialgebra. Let D be a dialgebra and set

$$D_{as} := D/Z(D), \qquad Z(D) := \langle a \dashv b - a \vdash b \mid a, b \in D \rangle,$$

where $Z(D)$ is an ideal in D, i.e. a k-submodule closed under multiplication for any element of D. Then D_{as} is an associative algebra, with product $[a] \cdot [b] := [a \dashv b] = [a \vdash b]$. For example, in the free dialgebra $D = \bar{D}ias(V)$ on a vector space V, as defined in [L5], the fusion maps any element $\underline{v} \dashv \underline{w} - \underline{v} \vdash \underline{w} \in D$ to $\underline{vw} - \underline{vw} = 0$. Therefore it induces an isomorphism with the free associative algebra $\bar{D}ias(V)_{as} \cong \bar{A}ss(V)$.

If A is an associative algebra, since $Z(A_{di}) = 0$, there is a natural isomorphism of algebras $(A_{di})_{as} \cong A$. Hence the functor di of (1.1) admits a left adjoint functor $as : \mathbf{Dias}_k \longrightarrow \mathbf{As}_k$.

When D has a bar-unit 1, the class [1] is a unit in D_{as}, hence $as : \mathbf{UDias}_k \longrightarrow \mathbf{UAs}_k$. In this case, the ideal $Z(D)$ is generated by the elements $a - a \vdash 1$, $a - 1 \dashv a$ for any $a \in D$. If, in addition, the bar-unit is also a unit on the pointed side of one product, this ideal is zero and $D = D_{as}$ is a unital associative algebra.

1.5 - Dialgebra induced by a bimodule morphism. In [L5] Loday exhibits a typical example of a dialgebra, constructed on a bimodule M over an associative algebra A. If there is a morphism of A-bimodules $f : M \longrightarrow A$, then on M there are dialgebra products

$$\begin{aligned} m \dashv m' &:= m \cdot f(m'), \\ m \vdash m' &:= f(m) \cdot m', \end{aligned} \qquad m, m' \in M.$$

We now invert this procedure and show that all dialgebra can be achieved in this way.

1.6 - Proposition. *Let D be a dialgebra, and D_{as} its canonical associative algebra. Then there exists a D_{as}-bimodule structure on D, and a morphism of D_{as} bimodules $f : D \longrightarrow D_{as}$ such that the dialgebra structure on D is recovered by $a \dashv b = a \cdot f(b)$ and $a \vdash b = f(a) \cdot b$.*

Proof. First define on D a structure of bimodule over D_{as}, by setting

$$\begin{aligned} D_{as} \otimes D &\longrightarrow D, \quad [a] \cdot b := a \vdash b, \\ D \otimes D_{as} &\longrightarrow D, \quad a \cdot [b] := a \dashv b. \end{aligned}$$

The identities 1, 2 and 4, 5 of (D) guarantee that the equality $[a \dashv b] = [a \vdash b]$ in D_{as} is coherent with the above actions. In respect to these actions, the canonical projection $D \longrightarrow D_{as}$ is clearly a bimodule morphism and it induces on D precisely the given dialgebra structure. $\square$

Before closing this section we give a definition which is frequently used in the sequel.

1.7 - Abelian dialgebras and abelianization. A dialgebra D is called *abelian* if it has zero products, i.e. $a \dashv b = 0 = a \vdash b$ for any $a, b \in D$. Any k-module can be seen as an abelian dialgebra, by setting all products to zero. Another example is the ideal $Z(D)$, which is an abelian sub-dialgebra of D. In fact, the identities 1, 2 of (D) show that $D \dashv Z(D) = 0$, so in particular $Z(D) \dashv Z(D) = 0$. Similarly, the identities 4, 5 show that $Z(D) \vdash Z(D) = 0$.

Given a dialgebra D, we call *abelianization of* D the abelian dialgebra

$$D_{ab} := D/D^2, \qquad D^2 := \langle a \dashv b, a \vdash b \mid a, b \in D \rangle,$$

where D^2 is the ideal generated by all products. For instance, in a free dialgebra $\bar{D}ias(V) = \bar{T}(V) \otimes V \otimes \bar{T}(V)$ the ideal $\bar{D}ias(V)^2$ consists of the whole dialgebra except the central elements, V. Hence there is an isomorphism of abelian dialgebras $\bar{D}ias(V)_{ab} \cong V$.

When D has a bar-unit 1, the ideal D^2 is generated by the elements a, $a \vdash 1$ and $1 \dashv a$ for any $a \in D$. Therefore $D^2 := \langle a, a \vdash 1, 1 \dashv a \mid a \in D \rangle \cong D$ and $D_{ab} = 0$.

2 Dialgebra cohomology and representations

In this section we define the dialgebra cohomology with coefficients in a representation, in such a way that the second cohomology group classifies the abelian extensions of dialgebras.

The boundary operator of the bar chain complex, in low dimension, only depends on the axioms of dialgebra. This means that, given a dialgebra D, the composition $d' \circ d' : D^{\otimes 3} \longrightarrow D^{\otimes 2} \longrightarrow D$ yields zero because of relations (D). Hence the module of 3-chains is made of 5 copies of $D^{\otimes 3}$, one for each identity of (D), and the module of 2-chains is made of 2 copies of $D^{\otimes 2}$, one for each operation. In [L5], Loday extends this complex by considering, in dimension n, a number of copies of $D^{\otimes n}$ equal to the Catalan number $c_n = \frac{2n!}{n!(n+1)!}$. The boundary operator d' can be easily described in terms of simplicial faces d_i, that is, $d' = \sum_{i=1}^{n-1}(-1)^i d_i$, if we regard the Catalan number as the cardinality of the set Y_n of planar binary trees with n internal vertices. Hence the dialgebra homology is denoted by HY_*.

2.1 - Representations. Let D be a dialgebra over a field k. A *representation* of D is a k-module M endowed with two right actions $\dashv, \vdash : M \otimes D \longrightarrow M$ and two left actions $\dashv, \vdash : D \otimes M \longrightarrow M$ satisfying the 15 axioms given by the 5 axioms (D) for each choice of one of the three elements a, b or c in M. We refer to these relations as (RD). If D has a bar-unit 1, a representation M is called *unital* if the identity $1 \vdash m = m = m \dashv 1$ holds for any $m \in M$.

If D_L is the canonical Leibniz algebra of D, on the k-module M we obtain a representation M_L of the Leibniz algebra D_L, as defined in [L-P], by introducing the new brackets

$$\begin{aligned}[m,a] &:= m \dashv a - a \vdash m,\\ [a,m] &:= a \dashv m - m \vdash a,\end{aligned} \qquad m \in M, \quad a \in D.$$

2.2 - Examples.

1. Any dialgebra D is clearly a representation of itself, and it is a unital representation if D has a bar-unit. Similarly, any ideal I is a representation of D.

2. Any k-module M becomes a representation of a dialgebra D by setting $m \star a := 0 =: a \star m$, for any $m \in M$ and $a \in D$. We call M the *trivial* representation of D. When D is bar-unital, the trivial representation can not be unital, unless $M = 0$.

3. Let D be a dialgebra and $D_{as} = D/Z(D)$ its canonical associative algebra. Then the ideal $Z(D)$ is a representation of the dialgebra $(D_{as})_{di}$, with actions
$$z \star [a] := z \star a \qquad \text{and} \qquad [a] \star z := a \star z, \qquad z \in Z(D), \quad [a] \in (D_{as})_{di}.$$
In fact the choise of the representative element $a \in D$ for the class $[a] \in (D_{as})_{di}$ is not relevant, because any other representant a' differs from a for an element z' of $Z(D)$, $a = a' + z'$, and since $Z(D)$ is an abelian dialgebra we have $z \star a = z \star a' + z \star z' = z \star a'$.

4. Given a dialgebra D with a representation M, the k-module $\mathrm{M}_r(M)$ of matrices with entries in M is a representation of the dialgebra $\mathrm{M}_r(D)$, with 'row $\times$ column' actions.

2.3 - Dialgebra cochain complex. Let D be a dialgebra over k and M a representation of D. For any $n \geq 0$, we call *module of n-cochains of D with coefficients in M* the k-module

$$CY^n(D,M) := \mathrm{Hom}_k(k[Y_n] \otimes D^{\otimes n}, M),$$

where Y_n is the set of planar binary trees with n internal vertices, see [L5]. In particular, since the sets Y_0 and Y_1 contain one tree, respectively $|$ and $\curlyvee$, the module of 0-cochains is $CY^0(D,M) = \mathrm{Hom}_k(k,M) \cong M$ and that of 1-cochains is $CY^1(D,M) = \mathrm{Hom}_k(D,M)$.

For any $n \geq 0$, we define the *coboundary operator* $\delta : CY^n(D,M) \longrightarrow CY^{n+1}(D,M)$ as the k-linear map $\delta = \sum_{i=0}^{n+1}(-1)^i \delta^i$ given by means of the following face maps:

$$(\delta^i f)(y \otimes (a_1, ..., a_{n+1})) := \begin{cases} a_1 \circ_0^y f(d_0(y) \otimes (a_2, ..., a_n)), & \text{for } i = 0, \\ f(d_i(y) \otimes (a_1, ..., a_i \circ_i^y a_{i+1}, ..., a_{n+1})), & \text{for } i = 1, ..., n, \\ f(d_{n+1}(y) \otimes (a_1, ..., a_n)) \circ_{n+1}^y a_{n+1}, & \text{for } i = n+1, \end{cases}$$

for any $y \in Y_{n+1}$, $a_1, ..., a_{n+1} \in D$ and $f : k[Y_n] \otimes D^{\otimes n} \longrightarrow M$, where the n-tree $d_i(y)$ is obtained by *deleting* the i^{th} leaf from y, and the maps $\circ_i : Y_{n+1} \longrightarrow \{\dashv, \vdash\}$ are defined by

$$\circ_0^y := \begin{cases} \dashv, & \text{if } y \text{ has the shape } \curlyvee, \\ \vdash, & \text{if } y \text{ has the shape } \curlyvee, \end{cases}$$

$$\circ_i^y := \begin{cases} \dashv, & \text{if the } i^{th} \text{ leaf of } y \text{ is oriented like } \backslash, \\ \vdash, & \text{if the } i^{th} \text{ leaf of } y \text{ is oriented like } /, \end{cases} \qquad i = 1, ..., n$$

$$\circ_{n+1}^y := \begin{cases} \vdash, & \text{if } y \text{ has the shape } \curlyvee, \\ \dashv, & \text{if } y \text{ has the shape } \curlyvee. \end{cases}$$

2.4 - Lemma. *The operators $\delta : CY^n(D, M) \longrightarrow CY^{n+1}(D, M)$ satisfy the identity $\delta \circ \delta = 0$.*

Proof. It suffices to verify that the face maps δ^i satisfy the cosimplicial relations $\delta^j\delta^i = \delta^i\delta^{j-1}$, for $i < j$. Since for $i = 1, ..., n-1$ the faces $\delta^i : CY^n(D, M) \longrightarrow CY^{n+1}(D, M)$ are the duals of the faces $d_i : CY_{n+1}(D) \longrightarrow CY_n(D)$, defined in [L5], which satisfy the simplicial relations $d_id_j = d_{j-1}d_i$, it suffices to show that these relations hold for $i = 0$ with j from 1 to $n + 2$, and for $j = n + 2$ with i from 0 to $n + 1$.

Let us verify that $\delta^j\delta^0 = \delta^0\delta^{j-1}$ for $j = 1, ..., n + 2$. Take a k-linear map $f : k[Y_n] \otimes D^{\otimes n} \longrightarrow M$ and an element $y \otimes \underline{a} = y \otimes (a_1, ..., a_{n+2}) \in k[Y_{n+2}] \otimes D^{\otimes n+2}$. Then for $j = 1$ we have on one side:

$$\delta^1\delta^0 f(y \otimes \underline{a}) = (a_1 \circ_1^y a_2) \circ_0^{d_1(y)} f(d_0d_1(y) \otimes \underline{a}') = \begin{cases} (a_1 \dashv a_2) \dashv m, & \text{if } y = \curlyvee \\ (a_1 \vdash a_2) \dashv m, & \text{if } y = \curlyvee \\ (a_1 \star a_2) \vdash m, & \text{otherwise} \end{cases}$$

where $m := f(d_0d_1 \otimes \underline{a}')$. Notice that the symbol $\star$ here means that for some trees we have the product $\dashv$, for some others we have $\vdash$, but because of equalities 4 and 5 in (D) the final product always agrees in the term $(a_1 \vdash a_2) \vdash m$. On the other side:

$$\delta^0\delta^0 f(y \otimes \underline{a}) = a_1 \circ_0^y [a_2 \circ_0^{d_0(y)} f(d_0d_0(y) \otimes \underline{a}')] = \begin{cases} a_1 \dashv (a_2 \star' m), & \text{if } y = \curlyvee \\ a_1 \vdash (a_2 \dashv m), & \text{if } y = \curlyvee \\ a_1 \vdash (a_2 \vdash m), & \text{otherwise} \end{cases}$$

where we have identified $f(d_0d_0(y) \otimes \underline{a}')$ with $f(d_0d_1(y) \otimes \underline{a}') = m$ because of the identity $d_0d_1 = d_0d_0$. Hence the equality of the two terms is a consequence of the axioms of the representation M. For $j > 1$, we have on one side:

$$\delta^j\delta^0 f(y \otimes \underline{a}) = a_1 \circ_0^{d_j(y)} f(d_0d_j(y) \otimes (a_2, ..., a_j \circ_j^y a_{j+1}, ..., a_{n+2}),$$

and on the other side:

$$\delta^0\delta^{j+1} f(y \otimes \underline{a}) = a_1 \circ_0^y f(d_{j-1}d_0(y) \otimes (a_2, ..., a_j \circ_{j-1}^{d_0(y)} a_{j+1}, ..., a_{n+2}).$$

Hence the equality of the two terms is a consequence of the identities $d_0 d_j = d_{j+1} d_0$, $\circ_0^{d_j(y)} = \circ_0^{y}$ and $\circ_{j-1}^{d_0(y)} = \circ_j^{y}$ for any $j > 1$.

In a similar way one verifies that $\delta^{n+2}\delta^i = \delta^i \delta^{n+1}$ for $i = 0, ..., n+1$. $\square$

2.5 - Definition (dialgebra cohomology). Let D be a dialgebra with a representation M. The *dialgebra cohomology of D with coefficients in M* is the homology of the cochain complex (2.3),

$$HY^n(D, M) := H_n(CY^*(D, M), \delta), \qquad n \geq 0.$$

In particular, when $M = D$, we use the notation $HHY^*(D) := HY^*(D, D)$.

2.6 - Derivations on dialgebras. Let D be a dialgebra over a field k, and M a representation of D. A *derivation on D with values in M* is a k-linear map $\mu : D \longrightarrow M$ such that

$$\mu(a \star b) = \mu(a) \star b + a \star \mu(b), \qquad a, b \in D.$$

Denote by $\mathrm{Der}(D, M)$, the k-module of all derivations on D with values in M. The *inner derivations of D in M* are the adjoint maps $ad_m = [-, m] : D \longrightarrow M$, for $m \in M$, and their set is $\mathrm{Inn}(D, M) = \{ad_m \in \mathrm{Der}(D, M) |$ for all $m \in M\}$. (Notice that, unlike for associative algebras, the map $[m, -] : D \longrightarrow M$ is *not* a derivation of D!)

2.7 - Elementary computations. Given a dialgebra D with a representation M, the zero-th coboundary of D in M is the k-linear map $\delta : M \longrightarrow \mathrm{Hom}(D, M)$, $\delta m(a) := a \dashv m - m \vdash a = [a, m]$. Hence $HY^0(D, M)$ is the submodule of invariants of M,

$$HY^0(D, M) = M^D = \{m \in M | \ [a, m] = 0 \ \text{for any } a \in D\}.$$

The first coboundary is the k-linear map $\delta : Hom_k(D, M) \longrightarrow Hom_k(k[Y_2] \otimes D^{\otimes 2}, M)$,

$$(\delta g)(\curlyvee \otimes (a, b)) := g(a) \dashv b - g(a \dashv b) + a \dashv g(b),$$

$$(\delta g)(\curlyvee \otimes (a, b)) := g(a) \vdash b - g(a \vdash b) + a \vdash g(b).$$

Thus the 1-cocycles of D with values in M are the derivations, while the 1-coboundaries are the inner derivations. Hence we have the classical result

$$HY^1(D, M) \cong \mathrm{Der}(D, M)/\mathrm{Inn}(D, M).$$

Notice that we established a bijection between the set Y_2 of 2-trees and the set $\{\dashv, \vdash\}$ of binary operations of a dialgebra: the trees $\curlyvee$, $\curlyvee$ correspond respectively to the products $\dashv$, $\vdash$.

2.8 - Abelian extensions of dialgebras. Let D be a dialgebra over k. By definition, an *abelian extension* of D is an exact sequence of dialgebras

$$0 \longrightarrow M \xrightarrow{i} D' \xrightarrow{\pi} D \longrightarrow 0,$$

which is split as a sequence of k-modules and where $M = \ker\pi$ is an abelian dialgebra. The first assumption forces the dialgebra D' to have underlying k-module $M \oplus D$, with inclusion $i(m) = (m, 0)$, projection $\pi(m, a) = a$, sections $\sigma(a) = (g(a), a)$ for arbitrary k-linear maps $g : D \longrightarrow M$, and products of generic form

$$(m, a) \star (n, b) := (m \star b + a \star n + f(\star; a, b), a \star b), \qquad m, n \in M, \quad a, b \in D,$$

where the k-linear map $f\colon k[\dashv, \vdash] \otimes D^{\otimes 2} \longrightarrow M$ must satisfy a 'cocycle condition' given in (2.9). The second assumption naturally endows M with the structure of a D-representation,

$$m \star a := i(m) \star a' \qquad \text{and} \qquad a \star m := a' \star i(m), \qquad m \in M, a \in D,$$

where $a' \in D'$ is such that $\pi(a') = a$. Hence the element a' must be of the form (m', a), for some $m' \in M$, where it is clear that any choise of $m' \neq 0$ does not affect the above definition.

Given a representation M of D, an *abelian extension of D by M* is an abelian extension such that the structure of D-representation induced on M coincides with the given one. Two abelian extensions of D by M, say D' and D'', are *equivalent* if there exists a dialgebra map $D' \longrightarrow D''$ which is compatible with the identity on M and on D. We denote by $\mathrm{Ext}(D, M)$ the set of equivalence classes of extensions of D by M.

An abelian extension of D by M is called *trivial* if it is split as a sequence of dialgebras. The prototype is the dialgebra $M \oplus D$ with products associated to the zero map, $f = 0$:

$$(m, a) \star (n, b) := (m \star b + a \star n, a \star b).$$

In fact the short exact sequence $0 \longrightarrow M \longrightarrow M \oplus D \longrightarrow D \longrightarrow 0$ is surely split, with section $\sigma : D \longrightarrow M \oplus D$ defined by $\sigma(a) := (0, a)$.

2.9 - Theorem. *Let D be a dialgebra over k and M a representation of D. Then the map which associates a 2-cocycle f to the abelian extension of D by M given by the dialgebra $M \oplus D$, with products defined by f, yields a natural isomorphism*

$$HY^2(D, M) \cong \mathrm{Ext}(D, M).$$

Proof. 1. Let $f : k[Y_2] \otimes D^{\otimes 2} \longrightarrow M$ be a k-linear map. Using the bijection $\{\dashv, \vdash\} \simeq Y_2$ described above, consider the products on $M \oplus D$ defined by f. Fix the following bijection: the axioms 1, 2, 3, 4, 5 of (D) correspond respectively to the trees of Y_3. Then the five axioms (D) for the dialgebra $M \oplus D$ produce five conditions on the map f which are equivalent to the condition $\delta(f) = 0$. Hence the products defined on $M \oplus D$ by f determine a structure of abelian extension of D by M if and only if f is a 2-cocycle.

2. The abelian extension $M \oplus D$, with products defined by f, is trivial if and only if there exists a section $\sigma : D \longrightarrow M \oplus D$ which is a morphism of dialgebras, i.e. $\sigma(a \star b) = \sigma(a) \star \sigma(b)$ for any $a, b \in D$. The section must be of the form $\sigma(a) = (g(a), a)$, for a k-linear map $g : D \longrightarrow M$. Then, by definition of the products on $M \oplus D$ associated to f, the map g must satisfy the condition

$$g(a \star b) = g(a) \star b + a \star g(b) + f(\star; a, b), \qquad a, b \in D.$$

This is essentially $\delta(g) = f$. Hence the extension is trivial if and only if f is a 2-coboundary.

3. It remains to show that homology classes of 2-cocycles correspond bijectively to equivalence classes of abelian extensions. In other words, that when a 2-cocycle f is modified with a coboundary $\delta(g)$ into a new 2-cocycle $f' = f + \delta(g)$, the two corresponding extensions are equivalent. This is immediately proved, with equivalence given by the map $M \oplus D \longrightarrow (M \oplus D)' : (m, a) \mapsto (m + g(a), a)$. The inverse follows from step 2. $\square$

2.10 - Characteristic element of a dialgebra. Let D be a dialgebra and $D_{as} = D/Z(D)$ its canonical associative algebra. The ideal $Z(D)$ is an abelian sub-dialgebra of D and a representation over $(D_{as})_{di}$. Hence the short exact sequence

$$0 \longrightarrow Z(D) \longrightarrow D \longrightarrow (D_{as})_{di} \longrightarrow 0$$

presents the dialgebra D as abelian extension of $(D_{as})_{di}$ by $Z(D)$. Its class in $HY^2((D_{as})_{di}, Z(D))$ is called *characteristic element* of the dialgebra D, and denoted by $ch(D)$.

3 Universal enveloping algebra of dialgebras

The duality between homology and cohomology groups is reflected on their modules of coefficients. For associative and Lie algebras these are both bimodules. For Leibniz algebras, the category of representations (coefficients for cohomology) is not equivalent to its dual category: the dual modules are called corepresentations (see [L-P]). The same happens for dialgebras.

To define the corepresentations, for a dialgebra D, we introduce a unital associative algebra $E(D)$ such that the category of representations of D is isomorphic to the category of left modules over $E(D)$. Then the corepresentations of D are the right $E(D)$-modules.

The universal enveloping algebra of dialgebras has a property of Poincaré-Birchoff-Witt type (see (3.8)), which plays a fundamental role in the computation of dialgebra homology as a derived functor (theorem (5.1)).

3.1 - Definition (universal enveloping algebra). Let D be a dialgebra. By definition, the *universal enveloping algebra* $E(D)$ is the quotient of the free unital associative algebra $T(Dx \oplus Du \oplus Dy \oplus Dv)$ on four copies of D, labeled by the symbols x, u, y, v, by the ideal corresponding to the following 15 relations

$$(ED) \quad \begin{cases} x(a) \cdot x(b) \overset{1}{=} x(a \dashv b) \overset{2}{=} x(a \vdash b), \\ y(a) \cdot y(b) \overset{3}{=} y(b \dashv a) \overset{4}{=} y(b \vdash a), \\ x(a) \cdot y(b) \overset{5}{=} y(b) \cdot x(a), \\ x(a) \cdot u(b) \overset{6}{=} u(a \vdash b), \\ u(a) \cdot u(b) \overset{7}{=} u(a) \cdot x(b) \overset{8}{=} u(a \dashv b), \\ y(a) \cdot v(b) \overset{9}{=} v(b \dashv a), \\ v(a) \cdot v(b) \overset{10}{=} v(a) \cdot y(b) \overset{11}{=} v(b \vdash a), \\ x(a) \cdot v(b) \overset{12}{=} v(b) \cdot x(a) \overset{13}{=} v(b) \cdot u(a), \\ y(a) \cdot u(b) \overset{14}{=} u(b) \cdot y(a) \overset{15}{=} u(b) \cdot v(a), \end{cases} \qquad a, b \in D.$$

If the dialgebra D has a bar-unit 1, the element $x(1) = y(1) = 1_E$ is the unit of $E(D)$, while $u(1)$ is only a right unit for $u(a)$ and $v(1)$ is only a right unit for $v(a)$, for any $a \in D$.

3.2 - Proposition. *A k-module M is a (unital) representation of a (bar-unital) dialgebra D if and only if it is a (unital) left module over $\bar{E}(D)$.*

Proof. Given a D-representation M, one defines a left action of $E(D)$ on M, and conversely, by setting

$$\begin{aligned} x(a) \cdot m &:= a \vdash m, & y(a) \cdot m &:= m \dashv a, \\ u(a) \cdot m &:= a \dashv m, & v(a) \cdot m &:= m \vdash a, \end{aligned} \qquad m \in M, \quad a \in D.$$

The same formulas hold for unital D-representations and unital left $E(D)$-modules. $\square$

3.3 - Symmetric and antisymmetric enveloping algebras. Let D be a dialgebra. Call *symmetric enveloping algebra* the quotient

$$S(D) := E(D)/I(D), \qquad I(D) := \langle x(a) - u(a), y(a) - v(a) \mid a \in D \rangle.$$

A representation M of D is called *symmetric* if it is a left module over $S(D)$, i.e. if the left action $E(D) \otimes M \longrightarrow M$ passes to the quotient $E(D) \longrightarrow S(D)$ ($\Longleftrightarrow I(D) \cdot M = 0$). This is equivalent to require that M satisfies the conditions

$$(SRD) \quad \begin{cases} m \dashv a - m \vdash a = 0, \\ a \dashv m - a \vdash m = 0, \end{cases} \qquad m \in M, \quad a \in D.$$

Conversly, call *antisymmetric enveloping algebra* the quotient

$$A(D) := E(D)/J(D), \qquad J(D) := \langle u(a), v(a) \mid a \in D \rangle,$$

and *antisymmetric* a D-representation M which is a left module over $A(D)$ ($\Longleftrightarrow$ $J(D) \cdot M = 0$), i.e. which satisfies the condition

$$(ARD) \qquad a \dashv m = 0 = m \vdash a, \qquad m \in M, \quad a \in D.$$

3.4 - Proposition. *For a dialgebra D, the category of symmetric D-representations, that of antisymmetric D-representations and the category of bimodules over D_{as} are all equivalent.*

Proof. Take a symmetric D-representation M and define an antisymmetric representation M' on the k-module M by setting $m \dashv' a := m \dashv a$, $m \vdash' a := 0$ and $a \vdash' m := a \vdash m$, $a \dashv' m := 0$. Similarly define a bimodule M_{as} over the associative algebra D_{as} by setting $m \cdot [a] := m \dashv a$ and $[a] \cdot m := a \vdash m$, where $[a]$ denotes the image of $a \in D$ in the projection $D \longrightarrow D_{as}$.

Both functors $M \mapsto M'$ and $M \mapsto M_{as}$ are faithful and full. For instance, if $f : M' \longrightarrow N'$ is a morphism of D-representations, where M' and N' are antisymmetric representations defined on symmetric representations M and N, then f is induced by the morphism $F : M \longrightarrow N$, $F(m) := f(m)$. Hence, with their inverse functors, they define equivalences of categories. □

3.5 - Examples.

1. The universal enveloping algebra $E(D)$ is itself, clearly, a representation of the dialgebra D, with actions given in (3.2) for $m \in E(D)$. From relations 5, 7, 10, 12, 13, 14 and 15 of (ED) follow immediately some additional properties of these actions (beside associativity), and in particular relations 7, 13, 15 and 10 imply that $J(D) \cdot I(D) = 0$, hence the ideal $I(D)$ of $E(D)$ is an antisymmetric representation of D.

2. **$E(A)$ for associative algebras.** For an associative algebra A, the classical enveloping algebra $A^e = A \otimes A^{op} \oplus A \oplus A^{op} \oplus k$ characterizes the category of bimodules over A. Regarding A as a dialgebra, the enveloping algebra $E(A) := E(A_{di})$ is generated by the set $\{x(a), u(a), y(a), v(a) | \ a \in A\}$ with the 15 relations (EA) obtained from (ED) by setting $\dashv = \vdash = \cdot$. In particular, these relations show that the maps $x, u : A \longrightarrow E(A)$ and $y, v : A^{op} \longrightarrow E(A)$ are morphisms of algebras. Then the unital morphism of algebras $A^e \longrightarrow E(A)$ defined by

$$A \ni a \mapsto x(a), \qquad A^{op} \ni b \mapsto y(b),$$
$$A \otimes A^{op} \ni a \otimes b \mapsto x(a)y(b),$$

yields an inclusion of A^e in $E(A)$. Hence *the algebra A^e is isomorphic to the unital subalgebra of $E(A)$ generated by the elements $x(a), y(a)$ for any $a \in A$.*

If A has a unit 1, its classical enveloping algebra is $A^e = A \otimes A^{op}$, with unit $1 \otimes 1$. In $E(A)$, the elements $u(1)$ and $v(1)$ are also both side units, thus $x(1) = u(1) = y(1) = v(1) = 1_E$.

3. **Universal enveloping algebra of a Leibniz algebra.** Let $\mathfrak{g}$ be a Leibniz algebra. The *universal enveloping algebra* $UL(\mathfrak{g})$ is the quotient of the free associative unital algebra $T(\mathfrak{g}^l \oplus \mathfrak{g}^r)$ by the ideal corresponding to the three following relations:

$$(UL) \quad \begin{cases} r([a,b]) \stackrel{1}{=} -r(a)r(b) + r(b)r(a), \\ l([a,b]) \stackrel{2}{=} -l(a)r(b) + r(b)l(a), \\ l(a)[l(b) + r(b)] \stackrel{3}{=} 0, \end{cases} \qquad a, b \in D.$$

In [L-P] it is shown that the category of $\mathfrak{g}$-representations is equivalent to the category of left modules over $UL(\mathfrak{g})$. (Stricktly talking, in [L-P] representations of $\mathfrak{g}$ are *right* modules over the algebra $UL(\mathfrak{g})$, which is then slightely different from ours.) The assignement

$$\begin{aligned} r(a) &\mapsto y(a) - x(a), \\ l(a) &\mapsto u(a) - v(a), \end{aligned} \qquad a \in D$$

yields a morphism of algebras $UL(D_L) \longrightarrow E(D)$.

3.6 - Graded enveloping algebra. Given a dialgebra D, the tensor algebra $T(Dx \oplus Du \oplus Dy \oplus Dv)$ is endowed with a length-filtration, where the degree $\deg(e)$ of a homogeneus element e is the length of the sequence of elements $x(a), u(b), y(c), v(d)$ componing e. In fact there is a canonical isomorphism of graded k-modules

$$T(Dx \oplus Du \oplus Dy \oplus Dv) \cong T(Dx) * T(Du) * T(Dy) * T(Dv),$$

where the symbol $*$ denotes the non-commutative tensor product[1]. This filtration induces on the quotient $E(D)$ a length-filtration given by the k-modules

$$F_k E(D) := \{e \in E(D) |\ \deg(e) \leq n\}, \qquad k \geq 0,$$

satisfying the standard properties $\bigcup_{k\geq 0} F_k E(D) = E(D)$, $F_k E(D) \subset F_{k+1} E(D)$ for any $k \geq 0$, $F_p E(D) \cdot F_q E(D) \subset F_{p+q} E(D)$ for any $p, q \geq 0$ and such that $F_0 E(D) = k$.

[1]The symbol $*$ is defined on two k-modules V and W by $V * W := V \otimes W \oplus W \otimes V$, [L4]. It can be extended to graded k-modules $T(V)$ and $T(W)$ as the graded k-module with homogeneous components

$$(T(V) * T(W))^{(n)} = \bigoplus_{\underline{\lambda} \vdash n} V^{\otimes \lambda_1} \otimes W^{\otimes \lambda_2} \otimes V^{\otimes \lambda_3} \otimes \cdots \otimes W^{\otimes \lambda_k},$$

where the direct sum is taken over all partitions $\underline{\lambda}$ of n (ordered sets of integers $(\lambda_1, ..., \lambda_k)$ such that $\sum_i \lambda_i = n$).

Denote by $GrE(D) = \bigoplus_{k\geq 0} Gr^k E(D)$ the graded algebra associated to this filtration, so $Gr^0 E(D) = k$ and $Gr^k E(D) = F_k E(D)/F_{k-1}E(D)$.

3.7 - Proposition. *Let D be a dialgebra with graded enveloping algebra $Gr^* E(D)$. Then,*

1. $Gr^1 E(D) \cong Dx \oplus Du \oplus Dy \oplus Dv$,

2. $Gr^2 E(D) \cong D^{\otimes 2}xy \oplus D^{\otimes 2}uv \oplus D^{\otimes 2}vu$,

3. $Gr^k E(D) = 0$ *for any* $k \geq 3$.

Proof. The assertion 1. is obvious.

2. An element of degree 2 in $E(D)$ is the class $[e]$ of an element of degree 2 of the tensor algebra $T(Dx \oplus Du \oplus Dy \oplus Dv)$, in the equivalence given by relations (ED). The element e must be of the form $e_1(a) \cdot e_2(b)$ for some elements $a, b \in D$, where each e_i, for $i = 1, 2$, is one of the four maps x, u, y, v. Among the 16 possible products $e_1(a) \cdot e_2(b)$ of this kind in the tensor algebra (for fixed $a, b \in D$), 7 products are identified with some others in the quotient algebra $E(D)$ because of relations (5), (7), (10), (12), (13), (14) and (15). Then, among the 9 distinct products which remain in $E(D)$, 6 products are equivalent to elements of degree 1 because of relations (1), (3), (6), (8), (9) and (11), yielding zero-classes in the quotient $F_2E(D)/F_1E(D)$. Hence in $F_2E(D)/F_1E(D)$ there remain only three distinct products, which can be chosen to be $x(a)y(b)$, $u(a)v(b)$ and $v(a)u(b)$ for any $a, b \in D$.

3. It suffices to show the identity $F_kE(D) = F_{k+1}E(D)$ for $k = 2$. Since the elements of degree 2 in $E(D)$ are of the form descrived above, it is easy to see by direct computations, using relations (ED), that any element of degree 2 multiplied from left or right for an element of degree 1 yields an element still of degree 2. For instance, the element $x(a)y(b)$ multiplied for $u(c)$ from left yields an element of degree 2 by relation (8). $\square$

3.8 - Corollary (Poincaré-Birchoff-Witt Isomorphism). *Let V denote the underlying k-module of a dialgebra D. Define on V a structure of abelian dialgebra, by setting $a \star b := 0$ for any $a, b \in V$. Then there is an isomorphism of associative unital algebras*

$$GrE(D) \cong E(V).$$

4 Dialgebra homology and corepresentations

In this section we introduce the dialgebra homology with coefficients as the dual theory of the cohomology defined in section 2.

4.1 - Corepresentations. Let D be a dialgebra with universal enveloping algebra $E(D)$. By definition, a *corepresentation* N of D is a right module over $E(D)$. Set

$$\begin{aligned} a \Vdash n &:= n \cdot x(a), & n \dashV a &:= n \cdot y(a), \\ a \dashV n &:= n \cdot u(a), & n \Vdash a &:= n \cdot v(a), \end{aligned} \qquad n \in N, \quad a \in D.$$

Then the k-module N is equipped with two right actions $\dashV, \Vdash : N \otimes D \longrightarrow N$ and two left actions $\dashV, \Vdash : D \otimes N \longrightarrow N$ satisfying the following 15 axioms:

$$(CD) \quad \begin{cases} n \dashV (a \dashv b) \overset{1}{=} (n \dashV b) \dashV a \overset{2}{=} n \dashV (a \vdash b), \\ (n \dashV a) \Vdash b \overset{3}{=} n \Vdash (b \dashv a), \\ (n \Vdash a) \Vdash b \overset{4}{=} n \Vdash (b \vdash a) \overset{5}{=} (n \Vdash a) \dashV b, \\ (a \dashV n) \dashV b \overset{6}{=} a \dashV (n \dashV b) \overset{7}{=} (a \dashV n) \Vdash b, \\ (a \Vdash n) \dashV b \overset{8}{=} a \Vdash (n \dashV b), \\ a \Vdash (n \Vdash b) \overset{9}{=} (a \Vdash n) \Vdash b \overset{10}{=} a \dashV (n \Vdash b), \\ a \dashV (b \dashV n) \overset{11}{=} (b \dashv a) \dashV n \overset{12}{=} a \Vdash (b \dashV n), \\ (a \vdash b) \dashV n \overset{13}{=} b \dashV (a \Vdash n), \\ (a \vdash b) \Vdash n \overset{14}{=} b \Vdash (a \Vdash n) \overset{15}{=} (a \dashv b) \Vdash n, \end{cases} \qquad n \in N, \quad a, b \in D.$$

If D has a bar-unit 1, a corepresentation N is called *unital* if $1 \Vdash n = n = n \dashV 1$ for all $n \in N$.

Notice that, unlike for representations, a dialgebra D is *not* a corepresentation of itself!

If D_L is the canonical Leibniz algebra of D, on the k-module N we obtain a corepresentation N_L of D_L, as defined in [L-P], by setting

$$[\![n, a]\!] := a \dashV n - n \Vdash a \qquad \text{and} \qquad [\![a, n]\!] := n \dashV a - a \Vdash n, \qquad n \in N, \quad a \in D.$$

4.2 - Symmetric and antisymmetric corepresentations. Let D be a dialgebra. A D-corepresentation N is called *symmetric* if it is a right module over the symmetric enveloping algebra $S(D)$. This is equivalent to require that the actions of $E(D)$ on N satisfy the conditions

$$(SCD) \quad \begin{cases} n \dashV a - n \Vdash a = 0, \\ a \dashV n - a \Vdash n = 0, \end{cases} \qquad n \in N, \quad a \in D.$$

The corepresentation N is called *antisymmetric* if it is a right module over the antisymmetric enveloping algebra $A(D)$, i.e. if it satisfies

$$(ACD) \qquad a \dashV n = 0 = n \Vdash a, \qquad n \in N, \quad a \in D.$$

If N is a symmetric corepresentation of D (resp. antysimmetric), then N_L is a symmetric corepresentation of D_L (resp. antisymmetric).

4.3 - Examples.

1. Given a dialgebra D over k, any k-module N becomes a D-corepresentation, called *trivial*, by setting $n \star a := 0 =: a \star n$ for any $n \in N$ and $a \in D$.

(Here $\star$ denotes both symbols $\dashv\!\!\dashv$ and $\vdash\!\!\vdash$.) If D is bar-unital, the trivial corepresentation can not be unital, unless $N = 0$.

2. Given a dialgebra D with a corepresentation N, the k-module $\mathrm{M}_r(N)$ of matrices with entries in N is a corepresentation of the dialgebra $\mathrm{M}_r(D)$, with 'row $\times$ column' actions.

3. The universal enveloping algebra $E(D)$ is itself, clearly, a corepresentation of the dialgebra D, with actions given in (4.1) for $n \in E(D)$. Moreover, relations 7, 15, 13, 10 of (ED) imply that $J(D) \cdot I(D) = 0$, hence the ideal $J(D)$ of $E(D)$ is a symmetric D-corepresentation.

4. **Isotypical decomposition.** Given a corepresentation N of a dialgebra D, the quotient

$$S(N) := N/NI(D), \qquad NI(D) := \{n \cdot e \in N \mid n \in N, e \in I(D)\},$$

is a symmetric corepresentation, since $S(N) \cdot I(D) = 0$. Similarly, the quotient of N

$$A(N) := N/NJ(D), \qquad NJ(D) := \{n \cdot e \in N \mid n \in N, e \in J(D)\},$$

is an antisymmetric corepresentation, because $A(N) \cdot J(D) = 0$. Moreover $NJ(D)$ is itself a symmetric corepresentation of D, hence N can be decomposed into the direct sum

$$N \cong A(N) \oplus NJ(D),$$

where $A(N)$ is antisymmetric and $NJ(D)$ is symmetric.

5. **Opposite corepresentations.** Let D be a dialgebra. Given a D-representation M, we can define two D-corepresentations on the k-module M. The first one, with actions

$$m \dashv\!\!\dashv a := a \vdash m =: m \vdash\!\!\vdash a,$$
$$a \dashv\!\!\dashv m := m \dashv a =: a \vdash\!\!\vdash m,$$

is by definition a symmetric corepresentation of D, and the second one, with actions

$$m \dashv\!\!\dashv a := a \vdash m, \qquad m \vdash\!\!\vdash a := 0,$$
$$a \vdash\!\!\vdash m := m \dashv a, \qquad a \dashv\!\!\dashv m := 0,$$

is antisymmetric. We call them *symmetric* and *antisymmetric opposite corepresentation* of M respectively, and denote them both by M^{op}. In particular, since D is a representation over itself, we can consider the symmetric or antisymmetric corepresentation D^{op}.

This procedure can be inverted. Given a corepresentation N, analogue formulas give the *symmetric* and *antisymmetric opposite representation* of N, denoted both by N^{op}.

6. Let D be a dialgebra, and let D^{op} be the symmetric corepresentation on D. Then D_L^{op} is a corepresentation of D_L, with actions

$$\begin{aligned} [\![a,b]\!] &= b \dashv\!\dashv a - a \Vdash b = a \dashv b - b \vdash a = [a,b], \\ [\![b,a]\!] &= a \dashv\!\dashv b - b \Vdash a = b \vdash a - a \dashv b = -[a,b], \end{aligned} \qquad a \in D_L^{op}, \quad b \in D_L.$$

It is clearly symmetric.

4.4 - Proposition. *The category of symmetric (or antisymmetric) representations of a dialgebra D, that of symmetric (or antisymmetric) corepresentations of D and the category of bimodules over the associative algebra D_{as} are all equivalent.*

Proof. After (3.4), we only need to show one of these equivalences, for instance that from antisymmetric representations to antisymmetric corepresentations. Take a morphism of corepresentations $f : M^{op} \longrightarrow M'^{op}$, where M^{op} and M'^{op} are the opposite of two representations. Then the k-linear map $f : M \longrightarrow M'$, on the underlying k-modules, is a morphism of representations. In fact,

$$f(m \dashv a) = f(a \Vdash m) = a \Vdash f(m) = f(m) \dashv a, \qquad m \in M, \quad a \in D,$$

and similarly $f(m \vdash a) = 0 = f(a \dashv m)$ and $f(a \vdash m) = a \vdash f(m)$. Hence the functor $M \mapsto M^{op}$ is full. Now take two representations M and M' such that their opposite corepresentations are isomorphic, $M^{op} \cong M'^{op}$. Clearly they are isomorphic as k-modules, $M \cong M'$, and also as D-representations, in fact,

$$m \dashv a = a \Vdash m = a \Vdash' m = m \dashv' a, \qquad m \in M, \quad a \in D,$$

and similarly $m \vdash a = 0 = m \vdash' a$, $a \dashv m = 0 = a \dashv' m$, and $a \vdash m = a \vdash' m$. Hence the functor $M \mapsto M^{op}$ is faithful. Finally, by definition, the functor $N \mapsto N^{op}$ from an antisymmetric corepresentation to its opposite antisymmetric representation is its inverse. □

4.5 - Dialgebra complex. Let D be a dialgebra over k and N be a corepresentation of D. For any $n \geq 0$, define the *module of n-chains of D with coefficients in N* as

$$CY_n(D,N) := k[Y_n] \otimes N \otimes D^{\otimes n},$$

and the boundary operator $d : CY_n(D,N) \longrightarrow CY_{n-1}(D,N)$ as the k-linear map $d = \sum_{i=0}^{n}(-1)^i d_i$ given by means of the face maps $d_i(y \otimes \underline{a}) = d_i(y) \otimes d_i^y(\underline{a})$, where the faces d_i on the tree y are defined in (2.3) and for any $\underline{a} = (a_0, a_1, ..., a_n) \in N \otimes D^{\otimes n}$ the terms

$$d_i^y(a_0, a_1, ..., a_n) := \begin{cases} (a_1 \circ_0^y a_0, a_2, ..., a_n), & \text{for } i = 0, \\ (a_0, a_1, ..., a_i \circ_i^y a_{i+1}, ..., a_n) & \text{for } i = 1, ..., n-1, \\ (a_0 \circ_n^y a_n, a_1, ..., a_{n-1}), & \text{for } i = n, \end{cases}$$

contain the operations $\circ_i^y$ defined in (2.3). (Notice the side on which the elements a_1 and a_n of D act on the element a_0 of N.) In particular, the operations $\circ_0^y$ and $\circ_n^y$ yield

$$a_1 \circ_0^y a_0 := \begin{cases} a_1 \dashv\!\!\mid a_0, & \text{if } y \text{ has the shape}, \\ a_1 \mid\!\!\vdash a_0, & \text{if } y \text{ has the shape}, \end{cases}$$

$$a_0 \circ_n^y a_n := \begin{cases} a_0 \mid\!\!\vdash a_n, & \text{if } y \text{ has the shape}, \\ a_0 \dashv\!\!\mid a_n, & \text{if } y \text{ has the shape}. \end{cases}$$

When $N = D^{op}$ is the symmetric corepresentation on D, the module of n-chains is

$$CY_n(D, D^{op}) = k[Y_n] \otimes D^{\otimes n+1},$$

and the face maps d_0 and d_n take the simple form

$$d_0(y \otimes (a_0, a_1, ..., a_n)) = d_0(y) \otimes (a_0 \dashv a_1, a_2, ..., a_n),$$
$$d_n(y \otimes (a_0, a_1, ..., a_n)) = d_n(y) \otimes (a_n \vdash a_0, a_1, ..., a_{n-1}),$$

for any n-tree y and $(a_0, a_1, ..., a_n) \in D^{\otimes n+1}$.

In order to prove that $(CY_*(D, N), d)$ is a chain complex, it suffices to show that the chain modules $CY_n(D, N)$ are dual to the cochain modules $CY^n(D, M)$, in the category of modules over $E(D)$, and that the faces d_i and δ^i correspond each other in this duality. To prove it, we introduce the suitable 'mean term' of the duality.

4.6 - Cotangent complex of a dialgebra. We call *module of cotangent n-chains* of a dialgebra D the module of n-chains of D with coefficients in the universal enveloping algebra $E(D)$,

$$EY_n(D) := CY_n(D, E(D)) = k[Y_n] \otimes E(D) \otimes D^{\otimes n}.$$

The first and last face maps, in this case, take the following form:

$$d_0(y \otimes (e, a_1, ..., a_n)) = d_0(y) \otimes \begin{cases} (e \cdot u(a_1), a_2, ..., a_n), & \text{if } y \text{ has the shape}, \\ (e \cdot x(a_1), a_2, ..., a_n), & \text{if } y \text{ has the shape}, \end{cases}$$

and similarly

$$d_n(y \otimes (e, a_1, ..., a_n)) = d_n(y) \otimes \begin{cases} (e \cdot v(a_n), a_0, ..., a_{n-1}), & \text{if } y \text{ has the shape}, \\ (e \cdot y(a_n), a_0, ..., a_{n-1}), & \text{if } y \text{ has the shape}, \end{cases}$$

for any n-tree y, any element $e \in E(D)$ and $a_i \in D$ for $i = 1, ..., n$. In particular, the first boundary operator $d : E(D) \otimes D \longrightarrow E(D)$ is explicitely defined as

$$d(e \otimes a) = e \cdot (u(a) - v(a)), \qquad e \in E(D), \qquad a \in D.$$

Remark that the cotangent complex in the context of associative algebras is the bar complex shifted by 1, while for commutative algebras it was defined by D. Quillen in [Q].

4.7 - Lemma. *Let D be a dialgebra with universal enveloping algebra $E(D)$.*

(i) For any D-representation M, the isomorphism of cochain modules

$$CY^n(D,M) \cong Hom_{E(D)}(EY_n(D),M)$$

commutes with the coboundary δ on $CY^*(D,M)$ and the coboundary d^* induced on $Hom_{E(D)}(EY_*(D),M)$ by the boundary operator d on $EY_*(D)$. Therefore $(EY_*(D),d)$ is a chain complex.

(ii) For any corepresentation N of D the isomorphism of chain modules

$$CY_*(D,N) \cong N \otimes_{E(D)} EY_*(D)$$

commutes with the boundaries. Hence $(CY_*(D,N),d)$ is also a chain complex.

Proof. (i) For any $n \geq 0$, the module of cotangent n-chains $EY_n(D) = k[Y_n] \otimes E(D) \otimes D^{\otimes n}$ is a left module over $E(D)$, with action given by the multiplication in the algebra $E(D)$. Therefore, in the dual space $Hom_{E(D)}(EY_n(D),M)$ of left $E(D)$-modules morphisms from $EY_n(D)$ to a D-representation M, it suffices to consider the value of any map f on the unit 1_E of $E(D)$, since

$$f(y \otimes e \otimes \underline{a}) = e \cdot f(y \otimes 1_E \otimes \underline{a})$$

for any $e \in E(D)$. Hence the canonical isomorphism of n-cochain modules

$$CY^n(D,M) = \mathrm{Hom}_k(k[Y_n] \otimes D^{\otimes n}, M) \cong Hom_{E(D)}(EY_n(D),M) = EY_n(D)^*$$

associates a k-linear map $f : k[Y_n] \otimes D^{\otimes n} \longrightarrow M$ to its $E(D)$-linear extension $\tilde{f}(y \otimes e \otimes \underline{a}) := e \cdot f(y \otimes \underline{a})$, and conversely an $E(D)$-linear map $g : k[Y_n] \otimes E(D) \otimes D^{\otimes n} \longrightarrow M$ to its restriction on $\{1_E\} \subset E(D)$, $\tilde{g}(y \otimes \underline{a}) := g(y \otimes 1_E \otimes \underline{a})$. Then the maps $d_i^* : Hom_{E(D)}(EY_n(D),M) \longrightarrow Hom_{E(D)}(EY_{n+1}(D),M)$ dual to the faces d_i defined above satisfy

$$(d_i^* f)(y \otimes 1_E \otimes \underline{a}) := f(d_i(y \otimes 1_E \otimes \underline{a})) = f(d_i(y) \otimes 1_E \otimes d_i(\underline{a})) = (\delta^i f)(y \otimes \underline{a}),$$

for any $i = 1, ..., n$, and similarly

$$\begin{aligned}(d_0^* f)(y \otimes 1_E \otimes \underline{a}) &= f(d_0(y) \otimes a_0 \circ_0^y 1_E \otimes \underline{a}') = l(a_1) \cdot f(d_0(y) \otimes \underline{a}') \\ &= a_0 \circ_0^y f(d_0(y) \otimes \underline{a}') = (\delta^0 f)(y \otimes \underline{a}),\end{aligned}$$

where $l(a_1) = x(a_1)$ or $u(a_1)$, and finally

$$\begin{aligned}(d_{n+1}^* f)(y \otimes 1_E \otimes \underline{a}) &= f(d_{n+1}(y) \otimes 1_E \circ_{n+1}^y a_{n+1} \otimes \underline{a}'') \\ &= r(a_{n+1}) \cdot f(d_{n+1}(y) \otimes \underline{a}'') \\ &= f(d_{n+1}(y) \otimes \underline{a}'') \circ_{n+1}^y a_{n+1} = (\delta^{n+1} f)(y \otimes \underline{a}),\end{aligned}$$

where $r(a_1) = y(a_{n+1})$ or $v(a_1)$. It follows that the faces d_i on $EY_*(D)$ satisfy the dual simplicial relations $d_i d_j = d_{j-1} d_i$ for any $i < j$, and $(EY_*(D), d)$ is a chain complex.

(ii) This follows, by applying the isomorphism of chain modules

$$CY_*(D, N) = k[Y_n] \otimes N \otimes D^{\otimes n} \cong k[Y_n] \otimes N \otimes_{E(D)} E(D) \otimes D^{\otimes n},$$

with straightforward computations on the faces. □

4.8 - Definition (dialgebra homology). Let D be a dialgebra and N a corepresentation of D. We call *dialgebra homology of D with coefficients in N* the homology of the chain complex defined in (4.5),

$$HY_n(D, N) := H_n(CY_*(D, N), d), \qquad n \geq 0.$$

We call *dialgebra homology of D with coefficients in itself* the homology

$$HHY_*(D) := HY_*(D, D^{op})$$

with coefficients in the symmetric corepresentation D^{op}. When A is an associative algebra, and M is an A-bimodule, we simply denote by

$$HY_*(A, M) := HY_*(A_{di}, M^{op})$$

the homology of A regarded as a dialgebra, where M^{op} is the symmetric corepresentation on M.

4.9 - Elementary computations. Given a dialgebra D with a corepresentation N, the first boundary of D in N is the k-linear map $d : N \otimes D \longrightarrow N$, $d(n, a) := a \dashV n - n \Vdash a$. Thus $HY_0(D, M)$ is the module of coinvariants of N by the Leibniz algebra D_L,

$$HY_0(D, N) = N/[\![N, D]\!],$$

where $[\![N, D]\!] := \{ [\![n, a]\!] = a \dashV n - n \Vdash a \mid n \in N, a \in D\}$. In particular, when $N = D^{op}$, by definition of the actions of D on D^{op} we recover the Leibniz bracket $d(a, b) = b \dashV a - a \Vdash b = a \dashv b - b \vdash a = [a, b]$ for any $a, b \in D$. Hence $[\![D^{op}, D]\!] = [D, D]$ and

$$HHY_0(D) = D/[D, D].$$

When $N = E(D)$ is the universal enveloping algebra of D, and D has a bar-unit 1, we can define a natural map $\pi : E(D) \longrightarrow D$ of left $E(D)$-modules as the $E(D)$-linear extension of the map which sends the unit of $E(D)$ into the bar-unit of D,

$$\pi : E(D) \longrightarrow D, \quad e \mapsto e \cdot 1,$$

where $e \cdot 1$ is the left action of $E(D)$ on D. The map π is then an augmentation for the cotangent complex, that is, $\pi \circ d = 0$, where $d : E(D) \otimes D \longrightarrow E(D)$ is the first boundary. In fact,

$$\pi \circ d(e \otimes a) = \pi(e \cdot (u(a) - v(a))) = e \cdot (u(a) - v(a)) \cdot 1 = e \cdot (a \dashv 1 - 1 \vdash a) = 0,$$

for any $e \in E(D)$ and $a \in D$. Hence

$$HY_0(D, E(D)) = M(D) := E(D)/\langle u(a) - v(a) |\ a \in D\rangle \cong \ker\pi.$$

4.10 - Bar-homology of dialgebras. When $N = k$ is the trivial D-corepresentation, the faces d_0 and d_n become zero on the module of n-chains of D. The boundary operator d then coincides with the boundary $d' = \sum_{i=1}^{n-1}(-1)^i d_i$ defined in [L5], and we recover the bar homology of D,

$$HY_*(D, k) = HY_*(D).$$

In particular, clearly, $HY_0(D, k) \cong k$.

For associative algebras it is well known that the Hochschild bar complex yields zero on homology, when the algebra has a unit (Proposition (1.1.12) in [L1]). The same holds for dialgebras, when they are equipped with a bar-unit.

4.11 - Theorem. *Let D be a dialgebra with bar-unit. Then the bar complex of D is acyclic, i.e.*

$$HY_n(D) = 0, \qquad n \geq 1.$$

Proof. Consider the identity and the zero maps on the chain complex $CY_*(D)$. If the dialgebra D has a bar-unit 1, we can define a homotopy between these two maps by setting

$$h : k[Y_n] \otimes D^{\otimes n} \longrightarrow k[Y_{n+1}] \otimes D^{\otimes n+1}, \qquad h(y \otimes \underline{a}) := s_0(y) \otimes (1, \underline{a}),$$

where $\underline{a} = (a_1, ..., a_n)$ is a generic element of $D^{\otimes n}$, and $s_0(y)$ is the $(n+1)$-tree obtained by bifurcating the 0^{th} leaf of y, i.e. by replacing the 0^{th} leaf $|$ with the branch Y. In order to show that

$$d'h + hd' = id,$$

we only need to verify that $d_1 h = id$ and $d_i h = h d_{i-1}$ for any $i = 2, ..., n+1$. In fact, for the first equality we have

$$d_1 h(y \otimes \underline{a}) = d_1 s_0(y) \otimes (1 \vdash a_1, ..., a_n) = y \otimes \underline{a},$$

because clearly $d_1 s_0$ deletes a leaf which has just been added, and for the second equality we have

$$\begin{aligned} d_i h(y \otimes \underline{a}) &= d_i s_0(y) \otimes (1, a_1, ..., a_{i-1} \circ_i^{s_0(y)} a_i, ..., a_n) \\ &= s_0 d_{i-1}(y) \otimes (1, a_1, ..., a_{i-1} \circ_{i-1}^{y} a_i, ..., a_n) = h d_{i-1}(y \otimes \underline{a}), \end{aligned}$$

if we observe that the i^{th} leaf of $s_0(y)$ is the leaf number $i-1$ of y, and that the two operators $d_i s_0$ and $s_0 d_{i-1}$ produce the same modification. □

The hypothesis that D has a bar-unit can be relieved to the assumption that there is an element $1 \in D$ such that $1 \vdash a = a$ for all $a \in D$. Dually, we could require that $a \dashv 1 = a$ and then use the homotopy $h(y \otimes \underline{a}) := s_n(y) \otimes (\underline{a}, 1)$, where s_n bifurcates the n^{th} leaf of y.

5 Dialgebra (co)homology as derived functor

In this section we prove our main theorem.

5.1 - Theorem. *Let D be a dialgebra and $E(D)$ its universal enveloping algebra. For any corepresentation N of D, there is an isomorphism*

$$HY_n(D,N) \cong \mathrm{Tor}_n^{E(D)}(M(D),N), \qquad n \geq 0,$$

where the left $E(D)$-module $M(D) := E(D)/\langle u - v\rangle$ is the cokernel of the first boundary operator.

Dually, for any representation M of D, there is an isomorphism

$$HY^n(D,M) \cong \mathrm{Ext}^n_{E(D)}(M, M(D)), \qquad n \geq 0.$$

Proof. Consider the cotangent chain complex $EY_n(D) = k[Y_n] \otimes E(D) \otimes D^{\otimes n}$. If N is any D-corepresentation, by (4.7) we have an isomorphism of complexes $CY_*(D,N) \cong N \otimes_{E(D)} EY_*(D)$. The chain modules $EY_n(D)$ are free over $E(D)$, hence projective. Then, in order to prove the Tor interpretation for dialgebra homology, we only need to show that $EY_*(D)$ is a resolution of $M(D)$. To this proof, which follows from corollary (5.10), is devoted the remaining part of this section. The second statement follows from the duality between homology and cohomology. □

5.2 - Definition (dialgebra chain bicomplex). Let D be a dialgebra over a field k, and N a D-corepresentation. For any $p,q \geq 0$, consider the set $Y_{p,q}$ of planar binary trees with p leaves oriented like \ and q leaves oriented like /. Define a k-linear map $d^h : CY_{p,q}(D,N) \longrightarrow CY_{p-1,q}(D,N)$, $d^h := \sum_{i=0}^{p+q+1}(-1)^i d_i^h$, by means of the face maps

$$d_i^h(y \otimes \underline{a}) := d_i^h(y) \otimes d_i^y(\underline{a}),$$

where $d_i^h(y)$ is the horizontal face acting on the tree, defined as $d_i(y)$, if $d_i(y) \in k[Y_{p-1,q}]$, and 0 otherwise, and the term $d_i^y(\underline{a})$ is defined in (4.5). Similarly, using the vertical faces on trees, we can introduce a k-linear map $d^v : CY_{p,q}(D,N) \longrightarrow CY_{p,q-1}(D,N)$.

5.3 - Lemma. *The family $(CY_{p,q}(D,N), d^h, d^v)$ forms a bicomplex, whose total complex is the shifted dialgebra complex $CY_{*-1}(D,N)$.*

Proof. By definition, the orientation of the faces on $CY_{p,q}(D,N)$ just depends on the orientation of the face maps in the bicomplex of trees. So the identities $d^hd^h = 0 = d^vd^v$ and $d^vd^h + d^hd^v = 0$, follow immediately from lemma (4.7) and the analogue result on the bicomplex of trees which is proved in [F4]. Finally, since

$$CY_n(D,N) = \bigoplus_{p+q+1=n} CY_{p,q}(D,N)$$

and $d = d^h + d^v$, it is evident that $CY_{*-1}(D,N) = \mathrm{Tot}(CY_{*,*}(D,N)$. $\square$

As a consequence, there is a spectral sequence abutting to the dialgebra homology.

5.4 - Theorem. *Let D be a dialgebra and N a D-corepresentation. There is a spectral sequence $E^2_{pq} \Longrightarrow HY_{p+q+1}(D,N)$ which converges to the dialgebra homology of D, with*

$$E^1_{pq} := H_q(CY_{p,*}(D,N), d^v),$$
$$E^2_{pq} := H_p(E^1_{*q}, (d^h)_*) = H_pH_q(CY_{*,*}(D,N)).$$

5.5 - Vertical towers. For any p-tree y, let $T_*(y) \subset k[Y_{p,*}]$ be the vertical tower defined in [F4]. Recall that the tower $T_*(y)$ is constructed on the base tree associated to y (which is the $(p,*)$-tree $\tilde{y}$ obtained by bifurcating all the leaves of y), by adding /-leaves in all possible distinct ways. In [F4] it is shown that the tower $T_*(y)$, on a p-tree y, is a multicomplex with dimension $d = 2p+1$ in the vertical complex $k[Y_{p,*}]$, and moreover that $k[Y_{p,*}]$ can be decomposed as the direct sum of the towers $T_*(y)$, for $y \in Y_p$.

Now consider a dialgebra D and a D-corepresentation N. For any p-tree y, we define a family of dialgebra chain submodules

$$CT^y_m(D,N) := T_m(y) \otimes N \otimes D^{\otimes p+q_y+m}, \qquad m \geq 0$$

where q_y denotes the number of /-leaves of y. It is easy to see that the family $CT^y_*(D,N)$ is a multi-complex with dimension $d = 2p+1$ in the vertical complex $CY_{p,*}(D,N)$.

5.6 - Lemma. *Let D be a dialgebra and N a D-corepresentation. For any $p \geq 0$, the vertical complex $(CY_{p,*}(D,N), d^v)$ decomposes as a direct sum of subcomplexes,*

$$CY_{p,*}(D,N) = \bigoplus_{y \in Y_p} CT^y_{*+q_y}(D,N).$$

Consequently, the E^1-plane of the spectral sequence (5.4) is

$$E^1_{pq} = \bigoplus_{y \in Y_p} H_q(CT^y_{*+q_y}(D,N), d^v).$$

Proof. Applying the definition of the towers one sees that this result does not involve the dialgebra structure of D. Hence it follows from the analogue proposition on towers of trees proved in [F4]. $\square$

5.7 - Proposition. *Let D be an abelian dialgebra, and $E(D)$ its universal enveloping algebra. Then $HY_0(D,E(D)) \cong E(D)/\langle u-v \rangle$ and*

$$HY_n(D,E(D)) = 0, \qquad n \geq 1.$$

Proof. Consider the bicomplex $EY_{p,q}(D) = k[Y_{p,q}] \otimes E(D) \otimes D^{\otimes p+q+1}$ which gives rise to the spectral sequence $E^2_{pq} \Longrightarrow HY_{p+q+1}(D, E(D))$. Since D is an abelian dialgebra, the only non-zero faces are the first and the last one, hence $d = d_0 + (-1)^n d_n$. They act on a generic element of $EY_{p,q}(D)$ as

$$d_0^y(e \otimes \underline{a}) = \begin{cases} e \cdot x(a_1) \otimes (a_2, .., a_n) \\ e \cdot u(a_1) \otimes (a_2, .., a_n) \end{cases}$$

$$d_n^y(e \otimes \underline{a}) = \begin{cases} e \cdot y(a_n) \otimes (a_1, .., a_{n-1}) \\ e \cdot v(a_n) \otimes (a_1, .., a_{n-1}) \end{cases}$$

dependently on the shape of the first and last leaves of the tree y (here $n = p + q + 1$). The horizontal and vertical boundary operators are such that $d = d^h + d^v$, hence the two non-zero faces are distributed in d^h and d^v in the four possible combinations.

1) Let us compute the E^1-plane,

$$E^1_{pq} = H_q(EY_{p,*}(D), d^v) = \bigoplus_{y \in Y_p} H_q(ET^y_{p,*}(D), d^v).$$

• For $p = 0$, both non-zero faces are obviously vertical, and because of the shape of the comb trees, they always act as multiplications by elements $x(a_1)$ and $v(a_n)$, hence the vertical tower for $p = 0$ is the complex

$$\cdots \longrightarrow E(D) \otimes D^{\otimes n} \xrightarrow{x \pm v} E(D) \otimes D^{\otimes n-1} \xrightarrow{x \mp v} \cdots \xrightarrow{x - v} E(D) \otimes D^{\otimes 2} \xrightarrow{x + v} E(D) \otimes D.$$

For $q = 0$, we get

$$E^1_{00} = H_0(EY_{0,*}(D), d^v) \cong \frac{E(D) \otimes D}{\langle x(a) \otimes b + v(b) \otimes a \rangle}.$$

For $q = 1$, the kernel of the map $e \otimes (a, b) \mapsto ex(a) \otimes b + ev(b) \otimes a$ is precisely the left $E(D)$-module of elements $\langle x(a) \otimes (b, c) - v(c) \otimes (a, b) \rangle$, hence

$$E^1_{01} = H_1(EY_{0,*}(D), d^v) = 0.$$

Furthermore, for $q > 1$, we get

$$E^1_{0q} \cong E^1_{01} \otimes D^{\otimes q-1} = 0.$$

• For $p \geq 1$, observe that on each copy $E(D) \otimes D^{\otimes p+q+1}$, labeled by a (p, q)-tree y, the two non-zero faces are necessarily distributed one in d^h and the other one in d^v. Hence, when we consider each $(2p+1)$-dimensional vertical tower, on each copy $E(D) \otimes D^{\otimes p+q+1}$ the vertical boundary is just the right multiplication on $E(D)$ by an element among $x(a_1)$, $u(a_1)$, $y(a_n)$, or $v(a_n)$. Let us give more details for $p = 1$.

In the 3-dimensional vertical tower, each $(1, q)$-tree labels the chain module $E(D) \otimes D^{\otimes q+2}$, and the boundaries, in the three directions I, II and III, are given as multiplications by the terms $x(a_1)$, $(-1)^n y(a_n)$ and $(-1)^n v(a_n)$, respectively. The total homology can be found with two spectral sequences [2]. The first one, $A^2_{rs} := H^{II}_r H^I_s \Longrightarrow H^{I,II}_{r+s}(EY_{1,*}(D))$, which abuts to the total homology of the bicomplex determined by the plane I, II, has stable plane $A^\infty_{rs} = A^2_{rs}$ and homology $H^{I,II}_z = \bigoplus_{r+s=z} A^2_{rs}$. The second one, $B^2_{tz} := H^{III}_t H^{I,II}_z \Longrightarrow H^{I,II,III}_{t+z}(EY_{1,*}(D), d^v)$, which converges to the total homology of the 3-complex, has stable plane $B^\infty_{tz} = B^2_{tz}$ described in fig. (5.1). Hence the higher terms E^1_{1q} are direct sums of copies of the terms $\frac{Eu}{Eyu} \otimes D^{\otimes q+2}$, $Evu \otimes D^{\otimes q+2}$, $\frac{Ev}{Exv} \otimes D^{\otimes q+2}$, $\frac{E}{E(x,y,v)} \otimes D^{\otimes q+2}$.

• For $p \geq 2$ we find a similar result, using $(2p+1)$ spectral sequences on each vertical tower.

2) When we compute the term $E^2_{pq} = H_p(E^1_{*q}, (d^h)_*) = H_p H_q(EY_{*,*}(D))$, we have a similar situation. Each (p,q)-tree y labels now a quotient or a sub-module of $E(D) \otimes D^{\otimes p+q+1}$. The induced boundary $(d^h)^*$ yields zero on the horizontal homolgy, except in degree $(0,0)$, hence

$$HY_n(D, E(D)) = 0, \qquad n \geq 2.$$

In degree $(0,0)$ we have

$$E^2_{00} = \frac{(E(D) \otimes D)/\langle x(a) \otimes b + v(b) \otimes a\rangle}{\langle u(a) \otimes b + y(b) \otimes a\rangle}.$$

3) Finally we observe the only term of the dialgebra chain complex which is not covered by the bicomplex, the boundary operator $E(D) \otimes D \longrightarrow E(D), e \otimes a \mapsto e(u(a) - v(a))$. The induced map $E^2_{00} \longrightarrow E(D)$ has zero kernel, hence

$$HY_1(D, E(D)) = 0,$$

and the cokernel is $HY_0(D, E(D)) \cong E(D)/\langle u - v\rangle$. □

5.8 - Filtration on the cotangent complex. Given a dialgebra D, the length-filtration $F_k E(D)$ on the universal enveloping algebra $E(D)$, defined in (3.6), induces a filtration on the cotangent complex $EY_*(D)$, given by the family of subcomplexes $F_k EY_*(D)$, for $k \geq 0$, with chain modules

$$F_k EY_n(D) := k[Y_n] \otimes F_{k-n} E(D) \otimes D^{\otimes n} \qquad n \geq 0.$$

[2]**Total homology of a multicomplex.** Let us discuss briefly how to compute the homology of a multicomplex with dimension $d > 1$. Denote by I, II, III, ..., the d directions of the boundary operators. Any two such directions determine a bicomplex whose total homology can be computed with a spectral sequence. Hence we have $d-1$ spectral sequences

$$H^I_r H^{II}_s \Longrightarrow H^{I,II}_{r+s}, \qquad H^{III}_t H^{I,II}_z \Longrightarrow H^{I,II,III}_{t+z}, \qquad \ldots,$$

whose final abutment is the total homology of the multicomplex. Each abutment is computed, as usual, with the help of the short exact sequences determined by its filtration.

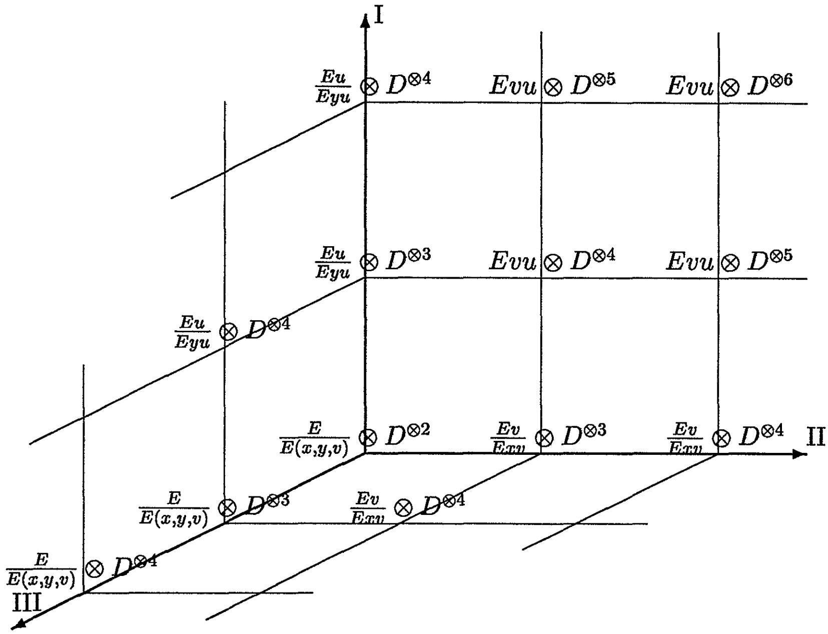

Figure 5.1: homology of the vertical tower with base $\curlyvee$.

Notations:

$\frac{E}{E(x,y,v)} = E(D)/\langle ex(a), ey(a), ey(a) \mid e \in E(D), a \in D\rangle$,

$\frac{Eu}{Eyu} = \frac{\{eu(a) \mid e \in E(D), a \in D\}}{\{ey(a)u(b) \mid e \in E(D), a,b \in D\}}$,

$Evu = \{ev(a)u(b) \mid e \in E(D), a, b \in D\}$,

In fact the boundary operator on $EY_*(D)$ respects the filtration,

$$dF_kEY_n(D) \subset k[Y_{n-1}] \otimes F_{(k-n)+1}E(D) \otimes D^{\otimes n-1} = F_kEY_{n-1}(D),$$

because the faces d_0 and d_n increase the length of the elements of $E(D)$. These complexes have the standard properties $F_kEY_*(D) \subset F_{k+1}EY_*(D)$, for any $k \geq 0$, $\bigcup_{k\geq 0} F_kEY_*(D) = EY_*(D)$, and moreover $F_0EY_n(D) = k$, for $n = 0$, and $F_0EY_n(D) = 0$, for $n \geq 1$.

Let $GrEY_*(D) = \bigoplus_{k\geq 0} Gr^kEY_*(D)$ be the graded complex where $Gr^0EY_*(D) = F_0EY_*(D)$ and $Gr^kEY_*(D) = F_kEY_*(D)/F_{k-1}EY_*(D)$ for $k \geq 1$.

5.9 - Proposition. *Let V denote the underlying k-module of a dialgebra D. Regard V as an abelian dialgebra, with $a \star b := 0$ for any $a, b \in V$. Then there is an isomorphism of complexes*

$$GrEY_*(D) \cong EY_*(V).$$

Proof. First we describe the isomorphism of k-modules $GrEY_n(D) \cong EY_n(V)$, for any $n \geq 0$. By definition we have

$$\begin{aligned} GrEY_n(D) &= \bigoplus_{k\geq 0}[F_kEY_n(D)/F_{k-1}EY_n(D)] \\ &\cong \bigoplus_{k\geq 0} k[Y_n] \otimes [F_{k-n}E(D)/F_{k-1-n}E(D)] \otimes D^{\otimes n} \\ &\cong k[Y_n] \otimes [\bigoplus_{k\geq 0} Gr^{k-n}E(D)] \otimes D^{\otimes n} \cong k[Y_n] \otimes E(V) \otimes V^{\otimes n} \end{aligned}$$

because of (3.8) and the isomorphism of k-modules $D \cong V$.

Now look at the boundary operators defined on the two complexes. The faces d_i on $GrEY_n(D)$ yield zero for $i = 1, ..., n-1$. In fact any element $y \otimes e \otimes \underline{a} \in GrEY_n(D)$ belongs to some component $Gr^kEY_n(D)$, and in particular e is an element of $F_{k-n}E(D)$. Then for $i = 1, ..., n-1$ the element $d_i(y \otimes e \otimes \underline{a}) = d_i(y) \otimes e \otimes (a_1, ..., a_i \circ_i^y a_{i+1}, ..., a_n)$ belongs to $Gr^kEY_{n-1}(D) = F_kEY_{n-1}(D)/F_{k-1}EY_{n-1}(D)$. This means that e is now seen as an element in the quotient $F_{k-n+1}E(D)/F_{k-n}E(D)$ and thus it is zero. Finally, the faces d_0 and d_n on $GrEY_n(D)$ do not involve the dialgebra structure of D, but only the product in the algebra $GrE(D)$, which is the same as in $E(V)$. □

5.10 - Corollary. *Let D be a dialgebra and $E(D)$ its universal enveloping algebra. Then $HY_0(D, E(D)) \cong E(D)/\langle u - v\rangle$ and*

$$HY_n(D, E(D)) = 0, \qquad n \geq 1.$$

Proof. Consider the cotangent complex of D, $EY_n(D) = k[Y_n] \otimes E(D) \otimes D^{\otimes n}$ and its graded complex $GrEY_*(D) = \bigoplus_{k\geq 0} Gr^kEY_*(D)$ where $Gr^0EY_*(D) = F_0EY_*(D)$ and $Gr^kEY_*(D) = F_kEY_*(D)/F_{k-1}EY_*(D)$ for $k \geq 1$.

Since the term $F_0EY_n(D) = k$, for $n = 0$, and it is zero for $n \geq 1$, we know that $H_n(F_0EY_*(D)) = 0$ for any $n \geq 1$. By proposition (5.9) we know that $GrEY_*(D) \cong EY_*(V)$, where V is the abelian dialgebra on the underlying k-module of D.

Now, by proposition (5.7) we have $H_n(GrEY_*(D)) = H_n(EY_*(V)) = 0$. From $H_n(F_0EY_*(D)) = 0$ and $H_n(F_1EY_*(D)/F_0EY_*(D)) = 0$ it follows that the term $H_n(F_1EY_*(D)) = 0$ for any $n \geq 1$. So on we proceed showing that the term $H_n(F_kEY_*(D)) = 0$ for any $k \geq 0$ and $n \geq 1$. Hence $H_n(\bigoplus_{k\geq 0} F_kEY_*(D)) = 0$ for any $n \geq 1$. □

6 Homology of bar-unital dialgebras

As we saw in (4.11), the bar homology of a bar-unital dialgebra is always zero. In this case, a bar-unit, for dialgebras, plays the same role as the unit for associative algebras. In this section we investigate this precise role, and its consequences on homology.

The dialgebra chain complex has boundary operator given by means of face maps. Hence the family of chain modules has an underlying structure of pre-simplicial module. For associative algebras, a unit permits us to define the degeneracy maps, and the Hochschild chain complex is in fact a simplicial module. For dialgebras, the bar-unit enables us to define similar maps. It turns out that the dialgebra chain modules are not simplicial, but the weaker properties of the degeneracy maps, called almost-simplicial, are enough to prove important theorems like Morita invariance of dialgebra homology for matrices and the isomorphism between dialgebra and Hochschild homology for unital associative algebras, proved in [F2].

6.1 - Almost-simplicial modules. An *almost-simplicial* module is a collection of k-modules $M = \{M_n\}_{n\geq 0}$ equipped with face maps $d_i : M_n \longrightarrow M_{n-1}$, for $i = 0, ..., n$, satisfying relations

$$(d) \qquad d_i d_j = d_{j-1} d_i, \qquad i < j,$$

and degeneracy maps $s_j : M_n \longrightarrow M_{n+1}$, for $j = 0, ..., n$, satisfying relations

$$(ds) \quad d_i s_j = \begin{cases} s_{j-1} d_i, & i < j, \\ id, & i = j, j+1, \\ s_j d_{i-1}, & i > j+1, \end{cases}$$

$$(s) \qquad s_i s_j = s_{j+1} s_i, \qquad i < j \qquad (\text{and not } i \leq j).$$

A simplicial module is an almost-simplicial module such that $s_i s_i = s_{i+1} s_i$, for any $i = 0, ..., n$.

Being pre-simplicial, an almost-simplicial module $M = \{M_n, d_i\}$ gives rise to a chain complex (M_*, d), with boundary operator $d = \sum_{i=0}^{n} (-1)^i d_i$. Being pseudo-simplicial[3], an almost-simplicial module satisfies the Eilenberg-Zilber Theorem described in [F2] and proved in [I].

6.2 - Degeneracies on dialgebra chain modules. Let D be a dialgebra with a bar-unit 1 and a corepresentation N. For any $n \geq 0$ and any $j = 0, ..., n$, define the degeneracy $s_j : CY_n(D, N) \longrightarrow CY_{n+1}(D, N)$ by

$$s_j(y \otimes (a_0, a_1, ..., a_n)) := s_j(y) \otimes (a_0, a_1, ..., a_j, 1, a_{j+1}, ..., a_n),$$

where a_0 is taken in N, $a_1, ...a_n$ in D, and for any $j = 0, ..., n$, the map $s_j : Y_n \longrightarrow Y_{n+1}$ *bifurcates* the j^{th} leaf of an n-tree y (that is, the tree $s_j(y)$ has the j^{th} leaf $|$ replaced with the branch $\curlyvee$).

[3]By definition, a *pseudo-simplicial* module is a family of k-modules endowed with faces and degeneracies satisfying relations (d) and (ds) but not necessarily relations (s), see [T-V] and [I].

6.3 - Lemma. *If the D-corepresentation N is unital, i.e. $1 \Vdash n = n = n \dashv\!\!\mid 1$ for all $n \in N$, the degeneracy maps satisfy relations (ds) and (s), for $i < j$. Hence the chain complex $CY_*(D,N)$ of a bar-unital dialgebra has an underlying structure of almost-simplicial module.*

Proof. (ds) The operations $d_i s_j$ on a tree y first adds a leaf replacing the leaf number j with the branch $\curlyvee$, and then deletes the leaf number i. So, when $i < j$, we obtain the same tree if we first delete the i^{th} leaf and then bifurcate the original j^{th} leaf, which is now labeled by $j-1$. Hence the two terms

$$d_i s_j(y \otimes \underline{a}) = d_i s_j(y) \otimes (a_0, ..., a_i \circ_i^{s_j(y)} a_{i+1}, ..., a_j, 1, a_{j+1}, ..., a_n)$$

and

$$s_{j-1} d_i(y \otimes \underline{a}) = s_{j-1} d_i(y) \otimes (a_0, ..., a_i \circ_i^{y} a_{i+1}, ..., a_j, 1, a_{j+1}, ..., a_n).$$

are equal because of the equality $a_i \circ_i^{s_j(y)} a_{i+1} = a_i \circ_i^{y} a_{i+1}$ for $i < j$. For $i = j$ we have:

$$d_i s_i(y \otimes \underline{a}) = d_i s_i(y) \otimes (a_0, ..., a_i \circ_i^{s_i(y)} 1, a_{i+1}, ..., a_n) = y \otimes \underline{a},$$

because the operator d_i evidently brings the tree $s_j(y)$ (with branch $\curlyvee$ labeled by $j, j+1$) back to the original tree and $\circ_i^{s_i(y)} = \dashv$. Similarly, for $i = j+1$ the identity $d_{j+1} s_j(y \otimes \underline{a}) = y \otimes \underline{a}$ follows from $\circ_{j+1}^{s_j(y)} = \vdash$. Finally, for $i > j+1$ the identity follows from the equality $a_{i-1} \circ_i^{s_j(y)} a_i = a_{i-1} \circ_i^{y} a_i$ for $i > j$, and the fact that we can invert the operations $d_i s_j$ and $s_{j-1} d_i$ after having observed that the leaf number i of the tree $s_j(y)$ is the leaf number $i-1$ in the tree y.

(s) Take $i < j$. The equality of the terms

$$s_i s_j(y \otimes \underline{a}) = s_i s_j(y) \otimes (a_0, ..., a_i, 1, a_{i+1}, ..., a_j, 1, a_{j+1}, ..., a_n)$$

and

$$s_{j+1} s_j(y \otimes \underline{a}) = s_{j+1} s_j(y) \otimes (a_0, ..., a_i, 1, a_{i+1}, ..., a_j, 1, a_{j+1}, ..., a_n)$$

follows from the fact that the tree $s_i s_j(y)$, obtained from y bifurcating first the leaf number j and then the leaf number i, produces the same tree by performing the two bifurcations in inverted order. (Observe that the j^{th} leaf of y is the leaf number $j+1$ of $s_i(y)$).

Notice that for $i = j$ the operator $s_i s_i$ replace the i^{th} leaf with the branch $\curlyvee$, while the operator $s_{i+1} s_i$ produces the branch $\curlyvee$, hence they can not coincide. $\square$

6.4 - Definition (normalized dialgebra homology). Given a bar-unital dialgebra D and a unital corepresentation N, we have the subcomplex of *degenerate elements* of $CY_*(D,N)$,

$$DY_n(D,N) := \{s_j(y, \underline{a}) \in CY_n(D,N) \mid \text{for any } (y, \underline{a}) \in CY_{n-1}(D,N) \text{ and any degeneracy } s_j\}.$$

The family of quotients $\bar{C}Y_n(D,N) := CY_n(D,N)/DY_n(D,N)$ is also a complex, that we call *normalized complex* of the dialgebra D. Its homology

$$\bar{H}Y_n(D,N) := H_n(\bar{C}Y_*(D,N),d)$$

is called *normalized homology of D*.

With no substantial differences from the case of associative algebras we have the following isomorphism.

6.5 - Proposition. *The complex of degenerate elements $DY_*(D,N)$ is acyclic, and the canonical projections $CY_n(D,N) \longrightarrow \bar{C}Y_n(D,N)$ induce an isomorphism on homology,*

$$HY_*(D,N) \cong \bar{H}Y_*(D,N).$$

The Eilenberg-Zilber Theorem for pseudo-simplicial modules, proved in [I], enabled us, in [F2], to prove the isomorphism between Hochschild and dialgebra homology for associative unital algebras,

$$HY_n(A,M) \cong H_n(A,M), \qquad n \geq 0.$$

From (5.1) it follows immediately that Hochschild homology of unital algebras can be interpreted as the derived functor of the tensor bifunctor $\otimes_{E(A)}$ on the category of modules over $E(A)$, and not only as derived functor of the tensor $\otimes_{A^e}$ on the category of modules over the universal enveloping algebra A^e.

6.6 - Corollary. *For any associative unital algebra A, and any unital A-bimodule M, we have isomorphisms*

$$H_*(A,M) \cong \mathrm{Tor}_*^{E(A)}(A,M) \qquad \textit{and} \qquad H^*(A,M) \cong \mathrm{Ext}^*_{E(A)}(A,M).$$

The Eilenberg-Zilber Theorem for pseudo-simplicial modules allows us to prove Morita invariance of dialgebra homology for matrices for any bar-unital dialgebra and *any* unital corepresentation, avoiding the restrictions needed in [R] to use the generalized trace map.

6.7 - Theorem (Morita Invariance). *Let D be a dialgebra and N a D-corepresentation. Consider, for any $r \geq 1$, the dialgebra of $r \times r$ matrices $M_r(D)$ with entries in D, and its corepresentation $M_r(N)$. If D and N are bar-unital, there is an isomorphism on dialgebra homology*

$$HY_n(M_r(D), M_r(N)) \cong HY_n(D,N), \qquad n \geq 0.$$

Proof. The proof is based on the isomorphism of k-vector spaces $\mathrm{M}_r(D) \cong \mathrm{M}_r(k) \otimes D$. Call (C^1_*, b^1) the Hochschild complex of the associative unital algebra $\mathrm{M}_r(k)$ with coefficients in itself, i.e. with the chain modules $C^1_n = \mathrm{M}_r(k)^{\otimes n+1}$ and the Hochschild boundary. We know that the complex C^1_* is associated to a simplicial module, that we denote by C^1, and that it is acyclic, i.e. $H_0(C^1_*) = k$ and $H_n(C^1_*) = 0$ for any $n \geq 1$.

Call (C^2_*, b^2) the dialgebra complex of D with coefficients in the corepresentation N, i.e. with the chain modules $C^2_n = k[Y_n] \otimes N \otimes D^{\otimes n}$ and the dialgebra boundary. When D and N are bar-unital, the complex C^2_* is associated to an almost-simplicial module that we denote by C^2. Then we can apply the Eilenberg-Zilber Theorem, [F2], and conclude that

$$H_n(\Delta(C^1_* \otimes C^2_*), b^1 \otimes b^2) = H_n(C^1 \times C^2) \cong H_n(C^2_*, b^2) = HY_n(D, N)$$

for any $n \geq 0$, where $\Delta(C^1_* \otimes C^2_*)$ is the diagonal complex of the tensor product of the two complexes.

Finally the result follows from the fact that the dialgebra complex of $\mathrm{M}_r(D)$ with coefficients in $\mathrm{M}_r(N)$ is isomorphic to the diagonal complex $(\Delta(C^1_* \otimes C^2_*), b^1 \otimes b^2)$. □

Remark that the same trick can be used to prove Morita Invariance for Hochschild homology.

6.8 - Theorem (Künneth formula). *Let D be a dialgebra with a corepresentation N, and let A be an associative algebra with a bimodule M. If all these spaces are (bar-)unital, we have an isomorphism on homology*

$$HY_*(A \otimes D, M \otimes N) \cong HY_*(A, M) \otimes HY_*(D, N).$$

Proof. As usual, the module of n-chains of the dialgebra $A \otimes D$,

$$\begin{aligned} CY_*(A \otimes D, M \otimes N) &= k[Y_n] \otimes M \otimes N \otimes A^{\otimes n} \otimes D^{\otimes n} \\ &= (M \otimes A^{\otimes n}) \otimes (k[Y_n] \otimes N \otimes D^{\otimes n}), \end{aligned}$$

splits into the product of two ps-modules: one is Hochschild chain module of A, and the other is the dialgebra chain module of D. By the Eilenberg-Zilber Theorem for pseudo-simplicial modules, we then have an isomorphism

$$HY_*(A \otimes D, M \otimes N) \cong H_*(A, M) \otimes HY_*(D, N),$$

where $H_*(A, M) \cong HY_*(A, M)$. □

7 HY-unital dialgebras

An associative algebra without unit is a dialgebra without bar-units, and gives rise to a chain complex which is not an almost-simplicial module. Hence, to handle its dialgebra homology, we can only make use of the spectral sequence. Even for associative algebras, the total homology of the vertical towers described in (5.5) is not easy to compute, and in general there is no relationship with the Hochschild homology of the algebra itself. The typical example is provided by the free non-unital associative algebra.

7.1 - Example (free associative algebra). Let $Ass(V) = \bar{T}(V) = V \oplus V^{\otimes 2} \oplus \cdots$ be the free associative non-unital algebra on a k-module V. It is well known that Hochschild homology vanishes on the free algebra from the second-order group, that is $HH_n(Ass(V)) = 0$, for $n \geq 2$. Instead, already in small dimension, the dialgebra homology *does not vanish* on the free associative algebra, and we find

$$\begin{aligned}
&HHY_0(\bar{T}(V)) \cong HH_0(\bar{T}(V)) = \oplus_{n \geq 1} V^{\otimes n}/(id - \tau) \quad \text{(coinvariants)},\\
&HHY_1(\bar{T}(V)) \cong HH_1(\bar{T}(V)) = \oplus_{n \geq 2} (V^{\otimes n})^{\tau} \quad \text{(invariants)},\\
&HHY_2(\bar{T}(V)) = V^{\otimes 3},\\
&HHY_3(\bar{T}(V)) = 2V^{\otimes 4}.
\end{aligned}$$

A class of algebras on which the computations are particularly simple, since almost all terms of the E^1-plane are already zero, is that of homologically unital algebras. For associative algebras, the property of having a unit is stronger then the property of having acyclic bar complex. An algebra A such that $H_*(A) = 0$, without necessarily having a unit, is called *homologically unital*, or *H-unital*. The importance of H-unital algebras is related to the extensions of unital algebras, since the kernel of such extensions can not be a unital algebra. It was proved by Wodzicki ([W], [L1]), that H-unitarity is equivalent to the excision property for Hochschild homology.

In [F2] we proved that for associative unital algebras there is an isomorphism between dialgebra and Hochschild homology with coefficients in unitary bimodules. We can now extend this result to the class of H-unital algebras.

7.2 - Theorem. *Let A be an associative H-unital algebra and M an A-bimodule. The dialgebra homology of A with values in M is isomorphic to the Hochschild homology of A, that is*

$$HY_n(A, M) \cong H_n(A, M), \qquad n \geq 0.$$

Proof. Consider the bicomplex $CY_{p,q}(A, M) = k[Y_{p,q}] \otimes M \otimes A^{\otimes p+q+1}$ and the spectral sequence converging to the dialgebra homology of A.

1) Let us compute the E^1-plane, $E^1_{pq} = \oplus_{y \in Y_p} H_q(CY^{\tilde{y}}_{p,*}(A, M), d^v)$.

• For $p = 0$, the vertical complex $CY_{0,q}(A, M)$ is isomorphic to the Hochschild complex of A shifted by 1. Hence

$$\begin{aligned}
&E^1_{00} = \operatorname{coker}(M \otimes A^{\otimes 2} \longrightarrow M \otimes A),\\
&E^1_{0q} \cong H_{q+1}(A, M), \qquad q \geq 1.
\end{aligned}$$

• For $p \geq 1$ remark that in the multi-complex $CY_{p,*}(A, M)$ any vertical boundary operator contains either the first face map, d_0, or the last one, d_{p+1+*}, or none of the two. So the first and the last face maps never appear together. This means that the homology along each direction of the vertical tower is in fact a bar-homology for A, so by assumption it is zero. Hence the total homology of each vertical tower is zero, and finally

$$E^1_{p0} \cong \bar{M} \otimes_A A \otimes_A \cdots \otimes_A A,$$
$$E^1_{pq} = 0, \qquad q \geq 1,$$

where $\bar{M} = M/AM$ is the module of coinvariants of M by the left action of A.

2) The only non-zero horizontal boundary which remains is

$$\cdots \longrightarrow E^1_{p0} \longrightarrow E^1_{p-10} \longrightarrow \cdots \longrightarrow E^1_{p0},$$

where $E^1_{p0} \cong \bar{M} \otimes_A A \otimes_A \cdots \otimes_A A$. The homology of this complex yields

$$\tilde{M} \otimes A/A^2 \otimes \cdots A/A^2,$$

where $\tilde{M} = \bar{M}/MA$ is the module of coinvariants of M by both left and right actions of A. This homology is zero since the algebra A is supposed to be H-unital. In conclusion the only non-zero terms of the E^2-plane are given by the vertical complex at $p = 0$, which is the Hochschild complex of A shifted by 1.

3) Finally, taking into account the homology of the external map,

$$\cdots \longrightarrow E^1_{00} \longrightarrow M,$$

we obtain the isomorphism $HY_n(A, M) \cong H_n(A, M)$ for any $n \geq 0$. □

The notion of homologically unital algebras can naturally be extended to the category of dialgebras.

7.3 - HY-unital dialgebras. A dialgebra D is called *HY-unital* if $HY_n(D) = 0$ for all $n \geq 1$. Theorem (4.11) implies that bar-unital dialgebras are HY-unital.

A standard example of an HY-unital dialgebra which has no bar-unit is provided by the inductive limit $\mathrm{M}(D) = \mathrm{M}_{r\to\infty}(D)$ of the inclusions of matrix dialgebras $\mathrm{M}_r(D) \hookrightarrow \mathrm{M}_{r+1}(D)$ with entries in a bar-unital dialgebra. The identity $HY_*(\mathrm{M}(D)) = 0$ is a consequence of Morita invariance of dialgebra homology with constant coefficients, namely $HY_*(\mathrm{M}(D)) \cong HY_*(D)$, which can be proved as we did for theorem (6.7).

From (7.2) it follows immediately that an associative algebra which is H-unital is also HY-unital. The converse is also true.

7.4 - Theorem. *An associative algebra A is H-unital if and only if A is HY-unital, that is*

$$H_*(A) = 0 \iff HY_*(A) = 0.$$

Proof. Assume that $HY_n(A) = 0$ for any $n \geq 1$. Then $H_0(A, k) \cong HY_0(A, k) = k$, where k is the trivial bimodule over A, and $H_1(A) \cong HY_1(A)$, which is zero by assumption. We prove that $H_n(A) = 0$ recursively on the order n.

Consider the bicomplex labeled by the oriented trees. We know on one side that $HY_n(A)$ is the total homology of this bicomplex, and on the other side that each vertical complex has homology given by means of Hochschild homology groups at smaller orders.

In particular, $HY_2(A)$ is the direct sum of $H_2(A)$, in the position $(0, 1)$ of the bicomplex, and some additional terms involving only $H_1(A)$ placed in the position $(1, 0)$. Since $HY_2(A)$ is zero by assumption, and $H_1(A) \cong HY_1(A) = 0$, we get also $H_2(A) = 0$.

Similarly, $HY_3(A)$ is the direct sum of $H_3(A)$, in the position $(0, 2)$ of the bicomplex, and some additional terms involving only $H_1(A)$ and $H_2(A)$, placed in the position $(1, 1)$ and $(2, 0)$. Hence $H_3(A) = 0$. In this way, proceding step by step, one gets $H_n(A) = 0$ for all $n > 0$. □

Aknowledgments

I warmly thank J.-L. Loday and T. Pirashvili for the so many nice conversations we had on the topics described here. I thank Loday and the *Institut de Recherche Mathématique Avancée* in particular for financial support and invitations during my Ph.D., while this paper was written.

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Un endofoncteur de la catégorie des opérades

Frédéric Chapoton

Introduction

On définit l'opérade Perm des digèbres commutatives et un morphisme d'opérades de Perm dans Com, dans la catégorie monoïdale symétrique des espaces vectoriels sur un corps $\mathbb{K}$ de caractéristique nulle. Si $\mathcal{P}$ est une opérade d'espaces vectoriels sur $\mathbb{K}$, on définit l'opérade $E(\mathcal{P}) = \mathcal{P} \otimes \mathrm{Perm}$. On a un morphisme d'opérades $N_{\mathcal{P}}$ de $E(\mathcal{P})$ dans $\mathcal{P}$ induit par le morphisme de Perm dans Com. E est un endofoncteur de la catégorie des opérades, muni d'une transformation naturelle N vers le foncteur Id. Cette construction existe pour d'autres catégories monoïdales symétriques, en particulier pour les opérades d'espaces topologiques et de complexes.

1 Construction de l'opérade Perm

Soit $\mathbb{K}$ un corps de caractéristique nulle et $\mathrm{Vect}_{\mathbb{K}}$ la catégorie monoïdale symétrique des espaces vectoriels sur $\mathbb{K}$. Pour tout entier positif ou nul n, si $\mathcal{P}$ est une opérade dans $\mathrm{Vect}_{\mathbb{K}}$, on note $\mathcal{P}(n)$ le $\mathbb{K}[S_n]$-module à droite associé à $\mathcal{P}$.

On pose $\mathrm{Perm}(n) = \mathbb{K}^n$ où $\mathbb{K}[S_n]$ agit à droite par permutation des vecteurs de la base canonique $(e^n_m)_m$ de $\mathbb{K}^n$, c'est à dire par $e^n_m\sigma = e^n_{\sigma^{-1}(m)}$.

On note ϕ la composition de l'opérade Perm :

$$\phi : \mathrm{Perm}(n) \otimes \mathrm{Perm}(j_1) \otimes \cdots \otimes \mathrm{Perm}(j_n) \longrightarrow \mathrm{Perm}(j_1 + \cdots + j_n), \tag{1}$$

définie par

$$\phi(e^n_m \otimes e^{j_1}_{i_1} \otimes \cdots \otimes e^{j_n}_{i_n}) = e^{j_1+\cdots+j_n}_{j_1+\cdots+j_{m-1}+i_m}. \tag{2}$$

Pour montrer qu'on obtient une opérade, il faut vérifier les axiomes d'associativité, d'unité et d'équivariance. Les diagrammes commutatifs correspondants apparaissent dans l'article de May [May97] et la vérification est facile.

On définit le morphisme d'opérades $N_{\mathrm{Perm}} : \mathrm{Perm} \to \mathrm{Com}$ de la façon suivante : $N_{\mathrm{Perm}}(e^n_m) = f_n$ où f_n est le vecteur de base de $\mathrm{Com}(n)$. Il résulte immédiatement de la formule (2) que N_{Perm} est bien un morphisme d'opérades.

On montrera par la suite que Perm est une opérade quadratique binaire. En admettant ceci pour l'instant, la définition de Perm montre qu'une Perm-algèbre est une algèbre associative telle que $xyz = xzy$ pour tous x,y,z.

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On peut d'autre part définir la notion de digèbre commutative comme suit : une digèbre D est commutative si le crochet de Leibniz induit sur D est nul, c'est à dire si $x \dashv y = y \vdash x$ pour tous $x,y \in D$. Si on note alors $xy = x \dashv y$, les axiomes de digèbre se ramènent à $x(yz) = (xy)z$ et $x(yz) = x(zy)$ pour tous $x,y,z \in D$, et on retrouve l'opérade Perm.

2 Fonctorialité et naturalité

Soient $\mathcal{P}$ et $\mathcal{Q}$ deux opérades dans $\mathrm{Vect}_{\mathbb{K}}$, on rappelle la définition de l'opérade $\mathcal{P} \otimes \mathcal{Q}$: on pose $(\mathcal{P} \otimes \mathcal{Q})(n) = \mathcal{P}(n) \otimes \mathcal{Q}(n)$ comme $\mathbb{K}[S_n]$-module à droite et la composition

$$\phi_{(\mathcal{P}\otimes\mathcal{Q})} : (\mathcal{P}\otimes\mathcal{Q})(n)\otimes(\mathcal{P}\otimes\mathcal{Q})(j_1)\otimes\cdots\otimes(\mathcal{P}\otimes\mathcal{Q})(j_n) \to (\mathcal{P}\otimes\mathcal{Q})(j_1+\cdots+j_n) \quad (3)$$

est définie par $\phi_{(\mathcal{P}\otimes\mathcal{Q})} = (\phi_{\mathcal{P}} \otimes \phi_{\mathcal{Q}}) \circ b_{\mathbb{K}}$, où

$$\begin{aligned} b_{\mathbb{K}} : \mathcal{P}(n) \otimes \mathcal{Q}(n) \otimes \mathcal{P}(j_1) \otimes \mathcal{Q}(j_1) \otimes \cdots \otimes \mathcal{P}(j_n) \otimes \mathcal{Q}(j_n) \\ \to \mathcal{P}(n) \otimes \mathcal{P}(j_1) \otimes \cdots \otimes \mathcal{P}(j_n) \otimes \mathcal{Q}(n) \otimes \mathcal{Q}(j_1) \otimes \cdots \otimes \mathcal{Q}(j_n) \end{aligned}$$

est un isomorphisme naturel de la catégorie symétrique $\mathrm{Vect}_{\mathbb{K}}$. L'opérade Com est une unité pour ce produit tensoriel. Ce produit tensoriel est introduit dans [GK94, 2.2.2].

Il résulte du fait que le produit tensoriel $\otimes$ est un bifoncteur de la catégorie des opérades que E est un foncteur et $N = \bullet \otimes N_{\mathrm{Perm}}$ une transformation naturelle de E vers le foncteur $\bullet \otimes \mathrm{Com} = \mathrm{Id}$.

3 Application aux opérades quadratiques

Dans la suite, on appelle module une suite V de $\mathbb{K}[S_n]$-modules à droite $V(n)$ pour $n \geq 1$. Si $\mathcal{P}$ est une opérade et R un sous-module de $\mathcal{P}$, on note $\langle R \rangle$ l'idéal de $\mathcal{P}$ engendré par R.

On note $F(V)$ l'opérade libre sur un module V. On rappelle que $F(V)$ est graduée par le nombre de sommets des arbres qui indicent les opérations. Cette graduation induit une filtration sur toute opérade $\mathcal{P}$ munie d'un module de générateurs. On notera $\mathcal{P}_V^k$ l'ensemble des éléments de $\mathcal{P}$ de filtration inférieure à k relativement à un module de générateurs V fixé. Par abus de notation, on notera de même si V n'engendre pas $\mathcal{P}$, sans introduire la sous-opérade engendrée par V.

Si $\mathcal{P}$ est une opérade alors $E(\mathcal{P})(n) = \mathcal{P}(n) \otimes \mathbb{K}^n$. On note $\mathcal{P}(n,m)$ pour $\mathcal{P}(n) \otimes \mathbb{K}e_m^n \subset E(\mathcal{P})(n)$ et, pour $x \in \mathcal{P}(n)$, x_m pour $x \otimes e_m^n$ lorsque cela ne prête pas à confusion.

Proposition 1 *Si $\mathcal{P}$ est une opérade quadratique, alors $E(\mathcal{P})$ est une opérade quadratique. Plus précisément, si $\mathcal{P}$ est décrite par le module de générateurs V et le module de relations quadratiques $R \subset F^2(V)$, alors $E(\mathcal{P})$ est décrite par le* *module de générateurs* $E(V)$ *et par le module de relations quadratiques* $\tilde{E}(R) + S(V) \subset F^2(E(V))$, *où* $\tilde{E}(R) \subset F(E(V))$ *est un relèvement de* $E(R) \subset E(F(V))$ *et* $S(V)$ *est le sous-module de* $F^2(E(V))$ *formé par les relations qui définissent* $E(F(V))$ *comme quotient de* $F(E(V))$.

On aura besoin du lemme suivant :

Lemme 1 *Soit* $\mathcal{P}$ *une opérade.*

1. *Si* V *est un sous-module de* $\mathcal{P}$ *qui engendre* $\mathcal{P}$ *comme opérade, alors* $E(V)$ *engendre* $E(\mathcal{P})$ *comme opérade.*
2. *Si* $\mathcal{I}$ *est un idéal de* $\mathcal{P}$ *et* V *un sous-module de* $\mathcal{I}$ *qui engendre* $\mathcal{I}$ *comme idéal de* $\mathcal{P}$, *alors* $E(V)$ *engendre* $E(\mathcal{I})$ *comme idéal de* $E(\mathcal{P})$.

Preuve du lemme : Par récurrence sur le nombre k d'opérations composées, on montre que, pour tout entier strictement positif n, pour tout entier i entre 1 et n, si $x \in P_V^k(n)$, alors $x_i \in E(\mathcal{P})^k_{E(V)}(n)$.

Pour $k = 0$, c'est clair ; pour $k = 1$, le résultat est donné par l'inclusion de V dans $\mathcal{P}$ et de $E(V)$ dans $E(\mathcal{P})$. Soient maintenant $k \geq 2$, $n \geq 1$, $1 \leq i \leq n$ et $x \in P_V^{k+1}(n)$. Alors, quitte a décomposer x en une somme, il existe $\sigma \in S_n, z \in V(p), y_1, \ldots, y_p \in \mathcal{P}_V^k$ tels que

$$x^\sigma = \phi_{\mathcal{P}}(z, y_1, \ldots, y_p), \tag{4}$$

où $\phi_{\mathcal{P}}$ désigne la composition de l'opérade $\mathcal{P}$.

Si $y_m \in \mathcal{P}_V^k(n_m)$ pour tout $1 \leq m \leq p$, on note r l'unique entier tel que $n_1 + \cdots + n_{r-1} < \sigma^{-1}(i) \leq n_1 + \cdots + n_r$. On pose $j_r = \sigma^{-1}(i) - (n_1 + \cdots + n_{r-1})$. On choisit arbitrairement $1 \leq j_m \leq n_m$ pour $m \neq r$. On a alors

$$(x^\sigma)_{\otimes} e^n_{\sigma^{-1}(i)} = \phi_{E(\mathcal{P})}(z_r, (y_1)_{j_1}, \ldots, (y_p)_{j_p}) \tag{5}$$

donc $x_i = (\phi_{E(\mathcal{P})}(z_r, (y_1)_{j_1}, \ldots, (y_p)_{j_p}))^{\sigma^{-1}}$ avec $z_r \in E(V)$ et $(y_m)_{j_m}$ dans $E(\mathcal{P})^k_{E(V)}$ par hypothèse de récurrence, donc x_i est dans $E(\mathcal{P})^{k+1}_{E(V)}$, ce qui termine la récurrence et montre que $E(\mathcal{P})$ est engendrée par $E(V)$. La preuve de la seconde partie du lemme est similaire. Ceci termine la démonstration du lemme.

Soit V un S-module, d'après le lemme 1, $E(V)$ engendre $E(F(V))$ comme opérade. On a donc un morphisme surjectif d'opérades $\Theta : F(E(V)) \to E(F(V))$. Soit $\mathcal{I}$ le noyau.

Lemme 2 *Dans la suite exacte*

$$0 \to \mathcal{I} \to F(E(V)) \xrightarrow{\Theta} E(F(V)) \to 0, \tag{6}$$

$\mathcal{I}$ *est quadratique en* $E(V)$, *engendré par le module* $S(V) = \mathcal{I} \cap F^2(E(V))$.

Preuve du lemme : L'opérade libre sur un module V est graduée par le nombre d'opérations de V composées. On a $F(V) = \oplus_{k \geq 0} F^k(V)$ avec $F^k(V) = \oplus_T \otimes_{s \in T} V(\mathrm{In}(s))$ où la somme est prise sur les classes d'isomorphismes d'arbres à k sommets aux feuilles numérotées et $\mathrm{In}(s)$ désigne l'ensemble des feuilles d'un sommet s de T.

Le foncteur E préserve cette graduation : on a $E(F(V)) = \oplus_{k\geq 0} E(F^k(V))$ avec $E(F^k(V)) = \oplus_{(T,f)} \otimes_{s\in T} V(\mathrm{In}(s))$ où f décrit l'ensemble des feuilles de T et s est un sommet de T. On a de même la description de $F(E(V)) = \oplus_{k\geq 1} F^k(E(V))$ avec $F^k(E(V)) = \oplus_T \otimes_{s\in T} E(V)(\mathrm{In}(s))$. Via l'isomorphisme naturel $E(V)(\mathrm{In}(s)) \simeq \oplus_{f\in \mathrm{In}(s)} V(\mathrm{In}(s))$, on peut donc écrire $F^k(E(V)) = \oplus_T \otimes_{s\in T} \oplus_{f\in \mathrm{In}(s)} V(\mathrm{In}(s))$. L'application Θ préserve cette graduation. Plus précisément, si $(T,(i_s),(v_s))$ désigne un élément de $F^k E(V))$, où i_s est une feuille du sommet s et v_s un élément de $V(\mathrm{In}(s))$, alors il existe une unique feuille $f((i_s)_s)$ de T raccordée à la racine par les feuilles i_s des sommets de T. Dans ce cas, on a $\Theta(T,(i_s),(v_s)) = (T,f((i_s)_s),(v_s))$ dans $E(F^k(V))$. On voit ici que la description de l'application Θ est en un sens indépendante du module V. Une classe d'isomorphisme d'arbre T et une feuille f de T étant fixées, l'application consiste à identifier toutes les copies de $\otimes_{s\in T} V(\mathrm{In}(s))$ dont l'indice, une famille de feuilles $(i_s)_s$, est tel que $f = f((i_s)_s)$.

En particulier, lorsque $k = 2$, l'application Θ consiste à identifier deux copies de $V(\mathrm{In}(s_1)) \otimes V(\mathrm{In}(s_2))$ lorsqu'elles sont indexées par deux arbres, munis d'une feuille marquée à chaque sommet, qui sont liés par une transformation qui consiste à changer la feuille marquée du sommet supérieur si celui-ci n'est pas greffé sur la feuille marquée du sommet inférieur. On note de type I cette transformation appliquée localement à un arbre à plus de deux sommets. Pour les besoins de la démonstration, on introduit une transformation de type II qui consiste à changer la feuille marquée de l'unique sommet qui a une arête commune avec la racine de l'arbre.

Pour montrer que $\mathcal{I}$ est quadratique, il suffit donc de montrer que, à (T,f) fixés, tous les arbres isomorphes à T munis d'une feuille marquée i_s à chaque sommet de sorte que $f = f((i_s)_s)$ sont liés par une chaîne de transformations locales élémentaires de type I.

Ce résultat s'obtient aisément par récurrence sur le nombre k de sommets des arbres, sur l'hypothèse double suivante :

- Pour toute classe d'équivalence T d'arbre numéroté à k sommets, pour toute feuille f de T, tous les arbres de type T munis d'une feuille marquée à chaque sommet reliant f à la racine sont liées par une chaîne de transformations élémentaires de type I.
- Pour toute classe d'équivalence T d'arbre numéroté à k sommets, tous les arbres de type T munis d'une feuille marquée à chaque sommet sont liés par une chaîne de transformations élémentaires de type I et II

On déduit de cette récurrence que $\mathcal{I}$ est engendré par $S(V)$, ce qui termine la démonstration du lemme.

Preuve de la proposition 1 : On remarque d'abord que si l'on a une suite exacte courte d'opérades,

$$0 \to \mathcal{I} \to \mathcal{P}' \to \mathcal{P} \to 0, \tag{7}$$

alors on a une suite exacte courte

$$0 \to E(\mathcal{I}) \to E(\mathcal{P}') \to E(\mathcal{P}) \to 0. \tag{8}$$

Dans le cas où $\mathcal{P}$ est quadratique, on a donc

$$0 \to E(\langle R\rangle) \to E(F(V)) \to E(\mathcal{P}) \to 0, \tag{9}$$

avec $R \subset F^2(V)$. Par la seconde partie du lemme 1, on a :

$$0 \to \langle E(R)\rangle \to E(F(V)) \to E(\mathcal{P}) \to 0. \tag{10}$$

Par le lemme 2, on a :

$$0 \to \langle \tilde{E}(R) + S(V)\rangle \to F(E(V)) \to E(\mathcal{P}) \to 0, \tag{11}$$

avec $S(V) \subset F^2(E(V))$ et $\tilde{E}(R) \subset F^2(E(V))$ un relèvement de $E(R)$. Donc $E(\mathcal{P})$ est quadratique, et la proposition est démontrée.

Définition 1 *On peut définir un foncteur quadratique dual $E^!$ de la restriction de E aux opérades quadratiques : on pose $E^!(\mathcal{P}) = E(\mathcal{P}^!)^!$ où $\mathcal{P}^!$ désigne l'opérade quadratique duale d'une opérade quadratique. Ce foncteur est muni d'une transformation naturelle $N^! : Id \to E^!$.*

La construction de E a été inspirée par la description dans [Lod97] et [LP98] de l'apparition naturelle des opérades Leib (resp. Dias) via les algèbres associatives (resp. de Lie) dans la catégorie monoïdale symétrique des applications linéaires munie d'un produit tensoriel non usuel. On peut voir que si $A \to M$ est une $\mathcal{P}$-algèbre dans cette catégorie monoïdale, alors A est naturellement une $E(\mathcal{P})$-algèbre dans la catégorie des espaces vectoriels.

Exemples : Par la remarque précédente, on a $E(\text{Ass}) = \text{Dias}$, $E(\text{Lie}) = \text{Leib}$. On a $E^!(\text{Com}) = \text{Leib}^!$. On peut résumer la situation par le diagramme commutatif suivant :

$$\begin{array}{ccccc}
\text{Leib}^! & \longleftarrow & \text{Dias}^! & \longleftarrow & \text{Perm}^! \\
\uparrow{\scriptstyle N^!} & & \uparrow{\scriptstyle N^!} & & \uparrow{\scriptstyle N^!} \\
\text{Com} & \longleftarrow & \text{Ass} & \longleftarrow & \text{Lie} \\
\uparrow{\scriptstyle N} & & \uparrow{\scriptstyle N} & & \uparrow{\scriptstyle N} \\
\text{Perm} & \longleftarrow & \text{Dias} & \longleftarrow & \text{Leib}
\end{array}$$

Remarque : Si on note $f_{\mathcal{P}}(x)$ la série de Poincaré de $\mathcal{P}$ définie par

$$f_{\mathcal{P}}(x) = \sum_{n=1}^{\infty} \dim(\mathcal{P}(n)) \frac{(-x)^n}{n!},$$

alors la série de Poincaré $f_{E(\mathcal{P})}$ est $E(f_{\mathcal{P}})$ où E est l'opérateur d'Euler $x\frac{d}{dx}$.

Remarque : Soit V une $\mathcal{P}$-algèbre et W une Perm-algèbre, c'est à dire la donnée de morphismes d'opérades $\mathcal{P} \to \text{End}(V)$ et $\text{Perm} \to \text{End}(W)$. On suppose V et W de dimensions finies. Alors on a un morphisme d'opérades

$$\mathcal{P} \otimes \text{Perm} \to \text{End}(V) \otimes \text{End}(W) \simeq \text{End}(V \otimes W),$$

donc $V \otimes W$ est une $E(\mathcal{P})$-algèbre. En particulier, le produit tensoriel d'une algèbre de Lie et d'une algèbre sur l'opérade Perm est naturellement une algèbre de Leibniz. En ce sens, le produit par une Perm-algèbre est une généralisation de l'extension des scalaires.

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Un théorème de Milnor-Moore pour les algèbres de Leibniz

François Goichot

Introduction

Les algèbres de Leibniz ont été introduites par Jean-Louis Loday ([L2], [L1]) comme analogue "non-commutatif" des algèbres de Lie. Le rôle que jouent pour les algèbres de Lie les algèbres associatives unitaires, est joué pour les algèbres de Leibniz par les digèbres ([L]). En particulier, toute algèbre de Leibniz $\mathcal{G}$ a une digèbre enveloppante $Ud(\mathcal{G})$ (ibid., 4.6).

Il est bien connu que l'algèbre enveloppante d'une algèbre de Lie a aussi une structure de cogèbre, compatible avec sa structure d'algèbre. On peut dès lors se demander si la digèbre enveloppante d'une algèbre de Leibniz pourrait de même avoir une "costructure", compatible avec sa structure de digèbre. Plus précisément, le théorème de Milnor-Moore, qui établit une équivalence entre la catégorie des algèbres de Lie et celle des algèbres de Hopf cocommutatives irréductibles ([MM], [Q]), a-t-il un analogue pour la catégorie des algèbres de Leibniz ?

L'objet de ce travail est d'apporter une réponse positive à ces questions. Nous montrons l'existence sur la digèbre $Ud(\mathcal{G})$ de deux "coproduits" satisfaisant des propriétés de compatibilité analogues à celles qui caractérisent les algèbres de Hopf. On a ainsi une bonne notion de "digèbre de Hopf", et nous prouvons alors l'analogue du théorème de Milnor-Moore : l'équivalence entre la catégorie des algèbres de Leibniz et celle des digèbres de Hopf cocommutatives et irréductibles.

Ces questions sont naturelles et ont déjà obtenu des réponses partielles : Loday et Pirashvili ([LP2]) établissent un théorème de Milnor-Moore pour les algèbres de Leibniz, mais dans le cadre de la catégorie des applications linéaires, et non des digèbres. Le théorème de B. Fresse ([F]) est beaucoup plus général, mais ne donne pas de résultat explicite non plus sur les digèbres.

Le plan de cet article est le suivant : nous rappelons d'abord au I les définitions et propriétés des algèbres de Leibniz et des digèbres dont nous aurons besoin. Nous établissons ensuite, au II, un système d'axiomes définissant les "digèbres de Hopf". Nous prouvons aux III et IV que la digèbre libre sur un espace vectoriel et la digèbre enveloppante d'une algèbre de Leibniz sont des digèbres de Hopf. Le IV contient aussi l'analogue pour les algèbres de Leibniz du théorème de Poincaré-Birkhoff-Witt ([Q], B.2.3), qui nous permet de prouver au V le "théorème de Milnor-Moore-Leibniz". En appendice, nous établissons une formule dans la digèbre libre, inspirée de Wigner, qui nous est nécessaire au III et permet par ailleurs d'obtenir l'équivalent pour les algèbres de Leibniz des théorèmes classiques de Friedrichs et de Specht-Wever.

J.-L. Loday et al: LNM 1763, pp. 111 - 133, 2001

Nous utilisons très largement les notations et résultats de [L]. Tout est sur un corps de base K de caractéristique 0. La catégorie des K-espaces vectoriels est notée $\mathbf{K-ev}$. Un monôme $v_1 \otimes v_2 \otimes \ldots \otimes v_n$ de l'algèbre tensorielle sera presque toujours abrégé en $v_1 v_2 \ldots v_n$. Un $\square$ signale la fin d'une démonstration.

Je remercie très vivement Jean-Louis Loday, qui m'a posé la question de départ, et m'a encouragé tout au long de ce travail.

I. Algèbres de Leibniz et digèbres.

1. Algèbres de Leibniz.

Définition I.1.1 ([L2]) *Une* **algèbre de Leibniz** *(sur K) est un K-espace vectoriel $\mathcal{G}$ muni d'un "crochet" $[-,-] : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ bilinéaire et vérifiant l'identité de Leibniz: pour tous x, y et z dans $\mathcal{G}$,*

$$[x,[y,z]] \;=\; [[x,y],z] \;-\; [[x,z],y]$$

Les morphismes d'algèbres de Leibniz sont les applications linéaires respectant le crochet. On note **Leib** la catégorie des algèbres de Leibniz. Si $\mathcal{G}$ est une algèbre de Leibniz, le quotient de $\mathcal{G}$ par l'idéal engendré par les éléments $[x,x]$, pour x dans $\mathcal{G}$, est une algèbre de Lie notée $\mathcal{G}_{Lie}$.

Définition-Proposition I.1.2. ([LP1], 1.3) *Soit V un K-espace vectoriel.*

L' **algèbre de Leibniz libre** *sur V est le K-espace vectoriel*

$$Leib(V) = \overline{T}V = V \oplus V^{\otimes 2} \oplus \ldots \oplus V^{\otimes n} \oplus \ldots ,$$

muni de l'unique crochet de Leibniz vérifiant:

$$x_1 x_2 \ldots x_n = [\ldots[[x_1,x_2],x_3],\ldots,x_n].$$

On a ainsi un foncteur $Leib() : \mathbf{K-ev} \longrightarrow \mathbf{Leib}$, adjoint à gauche de l'oubli.

L'algèbre de Leibniz libre est naturellement graduée : $Leib(V)_n = \overline{T}_n V = V^{\otimes n}$.

2. Digèbres.

Définition I.2.1. ([L], 2.1) *Une* **digèbre** *(sur K) est un K-espace vectoriel D muni de deux opérations $\dashv$: $D \otimes D \longrightarrow D$ et $\vdash$: $D \otimes D \longrightarrow D$, linéaires et vérifiant, pour tous x, y et z dans D :*

$$\left\{\begin{array}{lclcl} x \dashv (y \dashv z) & = & (x \dashv y) \dashv z & = & x \dashv (y \vdash z), \\ (x \vdash y) \dashv z & = & x \vdash (y \dashv z), & & \\ (x \dashv y) \vdash z & = & x \vdash (y \vdash z) & = & (x \vdash y) \vdash z \end{array}\right.$$

Un morphisme de digèbres $f : D \longrightarrow D'$ est une application linéaire telle que, pour tous x et y dans D, $f(x \dashv y) = f(x) \dashv f(y)$ $f(x \vdash y) = f(x) \vdash f(y)$. On note **Dias** la catégorie des digèbres. Si D est une digèbre, le quotient de D par l'idéal (au sens des digèbres, cf [L] 2.1) engendré par les éléments $x \dashv y - x \vdash y$, pour x et y dans D, est une algèbre associative notée D_{as}.

Définition-Proposition I.2.2. ([L], 2.4 et 2.5) *Soit V un K-espace vectoriel. La* **digèbre libre** *sur V est le K-espace vectoriel $Dias(V) = TV \otimes V \otimes TV$, muni des produits*

$$\begin{aligned} v_1 \dots \check{v}_i \dots v_n \dashv w_1 \dots \check{w}_j \dots w_p &= v_1 \dots \check{v}_i \dots v_n w_1 \dots w_j \dots w_p \\ v_1 \dots \check{v}_i \dots v_n \vdash w_1 \dots \check{w}_j \dots w_p &= v_1 \dots v_{i-1} v_i \dots v_n w_1 \dots \check{w}_j \dots w_p \end{aligned}$$

où $v_1 \dots \check{v}_i \dots v_n$ est la notation de [L] pour $v_1 \dots v_{i-1} \otimes v_i \otimes v_{i+1} \dots v_n \in TV \otimes V \otimes TV$. On a ainsi un foncteur $Dias()$: $\mathbf{K-ev} \longrightarrow$ **Dias**, *adjoint à gauche de l'oubli.*

La digèbre libre est naturellement graduée : $Dias(V)_n = \bigoplus_{p+q=n-1} V^{\otimes p} \otimes V \otimes V^{\otimes q}$, pour $n \geq 1$, en convenant que $V^{\otimes 0} = K$.

3. La digèbre enveloppante d'une algèbre de Leibniz.

Proposition I.3.1. ([L], 4.2) *Soit D une digèbre. Alors, pour le crochet $[x,y] := x \dashv y - y \vdash x$, le K-espace vectoriel D devient une algèbre de Leibniz, notée D_{Leib}.*

On a ainsi un foncteur $()_{Leib}$: **Dias** $\longrightarrow$ **Leib**.

Définition I.3.2. ([L], 4.6) *Soit $\mathcal{G}$ une algèbre de Leibniz. Sa* **digèbre enveloppante** *$Ud(\mathcal{G})$ est le quotient de $Dias(\mathcal{G})$ par l'idéal (au sens des digèbres) engendré par les $[x,y] - \check{x}y + y\check{x}$, avec x et y dans $\mathcal{G}$.*

Proposition I.3.3. ([L], 4.7) *Le foncteur Ud :* **Leib** $\longrightarrow$ **Dias** *est adjoint à gauche de $()_{Leib}$.*

II. Digèbres de Hopf.

Soit D une digèbre. Soit $A = D_{as} \oplus K$ l'algèbre associative unitaire qui lui est canoniquement associée : en notant $\pi : D \longrightarrow A$ la projection canonique sur D_{as}, et $\bar{x}$ l'image par π d'un élément x de D, le produit μ de A est par définition: $\mu(\bar{x} \otimes \bar{y}) = \pi(x \dashv y) = \pi(x \vdash y)$ (et 1_K neutre). L'algèbre A étant vue comme digèbre triviale, l'application π est alors un morphisme de digèbres.

L'application $\mu' : A \otimes D \longrightarrow A$, définie par $\mu' = \mu \circ (Id \otimes \pi)$ fait commuter les deux diagrammes suivants :

$$\begin{tikzcd} A\otimes A\otimes D \arrow[r, "\mu\otimes Id_D"] \arrow[d, "Id_A\otimes\mu'"'] & A\otimes D \arrow[d, "\mu'"] \\ A\otimes A \arrow[r, "\mu"'] & A \end{tikzcd}$$

$$\begin{tikzcd} A\otimes D\otimes D \arrow[r, "Id_A\otimes\dashv"] \arrow[d, "\mu'\otimes Id_D"'] & A\otimes D \arrow[d, "\mu'"] \\ A\otimes D \arrow[r, "\mu'"'] & A \end{tikzcd}$$

On note encore μ' l'application symétrique de la précédente : $D\otimes A \longrightarrow A$, le contexte suffisant à distinguer ces deux applications.

On a aussi $\dashv'$: $D\otimes A \longrightarrow D$, $d\otimes\overline{d}' \longmapsto d\dashv d'$, $d\otimes 1\longmapsto d$ (l'indépendance du représentant de d' découle des axiomes des digèbres), qui fait commuter les diagrammes :

$$\begin{tikzcd} D\otimes A\otimes A \arrow[r, "\dashv'\otimes Id_A"] \arrow[d, "Id_D\otimes\mu"'] & D\otimes A \arrow[d, "\dashv'"] \\ D\otimes A \arrow[r, "\dashv'"'] & D \end{tikzcd}$$

$$\begin{tikzcd} D\otimes D\otimes A \arrow[r, "\dashv\otimes Id_A"] \arrow[d, "Id_D\otimes\dashv'"'] & D\otimes A \arrow[d, "\dashv'"] \\ D\otimes D \arrow[r, "\dashv"'] & D \end{tikzcd}$$

Il s'agit dans les deux cas d'une conséquence directe de l'associativité de $\dashv$. Le premier de ces diagrammes signifie simplement que D est un A-module à droite.

On a bien sûr de même $\vdash'$: $A\otimes D \longrightarrow D$, $\overline{d}\otimes d' \longmapsto d\vdash d'$, $1\otimes d\longmapsto d$, qui fait commuter les deux diagrammes analogues aux précédents.

Nous allons maintenant introduire progressivement les cinq axiomes de structure des digèbres de Hopf sur D, avec les notations ci-dessus.

(DH1) L'algèbre A est une algèbre de Hopf. On note Δ son coproduit.

(DH2) Il existe $\Delta_1 : D \longrightarrow D\otimes A$ et $\Delta_2 : D\longrightarrow A\otimes D$ vérifiant :
$(\Delta_1\otimes Id_A)\circ\Delta_1 = (Id_D\otimes\Delta)\circ\Delta_1$, i.e. D est un A-comodule à droite.
$(Id_A\otimes\Delta_2)\circ\Delta_2 = (\Delta\otimes Id_D)\circ\Delta_2$, i.e. D est un A-comodule à gauche.
et $(\Delta_2\otimes Id_A)\circ\Delta_1 = (Id_A\otimes\Delta_1)\circ\Delta_2$, i.e. D est un A-bicomodule.

(DH3) Les quatre diagrammes suivants commutent :

$$\begin{tikzcd}
D\otimes D \arrow[r, "\Delta_1\circ\dashv"] \arrow[d, "\Delta_1\otimes(\Delta_1+\Delta_2)"'] & D\otimes A \\
D\otimes A\otimes(D\otimes A+A\otimes D) \arrow[r, "Id\otimes T\otimes Id"'] & D\otimes D\otimes A\otimes A\oplus D\otimes A\otimes A\otimes D \arrow[u, "\dashv\otimes\mu\ \oplus\ \dashv'\otimes\mu'"']
\end{tikzcd}$$

$$\begin{tikzcd}
D\otimes D \arrow[r, "\Delta_2\circ\dashv"] \arrow[d, "\Delta_2\otimes(\Delta_1+\Delta_2)"'] & A\otimes D \\
A\otimes D\otimes(D\otimes A+A\otimes D) \arrow[r, "Id\otimes T\otimes Id"'] & A\otimes D\otimes D\otimes A\oplus A\otimes A\otimes D\otimes D \arrow[u, "\mu'\otimes\dashv'\ \oplus\ \mu\otimes\dashv"']
\end{tikzcd}$$

$$\begin{tikzcd}
D\otimes D \arrow[r, "\Delta_1\circ\vdash"] \arrow[d, "(\Delta_1+\Delta_2)\otimes\Delta_1"'] & D\otimes A \\
(D\otimes A+A\otimes D)\otimes D\otimes A \arrow[r, "Id\otimes T\otimes Id"'] & D\otimes D\otimes A\otimes A\oplus A\otimes D\otimes D\otimes A \arrow[u, "\vdash\otimes\mu\ \oplus\ \vdash'\otimes\mu'"']
\end{tikzcd}$$

$$\begin{tikzcd}
D\otimes D \arrow[r, "\Delta_2\circ\vdash"] \arrow[d, "(\Delta_1+\Delta_2)\otimes\Delta_2"'] & A\otimes D \\
(D\otimes A+A\otimes D)\otimes A\otimes D \arrow[r, "Id\otimes T\otimes Id"'] & D\otimes A\otimes A\otimes D\oplus A\otimes A\otimes D\otimes D \arrow[u, "\mu'\otimes\vdash'\ \oplus\ \mu\otimes\vdash"']
\end{tikzcd}$$

où T est partout l'application d'échange : $T(x\otimes y)=y\otimes x$.

Définition II.1. *Soit D une digèbre vérifiant (DH1) à (DH3), et soit x dans D. On dit que x est* **primitif** *si $\Delta_1(x)=x\otimes 1$ dans $D\otimes A$ et $\Delta_2(x)=1\otimes x$ dans $A\otimes D$. On note $Prim(D)$ l'ensemble des éléments primitifs de D.*

Proposition II.2 *Si la digèbre D vérifie (DH1) à (DH3), alors $Prim(D)$ est une sous-algèbre de Leibniz de D_{Leib}.*

Démonstration. Soient x et y des éléments primitifs de D. Les deux premiers diagrammes de (DH3) donnent :

$$\begin{aligned}
(\Delta_1+\Delta_2)(x\dashv y) &= (\dashv\otimes\mu\oplus\dashv'\otimes\mu')(x\otimes y\otimes 1\otimes 1\ \oplus\ x\otimes 1\otimes 1\otimes y)\\
&\quad +(\mu'\otimes\dashv'\ \oplus\ \mu\otimes\dashv)(1\otimes y\otimes x\otimes 1\ \oplus\ 1\otimes 1\otimes x\otimes y)\\
&= (x\dashv y)\otimes 1+x\otimes\pi(y)+\pi(y)\otimes x+1\otimes(x\dashv y)
\end{aligned}$$

Les deux derniers diagrammes donnent de même :

$$(\Delta_1+\Delta_2)(y\vdash x) = (y\vdash x)\otimes 1+x\otimes\pi(y)+\pi(y)\otimes x+1\otimes(y\vdash x)$$

Par soustraction, $[x,y]=x\dashv y-y\vdash x$ est bien primitif. $\square$

(DH4) Le diagramme suivant commute :

$$\begin{array}{ccc} D & \xrightarrow{\Delta_1+\Delta_2} & D\otimes A + A\otimes D \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle \pi\otimes Id}\downarrow{\scriptstyle +Id\otimes\pi} \\ A & \xrightarrow[\Delta]{} & A\otimes A \end{array}$$

Proposition II.3. *Si la digèbre D vérifie (DH1) à (DH4), alors $\pi : D \longrightarrow A$ induit un morphisme d'algèbres de Lie : $\pi_* : (Prim(D))_{Lie} \longrightarrow Prim(A)$.*

Démonstration. Le morphisme de digèbres π induit par fonctorialité un morphisme d'algèbres de Leibniz $\pi_{Leib} : (D)_{Leib} \longrightarrow (A)_{Leib} = (A)_{Lie}$. L'axiome (DH4) implique immédiatement que l'image d'un primitif de D par π est un primitif de A ; on a donc par restriction $\pi_{Leib} : Prim(D) \longrightarrow Prim(A)$, qui est toujours un morphisme d'algèbres de Leibniz. Comme $Prim(A)$ est en fait une algèbre de Lie, ce morphisme se factorise par $(Prim(D))_{Lie}$, d'où π_*. □

(DH5) $\pi_* : (Prim(D))_{Lie} \longrightarrow Prim(A)$ est un isomorphisme (d'algèbres de Lie).

Définition II.4. *La digèbre D est une* **digèbre de Hopf** *si elle vérifie les axiomes (DH1) à (DH5). Elle est* **cocommutative** *si $T \circ \Delta_1 = \Delta_2$.*

On vérifie facilement, en utilisant (DH4), que si D est cocommutative, l'algèbre de Hopf associée A l'est aussi ; et on peut alors définir les primitifs par

$$PrimD = \{x \in D \mid \Delta_1 x = x \otimes 1\}.$$

Remarque. Dans l'axiome (DH1), on demande que A ait une structure d'algèbre de Hopf, et non seulement de bigèbre : la nécessité de l'existence de l'antipode apparaîtra dans la preuve du théorème de Milnor-Moore-Leibniz (V.2).

III. Les digèbres libres et enveloppantes.

L'objet de ce paragraphe est la construction de deux exemples de digèbres de Hopf: la digèbre libre sur un espace vectoriel, et la digèbre enveloppante d'une algèbre de Leibniz. A posteriori, la première sera bien sûr un cas particulier de la seconde. Mais nous construirons d'abord les "coproduits" sur la digèbre libre.

1. La digèbre libre.

Soit V un K-espace vectoriel, et soit $D = Dias(V)$ la digèbre libre sur V (I.2.2). L'algèbre unitaire associée à D est alors $A = TV$. On sait (cf. par exemple [L1], App.A.5) que A est une algèbre de Hopf : le produit est toujours la concaténation, et le coproduit est défini par:

$$\Delta(v_1 \dots v_i \dots v_n) = \sum_{\substack{p+q=n \\ \sigma\in S_{p,q}}} v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}$$

où $S_{p,q}$ est l'ensemble des (p,q)-battages de S_n: σ est dans $S_{p,q}$ si et seulement si on a $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ et $\sigma(p+1) < \cdots < \sigma(n)$. Par convention, le monôme vide (lorsque p ou q est nul) est 1. Enfin, l'antipode est induite par $S : v \longmapsto -v$ sur V.

Proposition III.1. *La digèbre $Dias(V)$ est une digèbre de Hopf cocommutative.*

Démonstration. Montrons d'abord que $Dias(V)$ vérifie (DH2). Soient

$$\begin{aligned}
\Delta_1 : \quad & Dias(V) \longrightarrow Dias(V) \otimes TV \\
& v_1 \ldots \check{v}_i \ldots v_n \longmapsto \sum_{\sigma^{-1}(i) \leq p} v_{\sigma(1)} \ldots \check{v}_i \ldots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \ldots v_{\sigma(n)} \\
\Delta_2 : \quad & Dias(V) \longrightarrow TV \otimes Dias(V) \\
& v_1 \ldots \check{v}_i \ldots v_n \longmapsto \sum_{\sigma^{-1}(i) > p} v_{\sigma(1)} \ldots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \ldots \check{v}_i \ldots v_{\sigma(n)}
\end{aligned}$$

les deux sommes portant comme ci-dessus sur les couples d'entiers naturels (p,q) tels que $p+q=n$, et les battages σ dans $S_{p,q}$. Ainsi par exemple :

$$\begin{aligned}
\Delta_1(\check{v}w) &= \check{v}w \otimes 1 + \check{v} \otimes w, & \Delta_2(\check{v}w) &= w \otimes \check{v} + 1 \otimes \check{v}w, \\
\Delta_1(v\check{w}) &= v\check{w} \otimes 1 + \check{w} \otimes v, & \Delta_2(v\check{w}) &= v \otimes \check{w} + 1 \otimes v\check{w}.
\end{aligned}$$

Pour p et q tels que $p+q=n$ et i entre 1 et n, posons $S^{1,i}_{p,q} = \{\sigma \in S_{p,q} \mid \sigma^{-1}(i) \leq p\}$ et $S^{2,i}_{p,q} = \{\sigma \in S_{p,q} \mid \sigma^{-1}(i) > p\}$ de sorte que $S_{p,q} = S^{1,i}_{p,q} \cup S^{2,i}_{p,q}$ et pour $k = 1$ et 2,

$$\Delta_k(v_1 \ldots \check{v}_i \ldots v_n) \;=\; \sum_{\substack{p+q=n \\ \sigma \in S^{k,i}_{p,q}}} v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}.$$

Pour le premier axiome de (DH2), il s'agit de calculer

$$
\begin{aligned}
&((Id_D\otimes\Delta)\circ\Delta_1)(v_1\ldots\check{v}_i\ldots v_n)\\
&\quad= (Id_D\otimes\Delta)\Big(\sum_{\substack{p+q=n\\ \sigma\in S^{1,i}_{p,q}}} v_{\sigma(1)}\cdots v_{\sigma(p)}\ \otimes\ v_{\sigma(p+1)}\cdots v_{\sigma(n)}\Big)\\
&\quad= \sum_{\substack{p+q=n\\ \sigma\in S^{1,i}_{p,q}}} v_{\sigma(1)}\cdots v_{\sigma(p)}\otimes\Big(\sum_{\substack{r+s=q\\ \rho\in S_{r,s}}} v_{\rho(\sigma(p+1))}\cdots v_{\rho(\sigma(p+r))}\otimes v_{\rho(\sigma(p+r+1))}\cdots v_{\rho(\sigma(n))}\Big)\\
&\quad= \sum_{\substack{p+r+s=n\\ \sigma\in S^{1,i}_{p,r+s},\rho\in S_{r,s}}} v_{\sigma(1)}\cdots v_{\sigma(p)}\otimes v_{\rho(\sigma(p+1))}\cdots v_{\rho(\sigma(p+r))}\otimes v_{\rho(\sigma(p+r+1))}\cdots v_{\rho(\sigma(n))}
\end{aligned}
$$

$$
\begin{aligned}
\text{et}&((\Delta_1\otimes Id_A)\circ\Delta_1)(v_1\ldots\check{v}_i\ldots v_n)\\
&\quad= (\Delta_1\otimes Id_A)\Big(\sum_{\substack{t+s=n\\ \mu\in S^{1,i}_{t,s}}} v_{\mu(1)}\cdots v_{\mu(p)}\ \otimes\ v_{\mu(p+1)}\cdots v_{\mu(n)}\Big)\\
&\quad= \sum_{\substack{t+s=n\\ \mu\in S^{1,i}_{t,s}}}\Big(\sum_{\substack{p+r=t\\ \tau\in S^{1,j}_{p,r}}} v_{\tau(\mu(1))}\cdots v_{\tau(\mu(p))}\otimes v_{\tau(\mu(p+1))}\cdots v_{\tau(\mu(p+r))}\Big)\otimes v_{\mu(t+1)}\cdots u_{\mu(n)}
\end{aligned}
$$

où j est le rang de $\check{v}_i$ dans le monôme $v_{\mu(1)}\ldots v_{\mu(p)}$, donc $j=\mu^{-1}(i)$.
Ainsi

$$
((\Delta_1\otimes Id_A)\circ\Delta_1)(v_1\ldots\check{v}_i\ldots v_n) =
\sum_{\substack{p+r+s=n\\ \mu\in S^{1,i}_{p+r,s},\tau\in S^{1,\mu^{-1}(i)}_{p,r}}} v_{\tau(\mu(1))}\cdots v_{\tau(\mu(p)}\ \otimes v_{\tau(\mu(p+1))}\cdots v_{\tau(\mu(p+r))}\ \otimes\ v_{\mu(p+r+1))}\cdots v_{\mu(n))}.
$$

Pour vérifier l'égalité de ces deux expressions, il suffit de remarquer que, (σ,ρ) étant donné dans $S^{1,i}_{p,r+s}\times S_{r,s}$, il existe un et un seul $(p+r,s)$-battage μ tel que $\mu(p+r+1)=(\rho\circ\sigma)(p+r+1)\ldots\mu(n)=(\rho\circ\sigma)(n)$ (car $\mu(1)\ldots\mu(p+r)$ sont alors "les entiers qui restent", dans l'ordre), et un et un seul (p,r)-battage τ tel que $(\tau\circ\mu)(1)=\sigma(1)\ldots(\tau\circ\mu)(p)=\sigma(p)$ et $(\tau\circ\mu)(p+1)=(\rho\circ\sigma)(p+1)\ldots(\tau\circ\mu)(p+r)=(\rho\circ\sigma)(p+r)$.

Les deux autres axiomes de (DH2) se vérifient de façon analogue.

Pour l'axiome (DH3) également, nous nous contenterons du premier diagramme : on a

$$
\begin{aligned}
(\Delta_1\circ\dashv)(v_1\ldots\check{v}_i\ldots v_n\otimes u_1\ldots\check{u}_j\ldots u_m) &= \Delta_1(v_1\ldots\check{v}_i\ldots v_nu_1\ldots u_j\ldots u_m)\\
&= \sum_{\substack{a+b=n+m\\ \rho\in S^{1,i}_{a,b}}} w_{\rho(1)}\cdots w_{\rho(a)}\otimes w_{\rho(a+1)}\cdots w_{\rho(n+m)}
\end{aligned}
$$

où $w_k = v_k$ si $k \leq n$, u_{k-n} si $k > n$. Par ailleurs,

$$
\begin{aligned}
& v_1 \ldots \check{v}_i \ldots v_n \otimes u_1 \ldots \check{u}_j \ldots u_m \overset{\Delta_1 \otimes (\Delta_1 + \Delta_2)}{\longmapsto} \\
& \quad \sum_{\substack{p+q=n \\ \sigma \in S^{1,i}_{p,q}}} v_{\sigma(1)} \ldots \check{v}_i \ldots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \ldots v_{\sigma(n)} \otimes \\
& \qquad \Big(\sum_{\substack{r+s=m \\ \tau \in S^{1,j}_{r,s}}} u_{\tau(1)} .. \check{u}_j .. u_{\tau(r)} \otimes u_{\tau(r+1)} .. u_{\tau(m)} + \sum_{\substack{r+s=m \\ \tau \in S^{2,j}_{r,s}}} u_{\tau(1)} .. u_{\tau(r)} \otimes u_{\tau(r+1)} .. \check{u}_j .. u_{\tau(m)} \Big) \\
& \overset{Id \otimes T \otimes Id}{\longmapsto} \sum_{\substack{p+q=n, r+s=m \\ \sigma \in S^{1,i}, \tau \in S^{1,j}}} v_{\sigma(1)} .. \check{v}_i .. v_{\sigma(p)} \otimes u_{\tau(1)} .. \check{u}_j .. u_{\tau(r)} \otimes v_{\sigma(p+1)} .. v_{\sigma(n)} \otimes u_{\tau(r+1)} .. u_{\tau(m)} \\
& \qquad + \sum_{\substack{p+q=n, r+s=m \\ \sigma \in S^{1,i}, \tau \in S^{2,j}}} v_{\sigma(1)} .. \check{v}_i .. v_{\sigma(p)} \otimes u_{\tau(1)} .. u_{\tau(r)} \otimes v_{\sigma(p+1)} .. v_{\sigma(n)} \otimes u_{\tau(r+1)} .. \check{u}_j .. u_{\tau(m)} \\
& \overset{\dashv \otimes \mu + \dashv' \otimes \mu'}{\longmapsto} \sum_{\substack{p+q=n, r+s=m \\ \sigma \in S^{1,i}, \tau \in S^{1,j}}} v_{\sigma(1)} .. \check{v}_i .. v_{\sigma(p)} u_{\tau(1)} .. u_j .. u_{\tau(r)} \otimes v_{\sigma(p+1)} .. v_{\sigma(n)} u_{\tau(r+1)} .. u_{\tau(m)} \\
& \qquad + \sum_{\substack{p+q=n, r+s=m \\ \sigma \in S^{1,i}, \tau \in S^{2,j}}} v_{\sigma(1)} .. \check{v}_i .. v_{\sigma(p)} u_{\tau(1)} .. u_{\tau(r)} \otimes v_{\sigma(p+1)} .. v_{\sigma(n)} u_{\tau(r+1)} .. u_j .. u_{\tau(m)} \\
& = \sum_{\substack{p+q=n, r+s=m \\ \sigma \in S^{1,i}_{p,q}, \tau \in S_{r,s}}} v_{\sigma(1)} \ldots \check{v}_i \ldots v_{\sigma(p)} u_{\tau(1)} \ldots u_{\tau(r)} \otimes v_{\sigma(p+1)} \ldots v_{\sigma(n)} u_{\tau(r+1)} \ldots u_{\tau(m)}.
\end{aligned}
$$

Pour prouver l'égalité de cette somme et de $(\Delta_1 \circ \dashv)(v_1 \ldots \check{v}_i \ldots v_n \otimes u_1 \ldots \check{u}_j \ldots u_m)$, on va construire une bijection entre les réunions disjointes

$$
\bigcup_{p+q=n, r+s=m} S^{1,i}_{p,q} \times S_{r,s} \qquad \text{et} \qquad \bigcup_{a+b=n+m} S^{1,i}_{a,b}
$$

Soient p, q, r, s tels que $p+q=n, r+s=m$, σ dans $S^{1,i}_{p,q}$, τ dans $S_{r,s}$. Posons $a = p+r, b = q+s$ et soit ρ la permutation

$$
\begin{pmatrix}
1 & .. & p & p+1 & .. & p+r=a & a+1 & .. & a+q & a+q+1 & .. & n+m \\
\sigma(1) & .. & \sigma(p) & n+\tau(1) & .. & n+\tau(r) & \sigma(p+1) & .. & \sigma(n) & n+\tau(r+1) & .. & n+\tau(m)
\end{pmatrix}
$$

ρ est clairement dans S_{n+m}. Comme $\sigma(1) < \ldots < \sigma(p) \leq n < n+\tau(1) < \ldots < n+\tau(r)$ et $\sigma(p+1) < \ldots < \sigma(n) \leq n < n+\tau(r+1) < \ldots < n+\tau(m)$, ρ est dans $S_{a,b}$. Enfin comme σ est dans $S^{1,i}_{p,q}$, on a $\sigma^{-1}(i) \leq p$, i.e. $i \in \{\sigma(1), \ldots, \sigma(p)\}$. A fortiori, $i \in \{\rho(1), \ldots, \rho(a)\}$, i.e. $\rho^{-1}(i) \leq a$, donc ρ est dans $S^{1,i}_{a,b}$.

Réciproquement, soit maintenant ρ dans $S^{1,i}_{a,b}$ avec $a+b=n+m$. Soit p le plus petit entier tel que $\rho(p+1) > n$. Notons que p est au plus égal à n puisque les entiers $\rho(1),\dots,\rho(p)$ sont tous distincts. Soient $q=n-p, r=a-p, s=b-q=m-r$,

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & p & p+1 & \cdots & n \\ \rho(1) & \rho(2) & \cdots & \rho(p) & \rho(a+1) & \cdots & \rho(a+q) \end{pmatrix}$$

$$\text{et } \tau = \begin{pmatrix} 1 & \cdots & r & r+1 & \cdots & m \\ \rho(p+1)-n & \cdots & \rho(a)-n & \rho(a+q+1)-n & \cdots & \rho(n+m)-n \end{pmatrix}$$

Par construction, σ est dans $S_{p,q}$ et τ dans $S_{r,s}$. Enfin, i est dans $\{\rho(1),\dots,\rho(a)\}$, mais en fait dans $\{\rho(1),\dots,\rho(p)\}$ puisque $i\le n$. Donc σ est dans $S^{1,i}_{p,q}$ (on laisse au lecteur les cas particuliers: p ou q nuls, etc).

L'axiome (DH4) est clairement vérifié. Reste à établir (DH5). On prouvera indépendamment dans l'appendice (Corollaire A.5) que $Prim(Dias(V)) \simeq Leib(V)$ comme algèbres de Leibniz. Il en résulte que $(Prim(Dias(V)))_{Lie} \simeq (Leib(V))_{Lie}$. Or, par composition d'adjoints à gauche, $(Leib(V))_{Lie} = Lie(V)$, et on sait par ailleurs (cf. [MM] ou [W]) que $Lie(V) \simeq Prim(TV)$, d'où l'isomorphisme cherché.

Montrons enfin que la digèbre de Hopf $Dias(V)$ est cocommutative. Soit, pour p fixé entre 1 et n, τ_p la permutation (cf. [K], III.2.4)

$$\tau_p = \begin{pmatrix} 1 & 2 & \cdots & n-p & n-p+1 & \cdots & n \\ p+1 & p+2 & \cdots & n & 1 & \cdots & p \end{pmatrix}$$

et soit $\varphi_p : S_n \longrightarrow S_n, \quad \sigma \longmapsto \sigma\circ\tau_p$. On a alors ([K]) $\varphi_p(S_{p,q}) = S_{q,p}$. Plus précisément, soit σ dans $S^{1,i}_{p,q}$ et $\sigma'=\varphi_p(\sigma)$. On a $\sigma'^{-1}(i) = \tau_p^{-1}(\sigma^{-1}(i))$ et $\sigma^{-1}(i)\le p$ donc $\sigma'^{-1}(i)\ge n-p+1$. Ainsi $\varphi_p(S^{1,i}_{p,q})\subset S^{2,i}_{p,q}$, et par symétrie il y a égalité. Alors :

$$\begin{aligned}(T\circ\Delta_2)(v_1\dots\check{v}_i\dots v_n) &= \sum_{\substack{p+q=n\\ \sigma\in S^{2,i}_{p,q}}} v_{\sigma(p+1)}\cdots v_{\sigma(n)} \otimes v_{\sigma(1)}\cdots v_{\sigma(p)}\\ &= \sum_{\substack{p+q=n\\ \sigma\in S^{2,i}_{p,q}}} v_{\sigma(\tau_p(1))}\cdots v_{\sigma(\tau_p(n-p))} \otimes v_{\sigma(\tau_p(q+1))}\cdots v_{\sigma(\tau_p(n))}\\ &= \sum_{\substack{p+q=n\\ \rho\in S^{1,i}_{p,q}}} v_{\rho(1)}\cdots v_{\rho(q)} \otimes v_{\rho(q+1)}\cdots v_{\rho(n)}\\ &= \Delta_1(v_1\dots\check{v}_i\dots v_n). \qquad \square\end{aligned}$$

2. La digèbre enveloppante d'une algèbre de Leibniz.

Rappelons que, si $\mathcal{G}$ une algèbre de Leibniz, on peut lui associer sa digèbre enveloppante $Ud(\mathcal{G})$ (cf. I.3).

Proposition III.2.1. *Pour toute algèbre de Leibniz $\mathcal{G}$, $Ud(\mathcal{G})$ est une digèbre de Hopf cocommutative.*

Démonstration. On sait que les algèbres $(Ud(\mathcal{G}))_{as} \oplus K$ et $U(\mathcal{G}_{Lie})$ sont isomorphes ([AG], 5.5 ou [L], 4.8). Il est bien connu que $U(\mathcal{G}_{Lie})$ est une algèbre de Hopf, les opérations étant induites par celles sur $T\mathcal{G}_{Lie}$ (cf. III.1); $Ud(\mathcal{G})$ vérifie donc (DH1).

Montrons que $\Delta_1 : Dias(\mathcal{G}) \longrightarrow Dias(\mathcal{G}) \otimes T\mathcal{G}$ définie au paragraphe précédent, passe aux quotients pour induire $\Delta_1 : Ud(\mathcal{G}) \longrightarrow Ud(\mathcal{G}) \otimes U(\mathcal{G}_{Lie})$. Il s'agit de vérifier que tous les éléments de l'idéal $\mathcal{I}$, noyau de $Dias(\mathcal{G}) \longrightarrow Ud(\mathcal{G})$, ont une image par Δ_1 dont la classe est nulle dans $Ud(\mathcal{G}) \otimes U(\mathcal{G}_{Lie})$. Soit d'abord $z = [x,y] - \check{x}y + y\check{x}$ un générateur de $\mathcal{I}$: z est primitif, comme différence des deux primitifs $[x,y]$ et $\check{x}y + y\check{x} = [\check{x},\check{y}]$; donc $\Delta_1(z) = z \otimes 1$ qui est bien nul dans $Ud(\mathcal{G}) \otimes U(\mathcal{G}_{Lie})$.

Dans le cas général maintenant: un élément quelconque de $\mathcal{I}$ est (une somme d'éléments) de la forme $u \vdash (z \dashv t) = (u \vdash z) \dashv t)$, $u \dashv (z \dashv t) = (u \dashv z) \dashv t$, $(u \vdash z) \vdash t = u \vdash (z \vdash t)$, $(u \dashv z) \vdash t$, ou $u \dashv (z \vdash t)$, avec z dans $\mathcal{I}$ et u, t dans $Dias(\mathcal{G})$. Nous utiliserons la notation de Sweedler pour les coproduits ; $\overline{x}$ désignera la classe dans $U(\mathcal{G}_{Lie})$ d'un élément x de $Dias(\mathcal{G})$; $\widetilde{y}$ désignera un représentant dans $Dias(\mathcal{G})$ d'un élément y de $U(\mathcal{G}_{Lie})$. Ainsi, on a $\Delta_1(t) = \sum_{(t)} t' \otimes t''$ avec les t' dans $Dias(\mathcal{G})$ et les t'' dans $T\mathcal{G}$; comme la digèbre de Hopf $Dias(\mathcal{G})$ est cocommutative, on a alors $\Delta_2(t) = \sum_{(t)} t'' \otimes t'$, et comme elle vérifie (DH3) :

$$\begin{aligned}
\Delta_1(z \dashv t) &= (\dashv \otimes \mu + \dashv' \otimes \mu')(Id \otimes T \otimes Id)(z \otimes 1 \otimes \sum_{(t)} t' \otimes t'' + t'' \otimes t') \\
&= (\dashv \otimes \mu + \dashv' \otimes \mu')(\sum_{(t)} z \otimes t' \otimes 1 \otimes t'' + \sum_{(t)} z \otimes t'' \otimes 1 \otimes t') \\
&= \sum_{(t)} (z \dashv t') \otimes t'' + \sum_{(t)} (z \dashv \widetilde{t''}) \otimes \overline{t'}
\end{aligned}$$

en convenant que, si t'' est dans $K = (T\mathcal{G})_0$, alors $z \dashv \widetilde{t''} = t''z$. On a donc bien $\overline{\Delta_1(z \dashv t)} = 0$.

On calcule de même :

$$\begin{aligned}
\Delta_1(z \vdash t) &= \sum_{(t)} (z \vdash t') \otimes t'' + \sum_{(t)} t' \otimes \overline{z}t'' \\
\Delta_1(t \dashv z) &= \sum_{(t)} (t' \dashv z) \otimes t'' + \sum_{(t)} t' \otimes t''\overline{z} \\
\Delta_1(t \vdash z) &= \sum_{(t)} (t' \vdash z) \otimes t'' + \sum_{(t)} (\widetilde{t''} \vdash z) \otimes \overline{t'}
\end{aligned}$$

et, par une nouvelle application de (DH3) pour $Dias(\mathcal{G})$, on en déduit l'image par Δ_1 des éléments de $\mathcal{I}$. Par exemple,

$$\Delta_1((u \dashv z) \vdash t) = ((\dashv \otimes \mu + \dashv' \otimes \mu')(Id \otimes T \otimes Id)((\Delta_1 + \Delta_2)(u \dashv z) \otimes \Delta_1(t))$$

$$= \sum_{(u),(t)} (u' \dashv z) \vdash t' \otimes u''t'' + (u' \vdash t') \otimes u''\overline{z}t'' + (\widetilde{u''} \vdash t') \otimes \overline{u' \dashv z}t'' + (\widetilde{u''\overline{z}} \vdash t') \otimes \overline{u'}t''$$

Or $\overline{u' \dashv z} = \overline{u'}\,\overline{z} = 0$, et $\widetilde{u''} \dashv z$ est un représentant de $u''\overline{z}$, donc $\overline{\Delta_1((u \dashv z) \vdash t)} = 0$. La vérification est analogue pour les éléments du type $u \vdash (z \dashv t)$, etc . On définit bien sûr de même $\Delta_2 : Ud(\mathcal{G}) \longrightarrow U(\mathcal{G}_{Lie}) \otimes Ud(\mathcal{G})$.

Ainsi, les deux diagrammes suivants, où π est la projection canonique $Dias(\mathcal{G}) \longrightarrow Ud(\mathcal{G}) = Dias(\mathcal{G})/\mathcal{I}$, respectivement $T\mathcal{G} \longrightarrow T\mathcal{G}_{Lie} \longrightarrow U(\mathcal{G}_{Lie})$, sont commutatifs :

$$\begin{tikzcd}
Ud(\mathcal{G}) \arrow[r, "\Delta_1"] & Ud(\mathcal{G}) \otimes U(\mathcal{G}_{Lie}) \\
Dias(\mathcal{G}) \arrow[u, "\pi"] \arrow[r, "\Delta_1"'] & Dias(\mathcal{G}) \otimes T\mathcal{G} \arrow[u, "\pi\otimes\pi"']
\end{tikzcd}$$

$$\begin{tikzcd}
U(\mathcal{G}_{Lie}) \arrow[r, "\Delta"] & U(\mathcal{G}_{Lie}) \otimes U(\mathcal{G}_{Lie}) \\
T\mathcal{G} \arrow[u, "\pi"] \arrow[r, "\Delta"'] & T\mathcal{G} \otimes T\mathcal{G} \arrow[u, "\pi\otimes\pi"']
\end{tikzcd}$$

Le premier l'est par définition de Δ_1 sur $Ud(\mathcal{G})$, le second par définition de Δ sur $U(\mathcal{G}_{Lie})$. On a bien sûr un diagramme commutatif analogue pour Δ_2. Les axiomes (DH2) à (DH4) pour $Ud(\mathcal{G})$, et la cocommutativité de $Ud(\mathcal{G})$, résultent alors immédiatement des mêmes propriétés pour $Dias(\mathcal{G})$.

Il reste enfin à prouver que $Ud(\mathcal{G})$ satisfait (DH5). Ce sera fait dans le prochain paragraphe. □

Proposition III.2.2. *Pour toute algèbre de Leibniz $\mathcal{G}$, on a un isomorphisme de cogèbres $Ud(\mathcal{G})^+_{as} \longrightarrow U(\mathcal{G}_{Lie})$.*

Démonstration. C'est ainsi que l'on a défini précédemment le coproduit dans $Ud(\mathcal{G})^+_{as}$. □

Proposition III.2.3. *Pour toute algèbre de Leibniz $\mathcal{G}$, on a un isomorphisme de $U(\mathcal{G}_{Lie})$-modules à droite entre $Ud(\mathcal{G})$ et $\mathcal{G} \otimes U(\mathcal{G}_{Lie})$.*

Démonstration. $Ud(\mathcal{G})$ et $\mathcal{G} \otimes U(\mathcal{G}_{Lie})$ sont des $U(\mathcal{G}_{Lie})$-modules à droite, respectivement par $\dashv'$ et par multiplication sur le second facteur. Le résultat est alors une conséquence immédiate de [L], 4.9. □

IV. Théorème de Poincaré-Birkhoff-Witt-Leibniz.

Si $\mathcal{L}$ est une algèbre de Lie, le théorème de Poincaré-Birkhoff-Witt ([Q], B.2.3) affirme l'existence d'un isomorphisme de cogèbres $e : S(\mathcal{L}) \longrightarrow U(\mathcal{L})$ entre l'algèbre symétrique et l'algèbre enveloppante. Il s'agit ici d'établir l'analogue de ce résultat pour les algèbres de Leibniz. Soit $\mathcal{G}$ une algèbre de Leibniz.

Théorème IV.1. *On a un isomorphisme $e' : \mathcal{G} \otimes S(\mathcal{G}_{Lie}) \longrightarrow Ud(\mathcal{G})$ de comodules à droite sur $S(\mathcal{G}_{Lie}) \simeq U(\mathcal{G}_{Lie})$. Plus précisément, e' fait commuter le diagramme:*

$$\begin{array}{ccc} Ud(\mathcal{G}) & \xrightarrow{\Delta_1} & Ud(\mathcal{G}) \otimes U(\mathcal{G}_{Lie}) \\ {\scriptstyle e'}\uparrow & & \uparrow{\scriptstyle e'\otimes e} \\ \mathcal{G} \otimes S(\mathcal{G}_{Lie}) & \xrightarrow[Id\otimes\Delta]{} & \mathcal{G} \otimes S(\mathcal{G}_{Lie}) \otimes S(\mathcal{G}_{Lie}) \end{array}$$

Démonstration. L'isomorphisme e' est composé de :

$$\mathcal{G} \otimes S(\mathcal{G}_{Lie}) \xrightarrow{Id\otimes e} \mathcal{G} \otimes U(\mathcal{G}_{Lie}) \xrightarrow{g} Ud(\mathcal{G})$$

où e est l'isomorphisme "de Poincaré-Birkhoff-Witt", et l'isomorphisme de K-espaces vectoriels g est celui de III.2.3 : pour x dans $\mathcal{G}$ et $u_1 \otimes \cdots \otimes u_n$ (classe d'un) élément de $T\mathcal{G}_{Lie}$, $\quad g(x \otimes (u_1 \otimes \cdots \otimes u_n)) = \overline{\check{x}u_1 \ldots u_n}$, classe dans $Ud(\mathcal{G})$ de l'élément $\check{x}u_1 \ldots u_n$ de $\mathcal{G} \otimes T\mathcal{G} \subset Dias(\mathcal{G})$. Le diagramme du théorème se décompose en :

$$\begin{array}{ccccc} \mathcal{G} \otimes S(\mathcal{G}_{Lie}) & \xrightarrow{Id\otimes e} & \mathcal{G} \otimes U(\mathcal{G}_{Lie}) & \xrightarrow{g} & Ud(\mathcal{G}) \\ {\scriptstyle Id\otimes\Delta}\downarrow & & {\scriptstyle Id}\downarrow{\scriptstyle \otimes\Delta} & & \downarrow{\scriptstyle \Delta_1} \\ \mathcal{G} \otimes S(\mathcal{G}_{Lie}) \otimes S(\mathcal{G}_{Lie}) & \xrightarrow[Id\otimes e\otimes e]{} & \mathcal{G} \otimes U(\mathcal{G}_{Lie}) \otimes U(\mathcal{G}_{Lie}) & \xrightarrow[g\otimes Id]{} & Ud(\mathcal{G}) \otimes U(\mathcal{G}_{Lie}) \end{array}$$

Le diagramme de gauche commute parce que e est un morphisme de cogèbres. Reste à prouver la commutativité de celui de droite. Or

$$\begin{aligned} (\Delta_1 \circ g)(u_1 \otimes (u_2 \ldots u_n)) &= \Delta_1(\check{u}_1 u_2 \ldots u_n) \\ &= \sum u_{\tau(1)} \ldots \check{u}_1 \ldots u_{\tau(p)} \otimes u_{\tau(p+1)} \ldots u_{\tau(n)} \end{aligned}$$

la somme portant sur les couples (p, q) tels que $p+q = n$ et sur les (p, q)-battages τ tels que $\tau^{-1}(1) \leq p$; en particulier, $p \geq 1$. Mais, si on pose $i = \tau^{-1}(1)$, un tel battage vérifie $\tau(1) < \cdots < \tau(i) = 1 < \ldots < \tau(p)$, ce qui n'est possible que si $i = 1$. Ainsi

$$(\Delta_1 \circ g)(u_1 \otimes (u_2 \ldots u_n)) = \sum \check{u}_1 u_{\tau(2)} \ldots u_{\tau(p)} \otimes u_{\tau(p+1)} \ldots u_{\tau(n)}$$

la somme portant sur les (p, q) tels que $p + q = n, p \geq 1$, et sur les τ fixant 1 et vérifiant $\tau(2) < \ldots < \tau(p)$ et $\tau(p+1) < \ldots < \tau(n)$, donc sur les $(p-1, q)$-battages de $\{2, \ldots, n\}$. Par ailleurs,

$$(Id \otimes \Delta)(u_1 \otimes (u_2 \ldots u_n) = u_1 \otimes \sum_{\substack{p=1\ldots n \\ \sigma\in S^*_{p,n-p}}} u_{\sigma(2)} \ldots u_{\sigma(p)} \otimes u_{\sigma(p+1)} \ldots u_{\sigma(n)}$$

où $S^*_{p,n-p} = \{\sigma \text{ permutation de } \{2,\dots,n\} \mid \sigma(2) < \dots < \sigma(p) \text{ et } \sigma(p+1) < \dots < \sigma(n)\}$, donc

$$((g\otimes Id)\circ(Id\otimes\Delta))(u_1\otimes(u_2\dots u_n)) = \sum_{\substack{p+q=n-1 \\ \sigma\in S^*_{p,q}}} \breve{u}_1 u_{\sigma(2)}\dots u_{\sigma(p)}\otimes u_{\sigma(p+1)}\dots u_{\sigma(n)}$$

On retrouve bien l'expression précédente. $\square$

Corollaire IV.2. *Les algèbres de Leibniz $Prim(Ud(\mathcal{G}))$ et $\mathcal{G}$ sont isomorphes.*

Démonstration. Posons $j = e'^{-1}$. Soit x un élément primitif de $Ud(\mathcal{G})$. D'après le théorème, $j(x)$ appartient à $Prim(\mathcal{G}\otimes S(\mathcal{G}_{Lie}))$, autrement dit $\Delta_1(j(x)) = j(x)\otimes 1$. Or un élément $y\otimes\omega$ de $\mathcal{G}\otimes S(\mathcal{G}_{Lie})$ est primitif si et seulement si $\Delta_1(y\otimes\omega) = y\otimes\omega\otimes 1$; comme $\Delta_1 = Id\otimes\Delta$ sur $\mathcal{G}\otimes S(\mathcal{G}_{Lie})$, cela équivaut à $y\otimes\Delta(\omega) = y\otimes\omega\otimes 1$, i.e. (sur un corps) $\Delta(\omega) = \omega\otimes 1$. Vu la structure de cogèbre de $S(\mathcal{G}_{Lie})$, ce n'est possible que si ω est dans K . Alors : $y\otimes 1 \in \mathcal{G}\otimes K \subset \mathcal{G}\otimes S(\mathcal{G}_{Lie})$, et l'image de $\mathcal{G}\otimes K$ par l'isomorphisme e' est $\mathcal{G}$. Donc x est bien dans $\mathcal{G}$. L'inclusion en sens inverse est claire. $\square$

Corollaire IV.3. *La digèbre $Ud(\mathcal{G})$ satisfait (DH5).*

Démonstration. Il s'agit de prouver que $(Prim(Ud(\mathcal{G})))_{Lie} \simeq Prim((Ud(\mathcal{G}))^+_{as})$. D'après IV.2 et III.2.2, cela équivaut à $\mathcal{G}_{Lie} \simeq Prim(U(\mathcal{G}_{Lie}))$, qui est connu. $\square$

V. Théorème de Milnor-Moore-Leibniz.

Définition. *Soit D une digèbre de Hopf. Elle est* **irréductible** *si l'algèbre de Hopf associée $A = D_{as}\oplus K$ est irréductible comme algèbre de Hopf ([S], VIII).*

On note **HopfDias₀** la catégorie des digèbres de Hopf cocommutatives et irréductibles (avec les morphismes naturels).

Théorème V.1. *Les foncteurs Ud :* **Leib** $\longrightarrow$ **HopfDias₀** *et $Prim$:* **HopfDias₀** $\longrightarrow$ **Leib** *réalisent une équivalence de catégories.*

Lemme V.2. *Tout objet de* **HopfDias₀** *est un module de Hopf (au sens de [S]p.83) sur son algèbre de Hopf associée.*

Démonstration du lemme. Soit D objet de **HopfDias₀**, et $A = D_{as}\oplus K$. Pour que D soit un A-module de Hopf, il faut et suffit que:

- D soit un A-module à droite : c'est le cas ici (par $\dashv'$)
- D soit un A-comodule à droite : c'est vrai aussi (par Δ_1)
- le diagramme suivant commute :

$$\begin{array}{ccc} D\otimes A & \xrightarrow{\Delta_1\circ\dashv'} & D\otimes A \\ {\scriptstyle \Delta_1\otimes\Delta}\downarrow & & \uparrow{\scriptstyle \dashv'\otimes\mu} \\ D\otimes A\otimes A\otimes A & \xrightarrow[Id\otimes T\otimes Id]{} & D\otimes A\otimes A\otimes A \end{array}$$

On le vérifie immédiatement à partir de (DH3) et (DH4). □

Démonstration du théorème. On sait déjà (IV.2) que $Prim \circ Ud \simeq Id$. Il reste à prouver que $Ud \circ Prim \simeq Id$. Soit D dans $\mathbf{HopfDias_0}$. Alors :

$$\begin{aligned} Ud(PrimD) &\simeq PrimD \otimes U((PrimD)_{Lie}) \quad \text{par III.2.3} \\ &\simeq PrimD \otimes U(PrimA) \quad \text{par (DH5)} \\ &\simeq PrimD \otimes A \end{aligned}$$

par le théorème de Milnor-Moore usuel, qui s'applique puisque A est irréductible et cocommutative. Et comme, d'après le lemme, D est un module de Hopf, le théorème 4.11 de [S] (qui s'applique car A est une algèbre de Hopf, et non seulement une bigèbre) donne un isomorphisme de modules de Hopf $\gamma : PrimD\otimes A \longrightarrow D$, $m \otimes a \longmapsto m.a$. Par composition, on a ainsi $\alpha : Ud(PrimD) \longrightarrow D$, qui
- est compatible avec les structures de A-modules à droite, puisque γ et l'isomorphisme de III.2.3 le sont ;
- est compatible avec les structures de A-comodules à droite, car γ l'est, et l'isomorphisme de III.2.3 aussi, ainsi qu'on l'a vu dans la preuve de IV.1 ; $Ud(Prim(D))$ et D étant toutes deux cocommutatives, α est donc aussi compatible avec les structures de A-comodules à gauche ;
- est clairement l'identité sur les primitifs.

Par ailleurs, l'inclusion $PrimD \longrightarrow D_{Leib}$ induit par propriété universelle un morphisme de digèbres $\beta : Ud(PrimD) \longrightarrow D$, qui est aussi l'identité sur les primitifs. On va montrer que $\alpha = \beta$. On peut se contenter de considérer les monômes (classes dans $Ud(PrimD)$ de) $\check{u}_0 u_1 \dots u_p$, car tout élément de $Ud(PrimD)$ est combinaison de tels monômes (cf. la preuve de [L], 4.9). Alors

$$\begin{aligned} \alpha(\check{u}_0 u_1 \dots u_p) &= \alpha(u_0 \dashv' \overline{u}_1 \dashv' \cdots \dashv' \overline{u}_p) \\ &= \alpha(u_0) \dashv' \overline{u}_1 \dashv' \dots \dashv' \overline{u}_p \quad \text{car } \alpha \text{ est un morphisme de } A\text{-modules} \\ &= \beta(u_0) \dashv' \overline{u}_1 \dashv' \dots \dashv' \overline{u}_p \quad \text{car } \alpha = \beta \text{ sur les primitifs} \\ &= \beta(u_0) \dashv' \overline{\beta(u_1)} \dashv' \dots \dashv' \overline{\beta(u_p)} \quad \text{car } \beta = Id \text{ sur les primitifs} \\ &= \beta(u_0) \dashv \beta(u_1) \dashv \dots \dashv \beta(u_p) \quad \text{par définition de } \dashv' \\ &= \beta(u_0 \dashv u_1 \dashv \dots \dashv u_p) \quad \text{car } \beta \text{ est un morphisme de digèbres} \\ &= \beta(\check{u}_0 u_1 \dots u_p). \end{aligned}$$

Ainsi $\alpha = \beta$, qui est donc un isomorphisme de digèbres de Hopf. □

Appendice : caractérisation à la Wigner de l'algèbre de Leibniz libre

Soit V un espace vectoriel sur K, corps de caractéristique 0.

Rappelons d'abord les résultats de Wigner. On considère l'algèbre de Lie libre $Lie(V)$ comme la sous-algèbre de Lie de TV engendrée par V, mais pour plus de clarté dans la suite on notera $i : Lie(V) \longrightarrow TV$ l'inclusion. Soit alors $\gamma : TV \longrightarrow Lie(V)$ l'application linéaire induite par :

$$v_1 \dots v_n \longmapsto [v_1, [v_2, [\dots [v_{n-1}, v_n]] \dots]$$

(et nulle sur $T_0V = K$). En notant μ et Δ le produit et le coproduit usuels sur TV, on a alors (cf. [W]) :

Proposition. *Pour tout x dans T_nV,* $(\mu \circ (i \otimes Id) \circ (\gamma \otimes Id) \circ \Delta)(x) = nx.$

Théorème. *Pour tout x dans T_nV, les trois propriétés suivantes sont équivalentes :*

$$\begin{aligned} i) &\quad x \in i(Lie(V)), \\ ii) &\quad x \in Prim(TV), \\ iii) &\quad \gamma(x) = nx. \end{aligned}$$

Soient maintenant $Leib(V) = \overline{T}V$ et $Dias(V) = TV \otimes V \otimes TV$ l'algèbre de Leibniz libre et la digèbre libre sur V (cf. I). Par propriété universelle de $Leib(V)$, l'inclusion canonique $V \longrightarrow Dias(V)$ se factorise par $i' : Leib(V) \longrightarrow Dias(V)$, morphisme d'algèbres de Leibniz. Nous allons d'abord expliciter i'.

Si σ est un élément de S_n, σ a une *descente* en i $(1 \leq i \leq n)$ si $\sigma(i) > \sigma(i+1)$ (cf. par exemple [L1], 4.5.5) . On notera $d(\sigma)$ le nombre de descentes de σ et Z_n l'ensemble des σ dans S_n dont les descentes sont consécutives à partir de 1: soit σ n'a pas de descente, soit σ a une seule descente et c'est en 1, soit σ a exactement deux descentes qui sont en 1 et en 2, etc.

Proposition A.1. *L'homomorphisme $i' : Leib(V) \longrightarrow Dias(V)$ est donné par :*

$$i'(u_1 \otimes \dots \otimes u_n) = \sum_{\sigma \in Z_n} (-1)^{d(\sigma)} u_{\sigma(1)} \otimes \dots \check{u}_1 \dots \otimes u_{\sigma(n)}.$$

Démonstration. Pour $n = 1$, c'est $i'(u) = \check{u}$ qui découle de la définition de i'. Supposons la formule vraie pour n. Alors, i' étant un morphisme d'algèbres de Leibniz,

$$\begin{aligned} i'(u_1 \dots u_n u_{n+1}) &= i'([[..[u_1, u_2], \dots u_n], u_{n+1}]) = [i'([[..[u_1, u_2], \dots u_n]), i'(u_{n+1})] \\ &= i'([..[u_1, u_2], \dots u_n]) \dashv \check{u}_{n+1} - \check{u}_{n+1} \vdash i'([..[u_1, u_2], \dots u_n]) \\ &= \sum_{\sigma \in Z_n} (-1)^{d(\sigma)} u_{\sigma(1)} \otimes \dots \check{u}_1 \dots \otimes u_{\sigma(n)} \otimes u_{n+1} \\ &\qquad - \sum_{\sigma \in Z_n} (-1)^{d(\sigma)} u_{n+1} \otimes u_{\sigma(1)} \otimes \dots \check{u}_1 \dots \otimes u_{\sigma(n)}. \end{aligned}$$

(a) Dans la première somme, posons

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n & n+1 \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) & n+1 \end{pmatrix}$$

Alors τ a les mêmes descentes que σ, qui sont consécutives à partir de 1. Donc τ est dans Z_{n+1}.
(b) Dans la deuxième somme, posons

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n & n+1 \\ n+1 & \sigma(1) & \cdots & \sigma(n-1) & \sigma(n) \end{pmatrix}$$

τ a une descente en 1, puis celles de σ, décalées ; elles sont donc encore consécutives à partir de 1 et τ est encore dans Z_{n+1}, avec $d(\tau) = d(\sigma) + 1$.

On a ainsi $i'(u_1 \otimes \cdots \otimes u_n \otimes u_{n+1}) = \sum_{\tau \in Z'_{n+1}} (-1)^{d(\tau)} u_{\tau(1)} \otimes \dots \check{u}_1 \cdots \otimes u_{\tau(n+1)}$, Z'_{n+1} étant le sous-ensemble de Z_{n+1} construit ci-dessus. Reste à montrer que $Z_{n+1} \subset Z'_{n+1}$. Soit τ dans Z_{n+1}. L'entier $\tau^{-1}(n+1)$ ne peut être que 1 ou $n+1$; en effet, si c'était un autre entier i, τ aurait une descente en i, et n'en aurait pas en $i-1$: c'est impossible pour un élément de Z_{n+1}. Mais alors on peut trouver un (et un seul) élément σ de Z_n donnant τ par la construction (a) ou (b) précédente. Ainsi $Z'_{n+1} = Z_{n+1}$, et on a prouvé la formule au rang $n+1$. □

Soit maintenant $\gamma' : Dias(V) \longrightarrow Leib(V)$ l'application linéaire définie comme suit: pour v dans V, on pose $\gamma'(\check{v}) = v$, et pour $n \geq 2$,

$$\gamma'(v_1 v_2 \dots \check{v}_i \dots v_n) = \begin{cases} [v_1, [v_2, [\dots [v_{n-1}, v_n]] \dots] & \text{si } i = 1 \\ -[\gamma'(v_2 \dots \check{v}_i \dots v_n), v_1] & \text{si } i \geq 2 \end{cases}$$

Ainsi on aura par exemple $\gamma'(u_1 \check{u}_2 u_3 u_4) = -[\gamma'(\check{u}_2 u_3 u_4), u_1] = -[[u_2, [u_3, u_4]], u_1]$. On vérifie facilement que, dans l'écriture de $\gamma'(v_1 v_2 \dots \check{v}_i \dots v_n)$ sous forme de crochets de Leibniz, $\check{v}_i$ sera l'élément "le plus à gauche". Ainsi

$$\gamma'(v_1 v_2 \dots \check{v}_n) = (-1)^{n+1} [[[\dots [v_n, v_{n-1}], v_{n-2}], \dots], v_1].$$

Proposition A.2. *Pour tout x dans $Dias(V)_n$, on a*

$$((\dashv' \oplus \vdash') \circ (i' \otimes Id \oplus i \otimes Id) \circ (\gamma' \otimes Id \oplus \gamma \otimes Id) \circ \Delta)(x) \;=\; nx$$

(les notations sont celles du II, avec $\Delta = \Delta_1 + \Delta_2$).

Démonstration. Posons $\phi = (\dashv' \oplus \vdash') \circ (i' \otimes Id \oplus i \otimes Id) \circ (\gamma' \otimes Id \oplus \gamma \otimes Id) \circ \Delta$. Pour $n = 1$, pour tout v dans V:

$$\begin{aligned}
\phi(v) &= ((\dashv' \oplus \vdash') \circ (i' \otimes Id \oplus i \otimes Id) \circ (\gamma' \otimes Id \oplus \gamma \otimes Id))(\check{v} \otimes 1 + 1 \otimes \check{v}) \\
&= ((\dashv' \oplus \vdash') \circ (i' \otimes Id \oplus i \otimes Id))(v \otimes 1 + 0 \otimes \check{v}) \\
&= (\dashv' \oplus \vdash')(\check{v} \otimes 1 + 0) \;=\; \check{v} \;=\; 1 \cdot \check{v}.
\end{aligned}$$

Supposons maintenant la propriété vraie au rang $(n-1)$. Par linéarité, il suffit de considérer le cas où $x = u_1 u_2 \ldots \check{u}_k \ldots u_n \quad \in Dias(V)_n$.

(a) Supposons d'abord k entre 2 et n. Alors :

$$\Delta(u_2 \ldots \check{u}_k \ldots u_n) \;=\; 1 \otimes u_2 \ldots \check{u}_k \ldots u_n + \sum_i \check{a}_i \otimes b_i + \sum_j c_j \otimes \check{d}_j$$

où $\check{a}_i, \check{d}_j$ (resp. b_i, c_j) sont des notations abrégées pour des monômes dans $Dias(V)$ (resp. TV). Et $\Delta(u_1 u_2 \ldots \check{u}_k \ldots u_n) \;=\; \Delta(\check{u}_1 \vdash u_2 \ldots \check{u}_k \ldots u_n)$ qui peut s'écrire (c'est un cas particulier de l'axiome (DH3) des digèbres de Hopf)

$$\begin{aligned}
&u_1 \otimes u_2 \ldots \check{u}_k \ldots u_n + 1 \otimes u_1 u_2 \ldots \check{u}_k \ldots u_n \\
&\qquad + \sum_i u_1 \check{a}_i \otimes b_i + \sum_i \check{a}_i \otimes u_1 b_i + \sum_j u_1 c_j \otimes \check{d}_j + \sum_j c_j \otimes u_1 \check{d}_j.
\end{aligned}$$

Par hypothèse de récurrence : $\phi(u_2 \ldots \check{u}_k \ldots u_n) = (n-1) u_2 \ldots \check{u}_k \ldots u_n$.
Or $((\gamma' \otimes Id + \gamma \otimes Id) \circ \Delta)(u_2 \ldots \check{u}_k \ldots u_n) \;=\; 0 \otimes u_2 \ldots \check{u}_k \ldots u_n + \sum_i \gamma'(\check{a}_i) \otimes b_i + \sum_j \gamma(c_j) \otimes \check{d}_j$, donc $\quad \phi(u_2 \ldots \check{u}_k \ldots u_n) \;=\; \sum_i (i' \circ \gamma')(\check{a}_i) b_i + \sum_j \gamma(c_j) \check{d}_j$.

De même

$$\begin{aligned}
\phi(u_1 u_2 \ldots \check{u}_k \ldots u_n) &= u_1 u_2 \ldots \check{u}_k \ldots u_n + 0 + \sum_i (i' \circ \gamma')(u_1 \check{a}_i) b_i \\
&\quad + \sum_i (i' \circ \gamma')(\check{a}_i) u_1 b_i + \sum_j \gamma(u_1 c_j) \check{d}_j + \sum_j \gamma(c_j) u_1 \check{d}_j.
\end{aligned}$$

Or, pour tout monôme $v_1 v_2 \ldots \check{v}_l \ldots v_n, \quad (l \geq 2)$, on a

$$\begin{aligned}
(i' \circ \gamma')(v_1 v_2 \ldots \check{v}_l \ldots v_n) &= -i'([\gamma'(v_2 \ldots \check{v}_l \ldots v_n), v_1]) \quad \text{par définition de } \gamma' \\
&= -[(i' \circ \gamma')(v_2 \ldots \check{v}_l \ldots v_n), \check{v}_1]
\end{aligned}$$

puisque i' est un morphisme d'algèbres de Leibniz et $i'(v_1) = \check{v}_1$. Donc

$$\begin{aligned}
\sum_i (i' \circ \gamma')(u_1 \check{a}_i) b_i &= -\sum_i [(i' \circ \gamma')(\check{a}_i), u_1] b_i \\
&= -\sum_i (i' \circ \gamma')(\check{a}_i) \dashv u_1 b_i \;+\; \sum_i (u_1 \vdash (i' \circ \gamma')(\check{a}_i)) \dashv b_i \\
&= -\sum_i (i' \circ \gamma')(\check{a}_i) u_1 b_i \;+\; u_1 \sum_i (i' \circ \gamma')(\check{a}_i)) b_i.
\end{aligned}$$

Par ailleurs (cf. [W]) :

$$\gamma(u_1c_j)+\gamma(c_j)u_1 \;=\; ad(u_1)(\gamma(c_j))+\gamma(c_j)u_1 \;=\; u_1\gamma(c_j)$$

dans $Lie(V)\subset T(V)$. En résumé :

$$\begin{aligned}
\phi(u_1u_2\ldots\check{u}_k\ldots u_n) &= u_1u_2\ldots\check{u}_k\ldots u_n + u_1\sum_i (i'\circ\gamma')(\check{a}_1)b_i + \sum_j u_1\gamma(c_j)\check{d}_j\\
&= u_1u_2\ldots\check{u}_k\ldots u_n + u_1\phi(u_2\ldots\check{u}_k\ldots u_n)\\
&= u_1u_2\ldots\check{u}_k\ldots u_n + u_1((n-1)u_2\ldots\check{u}_k\ldots u_n)\\
&= n\;\; u_1u_2\ldots\check{u}_k\ldots u_n.
\end{aligned}$$

(b) Reste à traiter le cas $k=1$. On a alors plus simplement

$$\Delta(\check{u}_1u_2\ldots u_n) \;=\; 1\otimes\check{u}_1u_2\ldots u_n + \check{u}_1\otimes u_2\ldots u_n + \sum_i \check{u}_1a_i\otimes b_i + \sum_i a_i\otimes\check{u}_1b_i$$

où $\Delta(u_2\ldots u_n) \;=\; \sum_i a_i\otimes b_i \quad \in TV\otimes TV$. Comme précédemment,

$$\phi(\check{u}_1u_2\ldots u_n) \;=\; 0+\check{u}_1u_2\ldots u_n + \sum_i (i'\circ\gamma')(\check{u}_1a_i)b_i + \sum_i \gamma(a_i)u_1b_i \quad ,\text{ et}$$

$$\begin{aligned}
(i'\circ\gamma')(\check{u}_1a_i) &= i'([u_1,\gamma(a_i)]) \quad \text{par définition de } \gamma'\\
&= [i'(u_1),\gamma(a_i)] \;=\; \check{u}_1\gamma(a_i)-\gamma(a_i)\check{u}_1.
\end{aligned}$$

Reste à remarquer que

$$\begin{aligned}
\sum_i \gamma(a_i)b_i &= (\mu\circ(\gamma\otimes Id))(\sum_i a_i\otimes b_i)\\
&= (\mu\circ(\gamma\otimes Id)\circ\Delta)(u_2\ldots u_n)\\
&= (n-1)u_2\ldots u_n \quad \text{(c'est le résultat de Wigner)}
\end{aligned}$$

d'où

$$\phi(\check{u}_1u_2\ldots u_n) \;=\; \check{u}_1u_2\ldots u_n + \check{u}_1\sum_i \gamma(a_i)b_i \;=\; n\check{u}_1u_2\ldots u_n. \quad \square$$

Théorème A.3. *Pour tout x dans $Dias(V)_n$, les trois propriétés suivantes sont équivalentes :*

$$\begin{aligned}
&i) \quad x\in i'(Leib(V)),\\
&ii) \quad x\in Prim(Dias(V)),\\
&iii) \quad (i'\circ\gamma')(x) \;=\; nx.
\end{aligned}$$

Démonstration. **i)⇒ii)** En degré 1 d'abord : $x = i'(y)$, avec y dans $Leib(V)_1 = (\overline{T}V)_1 = V$ et i' est l'identité en degré 1 donc $x = y \in V$; alors $\Delta x = x \otimes 1 + 1 \otimes x$. Ensuite, par construction, tout élément de $Leib(V)$ est un crochet d'éléments de V. $Prim(Dias(V))$ est une sous-algèbre de Leibniz de $(Dias(V))_{Leib}$ et i' est un morphisme d'algèbres de Leibniz. Donc l'implication est vraie en tout degré.

ii)⇒iii) Soit x primitif: $\Delta x = x \otimes 1 + 1 \otimes x \in Dias(V) \otimes TV + TV \otimes Dias(V)$. Alors, d'après la proposition 2, si x est en degré n :

$$\begin{aligned} nx &= ((\dashv' \oplus \vdash') \circ (i' \otimes Id \oplus i \otimes Id) \circ (\gamma' \otimes Id \oplus \gamma \otimes Id))(x \otimes 1 + 1 \otimes x) \\ &= ((\dashv' \oplus \vdash') \circ (i' \otimes Id \oplus i \otimes Id))(\gamma'(x) \otimes 1 + 0 \otimes x) \\ &= (\dashv' \oplus \vdash')(i'(\gamma'(x)) \otimes 1) \ = \ i'(\gamma'(x)). \end{aligned}$$

iii)⇒i) Il suffit de noter que $x = \frac{1}{n} i'(\gamma'(x))$ avec $\gamma'(x)$ dans $Leib(V)$. □

Proposition A.4. *Soit x dans $(LeibV)_n = \overline{T}_n V$. On a : $(\gamma' \circ i')(x) = nx$,*

et donc $i' : Leib(V) \longrightarrow Dias(V)$ est injective.

Remarque. L'analogue de ce résultat dans le cas des algèbres de Lie figure dans [Q] (App. B, 2.2).

Démonstration. En degré 1 : $(\gamma' \circ i')(v) = \gamma'(\check{v}) = v$. Supposons la propriété vraie jusqu'à $(n-1)$. Comme $u_1 u_2 \ldots u_n = u_1 \otimes \cdots \otimes u_n = [[\cdots[u_1, u_2], u_3], \cdots u_n]$ dans $Leib(V)$, on aura dans $Dias(V)$:

$$\begin{aligned} i'(u_1 \ldots u_n) &= i'([[\cdots[u_1, u_2], u_3], \cdots u_n]) \\ &= [i'([[\cdots[u_1, u_2], \cdots u_{n-1}], i'(u_n)] \\ &= i'([[\cdots[u_1, u_2], \cdots u_{n-1}] \dashv \check{u}_n \ - \ \check{u}_n \vdash i'([[\cdots[u_1, u_2], \cdots u_{n-1}] \\ &= i'(u_1 \ldots u_{n-1}) \dashv \check{u}_n \ - \ \check{u}_n \vdash i'(u_1 \ldots u_{n-1}) \\ &= \Big(\sum_{\sigma \in Z_{n-1}} (-1)^{d(\sigma)} u_{\sigma(1)} \otimes \ldots \check{u}_1 \cdots \otimes u_{\sigma(n-1)} \Big) \otimes u_n \\ &\qquad - u_n \otimes \Big(\sum_{\sigma \in Z_{n-1}} (-1)^{d(\sigma)} u_{\sigma(1)} \otimes \ldots \check{u}_1 \cdots \otimes u_{\sigma(n-1)} \Big) \end{aligned}$$

d'après la proposition 1. Il faut maintenant prendre l'image par γ': pour le terme de droite, c'est

$$
\begin{aligned}
\gamma'(\sum_{\sigma\in Z_{n-1}} &(-1)^{d(\sigma)} u_n \otimes u_{\sigma(1)} \otimes \ldots \check{u}_1 \cdots \otimes u_{\sigma(n-1)}) \\
&= \sum_{\sigma\in Z_{n-1}} (-1)^{d(\sigma)} \gamma'(u_n \otimes u_{\sigma(1)} \otimes \ldots \check{u}_1 \cdots \otimes u_{\sigma(n-1)}) \\
&= -\sum_{\sigma\in Z_{n-1}} (-1)^{d(\sigma)} [\gamma'(u_{\sigma(1)} \otimes \ldots \check{u}_1 \cdots \otimes u_{\sigma(n-1)}), u_n] \qquad \text{(déf. de } \gamma'\text{)} \\
&= -[\sum_{\sigma\in Z_{n-1}} (-1)^{d(\sigma)} \gamma'(u_{\sigma(1)} \otimes \ldots \check{u}_1 \cdots \otimes u_{\sigma(n-1)}), u_n] \\
&= -[\gamma'(\sum_{\sigma\in Z_{n-1}} (-1)^{d(\sigma)} u_{\sigma(1)} \otimes \ldots \check{u}_1 \cdots \otimes u_{\sigma(n-1)}), u_n] \\
&= -[\gamma'(i'(u_1 \ldots u_{n-1})), u_n] \\
&= -[(n-1)[\cdots [u_1, u_2], \cdots], u_{n-1}], u_n] \quad \text{par l'hypothèse de récurrence} \\
&= -(n-1)[[\cdots [u_1, u_2], \cdots], u_{n-1}], u_n] \;=\; -(n-1) u_1 u_2 \ldots u_n .
\end{aligned}
$$

Il reste donc à prouver que l'image par γ' du terme de gauche ci-dessus vaut $u_1 \ldots u_n$, autrement dit :

$$(*) \qquad \gamma'(i'(u_1 \ldots u_{n-1}) u_n) \;=\; u_1 \ldots u_n .$$

On le fait à nouveau par récurrence. Pour $n = 2$ c'est la définition de γ': $\gamma'(\check{u}_1 u_2) = [u_1, u_2] = u_1 u_2$. Pour $n = 3$, c'est l'identité de Leibniz :

$$\gamma'(\check{u}_1 u_2 u_3 - u_2 \check{u}_1 u_3) = [u_1, [u_2, u_3]] + [[u_1, u_3], u_2] = [[u_1, u_2], u_3] = u_1 u_2 u_3 .$$

Supposons que, pour tous u_i dans V, on ait $\gamma'(i'(u_1 \ldots u_{n-2}) u_{n-1}) = u_1 \ldots u_{n-2} u_{n-1}$. On écrit :

$$
\begin{aligned}
\gamma'(i'(u_1 \ldots u_{n-1}) u_n) &= \gamma'(\sum_{\sigma\in Z_{n-1}} (-1)^{d(\sigma)} u_{\sigma(1)} \otimes \ldots \check{u}_1 \cdots \otimes u_{\sigma(n-1)} \otimes u_n) \\
&= \gamma'(\sum_{\sigma(1)\neq n-1} \cdots + \sum_{\sigma(1)=n-1} \cdots) \\
&= \gamma'(\sum_{\sigma(n-1)=n-1} \cdots + \sum_{\sigma(1)=n-1} \cdots)
\end{aligned}
$$

(cf. la preuve de la prop. 1). Calculons séparément :

$$
\begin{aligned}
\gamma'(\sum_{\sigma(1)=n-1} &(-1)^{d(\sigma)} u_{\sigma(1)} \otimes \ldots \check{u}_1 \cdots \otimes u_{\sigma(n-1)} \otimes u_n) \\
&= -\sum_{\sigma(1)=n-1} (-1)^{d(\sigma)} [\gamma'(u_{\sigma(2)} \otimes \ldots \check{u}_1 \cdots \otimes u_{\sigma(n-1)} \otimes u_n), u_{n-1}] \\
&= -[\gamma'((\sum_{\sigma(1)=n-1} (-1)^{d(\sigma)} u_{\sigma(2)} \otimes \ldots \check{u}_1 \cdots \otimes u_{\sigma(n-1)}) \otimes u_n), u_{n-1}] \\
&= [\gamma'(i'(u_1 \ldots u_{n-2}) u_n), u_{n-1}]
\end{aligned}
$$

car, vu la définition de Z_{n-1} et Z_{n-2}, l'application

$$\begin{aligned}\{\sigma \in Z_{n-1} \mid \sigma(1) = n-1\} &\longrightarrow Z_{n-2}\\ \sigma &\longmapsto \tau = \begin{pmatrix} 1 & 2 & \cdots & n-2 \\ \sigma(2) & \sigma(3) & \cdots & \sigma(n-1)\end{pmatrix}\end{aligned}$$

est une bijection vérifiant $d(\tau) = d(\sigma) - 1$. Par l'hypothèse de récurrence, le dernier crochet vaut :

$$[u_1 \dots u_{n-2}u_n, u_{n-1}] \;=\; [[[\cdots[u_1,u_2],\cdots u_{n-2}],u_n],u_{n-1}].$$

D'autre part :

$$\begin{aligned}\gamma'\Big(\sum_{\sigma(n-1)=n-1} (-1)^{d(\sigma)} u_{\sigma(1)} \otimes \dots \check{u}_1 \cdots \otimes u_{\sigma(n-1)} \otimes u_n\Big) \;=\;&\\ \gamma'\Big(\sum_{\rho \in Z_{n-2}} (-1)^{d(\rho)} u_{\rho(1)} \otimes \dots \check{u}_1 \cdots \otimes u_{\rho(n-2)} \otimes u_{n-1} \otimes u_n\Big)&\end{aligned}$$

car on a encore une bijection

$$\begin{aligned}\{\sigma \in Z_{n-1} \mid \sigma(n-1) = n-1\} &\longrightarrow Z_{n-2}\\ \sigma &\longmapsto \sigma \text{ restreinte à } \{1,\dots,n-2\}\end{aligned}$$

qui respecte le nombre de descentes. Or, pour tout v dans V,

$$\gamma'\Big(\sum_{\rho \in Z_{n-2}} (-1)^{d(\rho)} u_{\rho(1)} \otimes \dots \check{u}_1 \cdots \otimes u_{\rho(n-2)} \otimes v\Big) \;=\; \gamma'(i'(u_1 \dots u_{n-2})v)$$

qui vaut $u_1 \dots u_{n-2}v \;=\; [[\cdots[u_1,u_2],\cdots u_{n-2}],v]$ par l'hypothèse de récurrence, et chaque $\gamma'(u_{\rho(1)} \dots \check{u}_1 \dots u_{\rho(n-2)}u_{n-1}u_n)$ s'obtient en remplaçant v par $[u_{n-1},u_n]$ dans l'expression de $\gamma'(u_{\rho(1)} \dots \check{u}_1 \dots u_{\rho(n-2)}v)$ (voir la définition de γ', en notant que " $\check{}$ n'est jamais sur v"). En résumé,

$$\begin{aligned}\gamma'(i'(u_1 \dots u_{n-1})u_n) &= [[[\cdots[u_1,u_2],\cdots u_{n-2}],u_n],u_{n-1}]\\ &\quad + [[[\cdots[u_1,u_2],\cdots u_{n-2}],[u_{n-1},u_n]]\\ &= [[[\cdots[u_1,u_2],\cdots u_{n-2}],u_{n-1}],u_n]\\ &\quad \text{par l'identité de Leibniz}\\ &= u_1u_2\dots u_n, \quad \text{ce qui termine la démonstration.} \quad \square\end{aligned}$$

Corollaire A.5. *Les algèbres de Leibniz Prim(Dias(V)) et Leib(V) sont isomorphes.*

Démonstration. D'après le théorème, $Prim(Dias(V)) \;=\; i'(Leib(V))$, qui d'après la proposition précédente est isomorphe à $Leib(V)$. $\square$

Remarque. Par analogie avec le cas classique (cf. [J]), dans le théorème ci-dessus, i)⇔ii) (ou le corollaire) est un "théorème de Friedrichs-Leibniz", et i)⇔iii) un "théorème de Specht-Wever-Leibniz".

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