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Chapter IV — Meyberg p. 25–27

(note the rest of Chapt III is subsumed by the second batch of exercises (#11–20))

def. triple system = vector space F & trilinear map
$$\langle\ ,\ ,\ \rangle : F \times F \times F \to F$$

def associative triple system of the first kind (not the preferred kind) satisfies

$$\langle 4.1\rangle \qquad \langle xy\langle uvw\rangle\rangle = \langle\langle xyu\rangle vw\rangle = \langle x\langle yuv\rangle w\rangle$$

$$L(x,y)z = \langle xyz\rangle$$
$$R(x,y)z = \langle zyx\rangle$$

$$\langle 4.2\rangle \qquad L(x,y)L(u,v) = L(\langle xyu\rangle, v) = L(x,\langle yuv\rangle)$$
$$\langle 4.3\rangle \qquad R(w,v)R(u,y) = R(\langle uvw\rangle, y) = R(w,\langle yuv\rangle)$$
$$\langle 4.4\rangle \qquad L(x,y)R(w,v) = R(w,v)L(x,y)$$

$$\left[\ \langle xy\langle zvw\rangle\rangle \overset{?}{=} \langle\langle xyz\rangle vw\rangle\ \right]$$

QED

defs $\tilde{E} = \operatorname{End} F \oplus (\operatorname{End} F)^{op}$

$L_0 = \operatorname{span}\{\lambda(x,y) : x,y \in F\}$ where $\lambda(x,y) = (L(x,y), R(y,x))$

$$(4.5) \qquad \lambda(x,y)\lambda(u,v) = \lambda(x,\langle yuv\rangle) = \lambda(\langle xyu\rangle, v)$$ misprint in Meyberg

Proof of Theorem 1 page 27 (= Exercise 1)

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F assoc. triple system of ~~first kind~~

$$\langle xy\langle uvz\rangle\rangle = \langle\langle xyu\rangle vz\rangle = \langle x\langle yuv\rangle z\rangle$$

$\tilde{E} = \operatorname{End}F \oplus (\operatorname{End}F)^{op}$ (an algebra with unit $(\mathrm{id}_F, \mathrm{id}_F)$)

$L_0 =$ linear span of $\{\lambda(x,y) : x, y \in F\} \subset \tilde{E}$

$\lambda(x,y) = (L(x,y), R(y,x)) \in \tilde{E}$

$L(x,y)z = \langle xyz\rangle \quad R(y,x)z = \langle zxy\rangle$

by (4.5) $\lambda(x,y)\lambda(u,v) = \lambda(x, \underset{cdz}{\langle yuv\rangle}) = \lambda(\underset{cd}{\langle xyu\rangle}, \underset{z}{v})$

L_0 is a subalgebra of E, let I ~~be the~~ $= (\mathrm{id}_F, \mathrm{id}_F)$ be the unit of $\tilde{E}$ and (Meyberg uses the letter E for I)

$L = \Phi I + L_0$, a subalgebra (with unit) of $\tilde{E}$

We can make the vector space F into a left $\tilde{E}$-module and a right $\tilde{E}$-module as follows: if $A = (A_1, A_2) \in \tilde{E}$, $x \in F$

$A = (A_1, A_2)$	
$A \circ x = A_1 x$	$x \circ A = A_2 x$
$E \times F \longrightarrow F$	$E \times F \longrightarrow F$
$(A, x) \longrightarrow A \circ x$	$(A, x) \longrightarrow x \circ A$
(left E-module)	right E-module

check that $A \circ (B \circ x) = (AB) \circ x$ check that $(x \circ A) \circ B = x \circ (AB)$

in fact $(A \circ x) \circ B = A \circ (x \circ B)$ This is lemma 1 on page 26

We now make the vector space $A = L \oplus F$ into an algebra by defining

$$(A,x)(B,y) = (AB + \lambda(x,y),\ A\circ y + \overset{x\circ B}{\cancel{B\circ x}})$$

Theorem 1 (page 27) A is an associative algebra with unit and $\exists$ lin. iso $f: F \longrightarrow A$ such that $f(\langle xyz\rangle) = f(x)f(y)f(z)$

~~Proof.~~ let us first $\surd$ Check the module properties at bottom of p. 1

$$A\circ(B\circ x) = A\circ(B_1 x) = \boxed{A_1B_1x}$$

$B = (B_1, B_2)$
$A = (A_1, A_2)$

note the reversed order

$$AB = (A_1, A_2)(B_1, B_2) = (A_1B_1, B_2A_2)$$

So $(AB)\circ x = \boxed{A_1B_1x}$ so F is a left E-module.

$$(x\circ A)\circ B = (A_2x)\circ B = \boxed{B_2A_2x}$$

$x\circ(AB) = x\circ(A_1B_1, B_2A_2) = \boxed{B_2A_2x}$ so F is a right E-module

$$\left.\begin{aligned}(A\circ x)\circ B &= (A_1x)\circ B = B_2A_1x\\ A\circ(x\circ B) &= A\circ(B_2x) = A_1B_2x\end{aligned}\right\}$$ these don't look equal; but they are!

To prove that $B_2A_1 = A_1B_2$ we use the fact that $A, B \in L$; hence we may assume that

$$A\cancel{x} = \cancel{\lambda(x,y)}\ \lambda(y,z) = (L(y,z), R(z,y))$$

and $B = \lambda(u,v) = (L(u,v), R(v,u))$

(which is (4.4)!)

Then by the line after (4.3) on page 26

namely $L(x,y)\,R(w,v) = R(w,v)\,L(x,y)$

we have $B_2 A_1 = R(v,u)\,L(y,z)$

and $A_1 B_2 = L(y,z)\,R(v,u)$

which are equal by

Hence E is an L-bimodule

(meaning left L-module and right-L-module satisfying

$$(A\circ x)\circ B = A\circ(x\circ B)$$

Now we can prove Theorem 1

A is a direct sum of vector spaces so it is a vector space

<u>bilinear product</u>?

$$\big((A,x)+(A',x')\big)(B,y) = (A+A',x+x')(B,y)$$

$$= \big((A+A')B + \lambda(x+x',y),\ (A+A')\circ y + B\circ(x+x')\big)$$

$$\big(AB + A'B + \lambda(x,y) + \lambda(x',y),\ A\circ y + A'\circ y + B\circ x + B\circ x'\big)$$

$$= \big(AB + \lambda(x,y) + A'B + \lambda(x',y),\ A\circ y + B\circ x + A'\circ y + B\circ x'\big)$$

$$= \big(AB+\lambda(x,y),\ A\circ y + B\circ x\big) + \big(A'B + \lambda(x',y),\ A'\circ y + B\circ x'\big)$$

$$= (A,x)(B,y) + (A',x')(B,y)$$

— similar for $(A,x)\big((B,y)+(B',y')\big) = (A,x)(B,y) + (A,x)(B',y')$

<u>associative algebra</u> ?

$$\big((A,x)(B,y)\big)(C,z) = \big(\overbrace{AB+\lambda(x,y)}^{A},\ \overbrace{A\circ y + x\circ B}^{x}\big)(C,z)$$

$$= \Big(\big(AB+\lambda(x,y)\big)C + \lambda(A\circ y + x\circ B,\ z),\ \big(AB+\lambda(x,y)\big)\circ z + (A\circ y + x\circ B)\circ C\Big)$$

$$\overset{?}{=}$$

$$(A,x)\big((B,y)(C,z)\big) = (A,x)\big(\overbrace{BC+\lambda(y,z)}^{B},\ \overbrace{B\circ z + y\circ C}^{y}\big)$$

$$= \Big(A\big(BC+\lambda(y,z)\big) + \lambda(x,\ B\circ z + y\circ C),\ A\circ(B\circ z + y\circ C) + \big(BC+\lambda(y,z)\big)\circ x\Big)$$

"Need"

$$\big(\overset{1}{AB} + \overset{3}{\lambda(x,y)}\big)C + \lambda(\overset{2}{A\circ y} + \overset{4}{x\circ B},\ z) \overset{?}{=} A\big(\overset{1}{BC} + \overset{2}{\lambda(y,z)}\big) + \lambda(x,\ \overset{4}{B\circ z} + \overset{3}{y\circ C})$$

and "Need"

$$\big(\overset{5}{AB} + \overset{8}{\lambda(x,y)}\big)\circ z + (\overset{7}{A\circ y} + \overset{6}{x\circ B})\circ C \overset{?}{=} \overset{5}{A\circ(B\circ z} + \overset{7}{y\circ C}) + x\circ(\overset{6}{BC} + \overset{8}{\lambda(y,z)})$$

We may assume $A = \lambda(a,b)$ $B = \lambda(c,d)$ $C = \lambda(e,f)$

where $a,b,c,d,e,f \in F$

$$A = \lambda(a,b) = (L(a,b),\ R(b,a))$$

$$B = \lambda(c,d) = (L(c,d),\ R(d,c))$$

$$C = \lambda(e,f) = (L(e,f),\ R(f,e))$$

Guess ③ $\lambda(x,y)\,C \overset{?}{=} \lambda(x, y\circ C)$

② $\lambda(A\circ y, z) \overset{?}{=} A\,\lambda(y,z)$

④ $\lambda(x\cdot B, z) \overset{?}{=} \lambda(x, B\circ z)$

proof of ③

$$\lambda(x,y)\,C = \lambda(x,y)\,\lambda(e,f) \overset{(4.5)}{=} \lambda(x, \langle yef\rangle)$$

$$\lambda(x, y\circ C) = \lambda(x, C_2 y) = \lambda(x, R(f,e)\,y) = \lambda(x, \langle yef\rangle)$$

proof of ②

$$\lambda(A\circ y, z) = \lambda(A_1 y, z) = \lambda(L(a,b)y, z) = \lambda(\langle aby\rangle, z)$$

$$A\,\lambda(y,z) = \cancel{(L(a,b), R(b,a))(L(y,z), R(z,y))}$$

$$\cancel{= (L(a,b)\,L(y,z), R(z,y)\,R(b,a))}$$

$$= \lambda(a,b)\,\lambda(y,z) \overset{(4.5)}{=} \lambda(\langle aby\rangle, z)$$

proof of ④

$$\lambda(x\circ B, z) = \lambda(B_2 x, z) = \lambda(R(d,c)x, z) = \lambda(\langle xcd\rangle, z)$$

$$\lambda(x, B\circ z) = \lambda(x, B_1 z) = \lambda(x, L(c,d)z) = \lambda(x, \langle cdz\rangle)$$

= by (4.5)

This proves the first "Need" on p. ④

proof of ⑧

$$\lambda(x,y)\circ z = (L(x,y), R(y,x))\circ z = L(x,y)\,z = \langle xyz\rangle$$

$$x\circ\lambda(y,z) = x\circ(L(y,z), R(z,y)) = R(z,y)\,x = \langle xyz\rangle \qquad \text{BINGO!}$$

This proves associativity

Finally, let $f : F \to A$ be defined

by $f(x) = (0, x)$

Then $f(x)\,f(y)\,f(z) = \underbrace{(0,x)(0,y)}(0,z)$

$$= (\lambda(x,y), 0)(0,z) = (0, \lambda(x,y)z) = (0, \langle xyz \rangle)$$

$$= f(\langle xyz \rangle)$$

$\blacksquare$