

## ON CLASSIFICATION OF COMPLEX FILIFORM LEIBNIZ ALGEBRAS

J. R. GÓMEZ<sup>1</sup>, B.A. OMIROV<sup>2</sup>

ABSTRACT. In this paper we prove that in classifying of complex filiform Leibniz algebras, for which its naturally graded algebra is non-Lie algebra, it suffices to consider some special basis transformations. Moreover, we establish a criterion whether given two such Leibniz algebras are isomorphic in terms of such transformations. The classification problem of filiform Leibniz algebras, for which its naturally graded algebras are non-Lie in an arbitrary dimension, is reduced to the investigation of the obtained conditions.

<sup>1</sup> Dpto. Matemática Aplicada I, Universidad de Sevilla, Avda. Reina Mercedes, s/n. 41012 Sevilla (Spain), e-mail: *jrgomez@us.es*

<sup>2</sup> Institute of Mathematics, 29, Do'rmon Yo'li str., 100125, Tashkent (Uzbekistan), e-mail: *omirovb@mail.ru*

**AMS Subject Classifications (2010):** 17A32, 17A36, 17B30.

**Key words:** Lie algebra, Leibniz algebra, filiform Leibniz algebra, natural gradation, classification, adapted basis.

## 1. INTRODUCTION

This paper is devoted to the study of Leibniz algebras, which have been introduced in [11], [12] and further investigated in many papers, including, for example [2], [6]-[8]. In fact, it is known that many properties of nilpotent Lie algebras can be extended to the Leibniz algebras [2], [3], [13].

For an arbitrary Leibniz algebra  $L$  with a basis  $\{e_0, e_1, \dots, e_n\}$  the table of multiplication is defined by the products of the basic elements. Namely, the products  $[e_i, e_j] = \sum_{k=0}^n \gamma_{ij}^k e_k$  completely determine products of arbitrary elements of the algebra. The constants  $\gamma_{ij}^k$  are called the structural constants of the algebra  $L$  at the basis  $\{e_0, e_1, \dots, e_n\}$ .

Thus the problem of classification of algebras can be reduced to the problem of finding a description of the structural constants up to a non-degenerate basis transformation. From the Leibniz identity we have polynomial equalities for the structural constants:

$$\sum_{l=0}^n (\gamma_{jk}^l \gamma_{il}^m - \gamma_{ij}^l \gamma_{lk}^m + \gamma_{ik}^l \gamma_{lj}^m) = 0.$$

But the straightforward description of structural constants is somewhat cumbersome and therefore usually one has to apply different methods of investigation.

Since the description of all nilpotent Leibniz algebras is unsolvable task (even in the case of Lie algebras) we reduce our discussion with restriction on their nilindex. The first step in this direction was done by M. Vergne in [16]. She classified naturally graded Lie algebras of maximal nilindex (filiform algebras) and presented a description of filiform Lie algebras into sum of naturally graded Lie algebra and its 2-cocycles. We should note that in the case of Leibniz algebras, unlike the Lie algebras, the notion of singly-generated algebra have sense (such nilpotent algebras called zero-filiform Leibniz algebras and evidently, they have maximal nilindex). In [2] the existence of only one zero-filiform Leibniz algebra in each dimension was shown and classification of naturally graded filiform (in Leibniz algebras case they have nilindex equal to the maximum minus one) is obtained. Also, the description of filiform Lie algebras were extended to the Leibniz algebras case.

Many authors have studied the classification of nilpotent Lie algebras for low dimensions. The lists of nilpotent Lie algebras up to dimension 8 can be found in [10] and the classification of filiform Lie algebras up to dimension 12 can be obtained from [5] and [9]. The extensions of the classification of filiform Lie algebras of dimension 6 and 7 to the case of Leibniz algebras were obtained in [1] and [15], respectively.

In fact, the classification algorithm of any variety of algebras with some conditions in fixed dimension consist of the following four steps:

- finding a basis (an adapted basis) in which the table of multiplication of an algebra have the most convenient form;
- to reduce the study of all transformations of the adapted basis to the simple ones;
- to find relations between parameters (structural constants) in initial and transformed basses;
- present the list of pairwise non-isomorphic algebras such that any algebra with the considered conditions is isomorphic to an algebra of the presented list.

A new interesting algorithm for classifying complex filiform Lie algebras is given in [4]. However, our algorithm for the special case studied here is different and enables us to get newer results.

In the case of filiform Lie algebras, the first two steps of the algorithm is already obtained [9]. In the present paper, we simplify the algorithm of classification for some filiform Leibniz algebras. In fact, using results of [2], where the families of filiform Leibniz algebra, for which its naturally graded algebra is non-Lie, are obtained (i.e. the first step of the algorithm for such algebras was done), we complete the next two of the mentioned steps. Therefore, now for the classification of such filiform Leibniz algebras in an arbitrary finite dimension, we can start from the analysis of the obtained conditions for structural constants and present the final list of the algebras. Moreover, from Theorem 4.4 we can conclude that description of such algebras in each dimension is an algorithmically solvable problem.

In [14] some properties of Leibniz filiform algebras, for which its naturally graded algebra is a Lie algebra were studied.

Throughout the paper the basic field is the field of complex numbers and in the tables of multiplication we shall omit the products which are equal to zero.

## 2. PRELIMINARIES

**Definition 2.1.** ([11]) A vector space  $L$  over a field  $F$  with a multiplication  $[-, -] : L \otimes L \rightarrow L$  is called a *Leibniz algebra* if it satisfies the following identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

Given an arbitrary Leibniz algebra  $L$ , we define the lower series sequence:

$$L^1 = L, L^{k+1} = [L^k, L], k \geq 1.$$

Now we define the main object of the paper.

**Definition 2.2.** A Leibniz algebra  $L$  is said to be *filiform* if  $\dim L^i = n - i$ , for  $2 \leq i \leq n$  and  $n = \dim L$ .

Note that the notion of filiform Leibniz algebras agrees with the notion of filiform Lie algebra [16].

**Definition 2.3.** Given a filiform Leibniz algebra  $L$ , put  $L_i = L^i/L^{i+1}$ ,  $1 \leq i \leq n - 1$ , and  $grL = L_1 \oplus L_2 \oplus \dots \oplus L_{n-1}$ . Then  $[L_i, L_j] \subseteq L_{i+j}$  and we obtain the graded algebra  $grL$ . If  $grL$  and  $L$  are isomorphic, denoted by  $grL = L$ , we say that *the algebra  $L$  is naturally graded*.

In the following theorem, we summarize the results of the works [2], [16].

**Theorem 2.4.** *Any complex  $(n+1)$ -dimensional naturally graded filiform Leibniz algebra is isomorphic to one of the following pairwise non isomorphic algebras:*

$$\begin{cases} [e_0, e_0] = e_2, \\ [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n - 1 \end{cases} \quad \begin{cases} [e_0, e_0] = e_2, \\ [e_i, e_0] = e_{i+1}, & 2 \leq i \leq n - 1 \end{cases}$$

$$\begin{cases} [e_i, e_0] = -[e_0, e_i] = e_{i+1}, & 1 \leq i \leq n - 1 \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = \delta(-1)^i e_n & 1 \leq i \leq n - 1. \end{cases}$$

where  $\delta \in \{0, 1\}$  for odd  $n$  and  $\delta = 0$  for even  $n$ .

It should be noted that the first two algebras are non-Lie Leibniz algebras and the third one is Lie algebra.

Due to the list of Theorem 2.4, we derive that the set of all complex filiform Leibniz algebras is decomposed into three disjoint families of algebras.

**Theorem 2.5.** *An arbitrary complex  $(n+1)$ -dimensional filiform Leibniz algebra  $L$  is isomorphic to one of the following algebras:*

$$\mu_1^{\bar{\alpha}, \theta} = \begin{cases} [e_0, e_0] = e_2, \\ [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1 \\ [e_0, e_1] = \alpha_3 e_3 + \alpha_4 e_4 + \dots + \alpha_{n-1} e_{n-1} + \theta e_n, \\ [e_j, e_1] = \alpha_3 e_{j+2} + \alpha_4 e_{j+3} + \dots + \alpha_{n+1-j} e_n, & 1 \leq j \leq n-2 \end{cases}$$

$$\mu_2^{\bar{\beta}, \gamma} = \begin{cases} [e_0, e_0] = e_2, \\ [e_i, e_0] = e_{i+1}, & 2 \leq i \leq n-1 \\ [e_0, e_1] = \beta_3 e_3 + \beta_4 e_4 + \dots + \beta_n e_n, \\ [e_1, e_1] = \gamma e_n, \\ [e_j, e_1] = \beta_3 e_{j+2} + \beta_4 e_{j+3} + \dots + \beta_{n+1-j} e_n, & 2 \leq j \leq n-2 \end{cases}$$

$$\mu_3^{\alpha, \beta, \gamma} = \begin{cases} [e_0, e_0] = \alpha e_n, \\ [e_1, e_1] = \beta e_n, \\ [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1 \\ [e_0, e_1] = -e_2 + \gamma e_n, \\ [e_0, e_i] = -e_{i+1}, & 2 \leq i \leq n-1 \\ [e_i, e_j] = -[e_j, e_i] \in \text{lin} \langle e_{i+j+1}, e_{i+j+2}, \dots, e_n \rangle, & 1 \leq i \leq n-3, \\ & 2 \leq j \leq n-1-i \\ [e_{n-i}, e_i] = -[e_i, e_{n-i}] = (-1)^i \delta e_n, & 1 \leq i \leq n-1 \end{cases}$$

where  $[ , ]$  is the multiplication in  $L$  and  $\{e_0, e_1, e_2, \dots, e_n\}$  is the basis of the algebra,  $\delta \in \{0, 1\}$  for odd  $n$  and  $\delta = 0$  for even  $n$ . Moreover, the table of multiplication of the family  $\mu_3^{\alpha, \beta, \gamma}$  should satisfy the Leibniz identity.

**Remark 2.6.** By Theorem 2.5 the first step of the algorithm is done, i.e. we find the basis in which the table of multiplication of filiform Leibniz algebra have the most convenient form. It is easy to see that algebras from  $\mu_1^{\bar{\alpha}, \theta}$ ,  $\mu_2^{\bar{\beta}, \gamma}$  are non-Lie and Lie algebras belong to the family  $\mu_3^{\alpha, \beta, \gamma}$ .

### 3. ON TRANSFORMATIONS OF COMPLEX FILIFORM LEIBNIZ ALGEBRAS.

Since an arbitrary filiform Leibniz algebra, up to an isomorphism, belongs to one of the families of Theorem 2.5, we conclude that in order to investigate the isomorphisms inside the families, we need to study the behavior of the parameters (structural constants) under the action of the non-degenerate change of basis. Further throughout the paper we shall consider only the first two families of Theorem 2.5.

Let  $L$  be a complex filiform  $(n+1)$ -dimensional Leibniz algebra which is obtained from the naturally graded filiform non-Lie Leibniz algebras.

**Definition 3.1.** A basis  $\{e_0, e_1, \dots, e_n\}$  of an algebra is said to be *adapted* if the multiplication of the algebra has the form  $\mu_1^{\bar{\alpha}, \theta}$  or  $\mu_2^{\bar{\beta}, \gamma}$ .

Let  $L$  be a Leibniz algebra defined on a vector space  $V$  and  $\{e_0, e_1, \dots, e_n\}$  is the adapted basis of the algebra  $L$ .

**Definition 3.2.** A basis transformation  $f \in GL(V)$  is said to be *an adapted for the multiplication of the algebra  $L$*  if a basis  $\{f(e_0), f(e_1), \dots, f(e_n)\}$  is adapted.

The closed subgroup of the group  $GL(V)$  consisting of adapted transformations will be denoted by  $GL_{ad}(V)$ .

From the following equalities:

$$\sum_{i=k}^n a(i) \sum_{j=i+p}^n b(i, j) e_j = \sum_{t=k}^{n-p} \sum_{i=k}^t a(i) b(i, t+p) e_{t+p} = \sum_{j=k+p}^n \sum_{i=k}^{j-p} a(i) b(i, j) e_j,$$

we obtain the equality

$$\sum_{i=k}^n a(i) \sum_{j=i+p}^n b(i, j) e_j = \sum_{j=k+p}^n \sum_{i=k}^{j-p} a(i) b(i, j) e_j, \quad 0 \leq p \leq n-k, 3 \leq k \leq n. \quad (1)$$

**Proposition 3.3.** *Let  $f \in GL_{ad}(V)$ .*

a) *If the algebra  $L$  belongs to the family  $\mu_1^{\bar{\alpha}, \theta}$ , then  $f$  has the following form:*

$$\begin{cases} f(e_0) = \sum_{i=0}^n a_i e_i, \\ f(e_1) = (a_0 + a_1)e_1 + \sum_{i=2}^{n-2} a_i e_i + (a_{n-1} + a_1(\theta - \alpha_n))e_{n-1} + b_n e_n, \\ f(e_{i+1}) = [f(e_i), f(e_0)], \quad 1 \leq i \leq n-1 \\ f(e_2) = [f(e_0), f(e_0)]. \end{cases}$$

b) *If the algebra  $L$  belongs to the family  $\mu_2^{\bar{\beta}, \gamma}$ , then  $f$  has the following form:*

$$\begin{cases} f(e_0) = \sum_{i=0}^n a_i e_i, \\ f(e_1) = b_1 e_1 - \frac{a_1 b_1 \gamma}{a_0} e_{n-1} + b_n e_n \\ f(e_{i+1}) = [f(e_i), f(e_0)], \quad 2 \leq i \leq n-1 \\ f(e_2) = [f(e_0), f(e_0)]. \end{cases}$$

*Proof.* Let  $f \in GL_{ad}(V)$ . We set  $f(e_0) = \sum_{i=0}^n a_i e_i$  and  $f(e_1) = \sum_{j=0}^n b_j e_j$ .

**Case a).** Consider the product  $f(e_2) = [f(e_0), f(e_0)]$ . Using the equality (1) we have

$$\begin{aligned} [f(e_0), f(e_0)] &= a_0(a_0 + a_1)e_2 + a_0 \sum_{i=3}^n a_{i-1}e_i + a_0 a_1 \left( \sum_{i=3}^{n-1} \alpha_i e_i + \theta e_n \right) + a_1^2 \sum_{i=3}^n \alpha_i e_i + \\ a_1 \sum_{i=2}^{n-2} a_i \sum_{k=i+2}^n \alpha_{k+1-i} e_k &= a_0(a_0 + a_1)e_2 + a_0 \sum_{i=3}^n a_{i-1}e_i + a_1(a_0 + a_1) \sum_{i=3}^{n-1} \alpha_i e_i + a_1(a_0\theta + a_1\alpha_n)e_n + \\ a_1 \sum_{i=4}^n a_{i-2} \sum_{k=i}^n \alpha_{k+3-i} e_k &= a_0(a_0 + a_1)e_2 + a_0 \sum_{i=3}^n a_{i-1}e_i + a_1(a_0 + a_1) \sum_{i=3}^{n-1} \alpha_i e_i + a_1(a_0\theta + a_1\alpha_n)e_n + \\ a_1 \sum_{i=4}^n \sum_{k=i}^k (a_{i-2} \alpha_{k+3-i} e_k) &= a_0(a_0 + a_1)e_2 + (a_0 a_2 + a_1(a_0 + a_1)\alpha_3)e_3 + \sum_{t=4}^{n-1} (a_0 a_{t-1} + a_1(a_0 + a_1)\alpha_t + \\ & a_1 \sum_{i=4}^t a_{i-2} \alpha_{t+3-i}) e_t + (a_0 a_{n-1} + a_1(a_0\theta + a_1\alpha_n) + a_1 \sum_{i=4}^n a_{i-2} \alpha_{n+3-i}) e_n = f(e_2). \end{aligned}$$

Since  $[f(e_0), f(e_0)] \in L^2$ , we get  $a_0(a_0 + a_1) \neq 0$ .

Consider the product

$$[f(e_0), f(e_1)] = b_0(a_0 + a_1)e_2 + \sum_{i=3}^n c_i e_i.$$

Since  $[f(e_0), f(e_1)] \notin L^2$  and  $a_0 + a_1 \neq 0$ , we conclude that  $b_0 = 0$ .

The properties of the adapted transformation deduce  $f(e_2) = [f(e_1), f(e_0)]$ .

The product  $[f(e_1), f(e_0)]$  in the basis  $\{e_0, e_1, \dots, e_n\}$  has the following form:

$$\begin{aligned} [f(e_1), f(e_0)] &= a_0 b_1 e_2 + (a_0 b_2 + a_1 b_1 \alpha_3) e_3 + \sum_{t=4}^{n-1} (a_0 b_{t-1} + a_1 b_1 \alpha_t + a_1 \sum_{i=4}^t b_{i-2} \alpha_{t+3-i}) e_t + \\ & (a_0 b_{n-1} + a_1 b_1 \alpha_n + a_1 \sum_{i=4}^n b_{i-2} \alpha_{n+3-i}) e_n. \end{aligned}$$

Comparing the coefficients at the basis elements we get the conditions to coefficients of the transformation  $f$ :

$$\begin{cases} a_0 + a_1 = b_1, & a_2 = b_2, \\ a_0 a_{t-1} + a_1 \sum_{i=4}^t a_{i-2} \alpha_{t+3-i} = a_0 b_{t-1} + a_1 \sum_{i=4}^t b_{i-2} \alpha_{t+3-i}, \\ a_0 a_{n-1} + a_1(a_0\theta + a_1\alpha_n) + a_1 \sum_{i=4}^n a_{i-2} \alpha_{n+3-i} = a_0 b_{n-1} + a_1 b_1 \alpha_n + a_1 \sum_{i=4}^n b_{i-2} \alpha_{n+3-i}. \end{cases}$$

From these conditions we have

$$\begin{cases} b_1 = a_0 + a_1, \\ b_i = a_i, \quad 2 \leq i \leq n-2 \\ b_{n-1} = a_{n-1} + a_1(\theta - \alpha_n). \end{cases}$$

Case b) is proved by a similar way.  $\square$

Similarly to [9], we introduce the notion of elementary transformations for algebras from families  $\mu_1^{\bar{\alpha}, \theta}$  and  $\mu_2^{\bar{\beta}, \gamma}$ .

**Definition 3.4.** The following types of the adapted transformations are said to be *elementary*:

$$\text{first type} - \tau(a, b, k) = \begin{cases} f(e_0) = e_0 + ae_k \\ f(e_1) = e_1 + be_k \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \leq i \leq n-1, 2 \leq k \leq n \\ f(e_2) = [f(e_0), f(e_0)] \end{cases}$$

$$\text{second type} - \vartheta(a, b) = \begin{cases} f(e_0) = ae_0 + be_1 \\ f(e_1) = (a+b)e_1 + b(\theta - \alpha_n)e_{n-1}, & a(a+b) \neq 0 \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \leq i \leq n-1, \\ f(e_2) = [f(e_0), f(e_0)] \end{cases}$$

$$\text{third type} - \sigma(b, n) = \begin{cases} f(e_0) = e_0 \\ f(e_1) = e_1 + be_n, \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 2 \leq i \leq n-1, \\ f(e_2) = [f(e_0), f(e_0)] \end{cases}$$

$$\text{fourth type} - \eta(a, k) = \begin{cases} f(e_0) = e_0 + ae_k \\ f(e_1) = e_1 \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 2 \leq i \leq n-1, 2 \leq k \leq n, \\ f(e_2) = [f(e_0), f(e_0)] \end{cases}$$

$$\text{fifth type} - \delta(a, b, d) = \begin{cases} f(e_0) = ae_0 + be_1 \\ f(e_1) = de_1 - \frac{bd\gamma}{a}e_{n-1}, & ad \neq 0 \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 2 \leq i \leq n-1, \\ f(e_2) = [f(e_0), f(e_0)] \end{cases}$$

where  $a, b, d \in \mathbb{C}$ .

Let  $f$  be an arbitrary element of the group  $GL_{ad}(V)$ , then  $f$  can be expressed as superposition of the elementary transformations.

**Proposition 3.5.**

i) Let  $f$  has the form a) of Proposition 3.3. Then

$$f = \tau(a_n, b_n, n) \circ \tau(a_{n-1}, a_{n-1}, n-1) \circ \dots \circ \tau(a_2, a_2, 2) \circ \vartheta(a_0, a_1)$$

ii) Let  $f$  have the form b) of Proposition 3.3. Then

$$f = \sigma(b_n, n) \circ \eta(a_n, n) \circ \eta(a_{n-1}, n-2) \circ \dots \circ \eta(a_2, 2) \circ \delta(a_0, a_1, b_1)$$

*Proof.* Straightforward.  $\square$

For the above decompositions the following is true:

**Proposition 3.6.**

1) A basis transformation

$$g = \tau(a_n, b_n, n) \circ \tau(a_{n-1}, a_{n-1}, n-1) \circ \dots \circ \tau(a_2, a_2, 2)$$

does not change the structural constants of an algebra of the family  $\mu_1^{\bar{\alpha}, \theta}$ .

2) A basis transformation

$$\varphi = \sigma(b_n, n) \circ \eta(a_n, n) \circ \eta(a_{n-1}, n-2) \circ \dots \circ \eta(a_2, 2)$$

does not change the structural constants of an algebra of the family  $\mu_2^{\bar{\beta}, \gamma}$ .

*Proof.* Let us prove the first assertion.

Consider a basis transformation  $\tau(a, b, k)$  :

$$\tau(a, b, k) = \begin{cases} f(e_0) = e_0 + ae_k, \\ f(e_1) = e_1 + be_k, & 2 \leq k \leq n \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \leq i \leq n-1 \\ f(e_2) = [f(e_0), f(e_0)] \end{cases}$$

For  $2 \leq k \leq n-1$  we put  $a = b$  and consider the products which involve the parameters:

$$[f(e_0), f(e_1)] = \sum_{i=3}^{n-k+1} \alpha_i(e_i + ae_{k+i-1}) + \sum_{i=n-k+2}^{n-1} \alpha_i e_i + \theta e_n = \sum_{i=3}^{n-1} \alpha_i f(e_i) + \theta f(e_n),$$

$$[f(e_1), f(e_1)] = \sum_{i=3}^n \alpha_i e_i + a \sum_{i=3}^{n-k+1} \alpha_i e_{k+i-1} = \sum_{i=3}^{n-k+1} \alpha_i(e_i + ae_{k+i-1}) + \sum_{i=n-k+2}^n \alpha_i e_i = \sum_{i=3}^n \alpha_i f(e_i).$$

Therefore, basis transformations  $\tau(a, a, k)$ ,  $2 \leq k \leq n-1$  for any  $a$  do not change the parameters  $\alpha_i$ ,  $\theta$ .

Analogously, one can check that  $\tau(a, b, n) \in GL_{ad}(V)$  does not change parameters  $\alpha_i$ ,  $\theta$  for any value of  $a$ .

Since a superposition of adapted transformations is again an adapted transformation, we conclude that transformation

$$g = \tau(a_n, b_n, n) \circ \tau(a_{n-1}, a_{n-1}, n-1) \circ \cdots \circ \tau(a_2, a_2, 2)$$

does not change the structural constants of family  $\mu_1^{\bar{\alpha}, \theta}$ .

The proof of the second assertion of the proposition is carried out in a similar way.  $\square$

Thus, the problem of the study of all basis transformations is reduced to the problem of investigation of the second and the fifth types of elementary transformations for the families  $\mu_1^{\bar{\alpha}, \theta}$  and  $\mu_2^{\bar{\beta}, \gamma}$ , respectively.

#### 4. A CRITERION OF ISOMORPHISMS OF COMPLEX FILIFORM NON-LIE LEIBNIZ ALGEBRAS.

For an arbitrary element  $a$  of the Leibniz algebra  $L$ , denote the operator of right multiplication by  $R_a(x)$  (i.e.  $R_a(x) = [x, a]$ ).

Set

$$R_a^m(x) := \underbrace{[[\dots[x, a], a], \dots, a]}_{m\text{-times}} \text{ and } R_a^0(x) := x.$$

It should be noted that for an algebra from the first two families of Theorem 2.5 the following equality holds true:

$$[[e_s, e_1], e_0] = [e_{s+1}, e_1], \quad 2 \leq s \leq n. \quad (2)$$

Let  $L$  be an algebra of the family  $\mu_1^{\bar{\alpha}, \theta}$  (respectively, of the family  $\mu_2^{\bar{\beta}, \gamma}$ ), then from (2) we derive that for  $m \in \mathbb{N}$ ,  $0 \leq p \leq n$  (respectively,  $0 \leq p \leq n$ ,  $p \neq 1$ ) the following equality holds:

$$R_{e_1}^m(e_p) = R_{e_0}^{p-1}(R_{e_1}^m(e_0)). \quad (3)$$

In order to prove the main theorem we need the following lemma.

**Lemma 4.1.** *Let  $L$  be a filiform Leibniz algebra of the first two families from Theorem 2.5. Then for  $2 \leq m \leq \frac{n-1}{2}$  the following equality holds*

$$R_{e_1}^m(e_0) = \sum_{i_m=2m+1}^n \sum_{i_{m-1}=2m+1}^{i_m} \cdots \sum_{i_1=2m+1}^{i_2} \alpha_{i_m+3-i_{m-1}} \cdots \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+3-2(m-1)} e_{i_m},$$

where  $\eta_i = \begin{cases} \alpha_i, & \text{when } L \text{ belongs to the first family} \\ \beta_i, & \text{when } L \text{ belongs to the second family.} \end{cases}$

*Proof.* Let  $\eta_i = \alpha_i$ , the case  $\eta_i = \beta_i$  is proved similarly.

We shall use induction by  $m$ . Using equality (1), for  $m = 2$  we have

$$R_{e_1}^2(e_0) = \left[ \sum_{i=3}^{n-1} \alpha_i e_i + \theta e_n, e_1 \right] = \sum_{i=5}^n \alpha_{i-2} [e_{i-2}, e_1] = \sum_{i=5}^n \alpha_{i-2} \sum_{j=i}^n \alpha_{j+3-i} e_1 = \sum_{j=5}^n \sum_{i=5}^j \alpha_{i-2} \alpha_{j+3-i} e_1.$$

Assume that equality of the lemma for  $m$  is true. Then the following equalities

$$R_{e_1}^{m+1}(e_0) = [R_{e_1}^m(e_0), e_1] = \left[ \sum_{i_k=2m+1}^n \sum_{i_{m-1}=2m+1}^{i_m} \cdots \sum_{i_1=2m+1}^{i_2} \alpha_{i_m+3-i_{m-1}} \cdots \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+3-(2m+1)} e_{i_m}, e_1 \right] =$$

$$\begin{aligned}
& \left[ \sum_{i_m=2m+3}^n \sum_{i_{m-1}=2m+3}^{i_m} \cdots \sum_{i_1=2m+3}^{i_2} \alpha_{i_m+3-i_{m-1}} \cdots \alpha_{i_2+3-i_1} \alpha_{i_1+3-(2m+3)} e_{i_m-2}, e_1 \right] = \\
& \sum_{i_m=2m+3}^n \sum_{i_{m-1}=2m+3}^{i_m} \cdots \sum_{i_1=2m+3}^{i_2} \alpha_{i_m+3-i_{m-1}} \cdots \alpha_{i_2+3-i_1} \alpha_{i_1+3-(2m+3)} \sum_{i_{m+1}=i_m}^n \alpha_{i_{m+1}+3-i_m} e_{i_{m+1}} = \\
& \sum_{i_m=2m+3}^n \sum_{i_{m+1}=i_m}^n \sum_{i_{m-1}=2m+3}^{i_m} \cdots \sum_{i_1=2m+3}^{i_2} \alpha_{i_m+3-i_{m-1}} \cdots \alpha_{i_2+3-i_1} \alpha_{i_1+3-(2m+3)} e_{i_{m+1}} = \\
& \sum_{i_{m+1}=2m+3}^n \sum_{i_m=2m+3}^{i_{m+1}} \sum_{i_{m-1}=2m+3}^{i_m} \cdots \sum_{i_1=2m+3}^{i_2} \alpha_{i_m+3-i_{m-1}} \cdots \alpha_{i_2+3-i_1} \alpha_{i_1+3-(2m+3)} e_{i_{m+1}}
\end{aligned}$$

prove the equality of the lemma for  $m+1$  and consequently completes the proof.  $\square$

Since the study of adapted transformations for the family  $\mu_1^{\bar{\alpha}, \theta}$  is reduced to the study of the following basis transformation

$$\begin{cases} e'_0 = Ae_0 + Be_1, & A(A+B) \neq 0, \\ e'_1 = (A+B)e_1 + B(\theta - \alpha_n)e_{n-1}, \end{cases} \quad (4)$$

we need the expressions for a new basis. Namely, we have

**Corollary 4.2.**

$$\begin{aligned}
e'_2 &= A(A+B)e_2 + B(A+B) \sum_{i=3}^{n-1} \alpha_i e_i + B(A\theta + B\alpha_n), \\
e'_k &= (A+B) \left( \sum_{i=0}^{k-2} C_{k-1}^{k-1-i} A^{k-1-i} B^i R_{e_1}^i(e_{k-i}) + B^{k-1} R_{e_1}^{k-1}(e_0) \right), \quad (5)
\end{aligned}$$

where  $3 \leq k \leq n$  and  $C_s^t = \frac{s!}{(s-t)!t!}$ .

*Proof.* We will prove the corollary by induction on  $k$ . For  $k=2, 3$  we have

$$e'_2 = [e'_1, e'_0] = A(A+B)e_2 + B(A+B) \sum_{i=3}^{n-1} \alpha_i e_i + B(A\theta + B\alpha_n),$$

$$\begin{aligned}
e'_3 &= [e'_2, e'_0] = A^2(A+B)e_3 + 2AB(A+B)[e_2, e_1] + B^2(A+B)[[e_0, e_1], e_1] = \\
& (A+B)(A^2e_3 + 2ABR_{e_1}(e_2) + B^2R_{e_1}^2(e_0)).
\end{aligned}$$

Suppose that equality (5) is true for  $k$ . Taking into account the equality (2) and the following equalities

$$\begin{aligned}
e'_{k+1} &= [e'_k, e_0] = [(A+B) \left( \sum_{i=0}^{k-2} C_{k-1}^{k-1-i} A^{k-1-i} B^i R_{e_1}^i(e_{k-i}) + B^{k-1} R_{e_1}^{k-1}(e_0) \right), Ae_0 + Be_1] = \\
& (A+B) \left( \sum_{i=0}^{k-2} C_{k-1}^{k-1-i} A^{k-i} B^i R_{e_1}^i(e_{k+1-i}) + AB^{k-1} R_{e_1}^{k-1}(e_2) + \sum_{i=0}^{k-2} C_{k-1}^{k-1-i} A^{k-1-i} B^{i+1} R_{e_1}^{i+1}(e_{k-i}) + \right. \\
& B^k R_{e_1}^k(e_0) \left. \right) = (A+B) \left( \sum_{i=0}^{k-2} C_{k-1}^{k-1-i} A^{k-i} B^i R_{e_1}^i(e_{k+1-i}) + AB^{k-1} R_{e_1}^{k-1}(e_2) + \sum_{i=1}^{k-1} C_{k-1}^{k-i} A^{k-i} B^i R_{e_1}^i(e_{k+1-i}) + \right. \\
& B^k R_{e_1}^k(e_0) \left. \right) = (A+B) \left( \sum_{i=1}^{k-2} (C_{k-1}^{k-1-i} + C_{k-1}^{k-i}) A^{k-i} B^i R_{e_1}^i(e_{k+1-i}) + C_{k-1}^{k-1} A^k e_{k+1} + C_{k-1}^1 AB^{k-1} R_{e_1}^{k-1}(e_2) + \right. \\
& AB^{k-1} R_{e_1}^{k-1}(e_2) + B^k R_{e_1}^k(e_0) \left. \right) = (A+B) \left( \sum_{i=1}^{k-2} C_k^{k-i} A^{k-i} B^i R_{e_1}^i(e_{k+1-i}) + C_k^k A^k e_{k+1} + C_k^1 AB^{k-1} R_{e_1}^{k-1}(e_2) + \right. \\
& B^k R_{e_1}^k(e_0) \left. \right) = (A+B) \left( \sum_{i=0}^{k-1} C_k^{k-i} A^{k-i} B^i R_{e_1}^i(e_{k+1-i}) + B^k R_{e_1}^k(e_0) \right)
\end{aligned}$$

we complete the proof of the equality (5) for  $k+1$ .  $\square$

Similarly, for the family  $\mu_2^{\bar{\beta}, \gamma}$  applying the transformation of the type

$$\begin{cases} e'_0 = Ae_0 + Be_1, & AD \neq 0, \\ e'_1 = De_1 - \frac{BD\gamma}{A}e_{n-1}, \end{cases}$$

one can prove the following corollary.

**Corollary 4.3.** *For arbitrary  $3 \leq k \leq n$ ,*

$$e'_k = A \left( \sum_{i=0}^{k-2} C_{k-1}^{k-1-i} A^{k-1-i} B^i R_{e_1}^i (e_{k-i}) + B^{k-1} R_{e_1}^{k-1} (e_0) \right).$$

*Proof.* The proof is carried out in a similar way as the proof of Corollary 4.2. □

We shall denote an algebra from family  $\mu_1^{\bar{\alpha}, \theta}$  (respectively,  $\mu_2^{\bar{\beta}, \gamma}$ ) as  $L(\alpha_3, \alpha_4, \dots, \alpha_n, \theta)$  (respectively,  $L(\beta_3, \beta_4, \dots, \beta_n, \gamma)$ ).

**Theorem 4.4.** *a) Two algebras  $L(\alpha_3, \alpha_4, \dots, \alpha_n, \theta)$  and  $L'(\alpha'_3, \alpha'_4, \dots, \alpha'_n, \theta')$  are isomorphic if and only if there exist  $A, B \in \mathbb{C}$  such that  $A(A+B) \neq 0$  and the following conditions hold:*

$$\begin{aligned} \alpha'_3 &= \frac{(A+B)}{A^2} \alpha_3 \\ \alpha'_t &= \frac{1}{A^{t-1}} \left( (A+B)\alpha_t - \sum_{k=3}^{t-1} (C_{k-1}^{k-2} A^{k-2} B \alpha_{t+2-k} + C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^t \alpha_{t+3-i_1} \alpha_{i_1+1-k} + \right. \\ &+ C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^t \sum_{i_1=k+3}^{i_2} \alpha_{t+3-i_2} \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1-k} + \dots + \\ &+ C_{k-1}^1 A B^{k-2} \sum_{i_{k-3}=2k-2}^t \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_{k-3}} \alpha_{t+3-i_{k-3}} \cdot \alpha_{i_{k-3}+3-i_{k-4}} \cdot \dots \cdot \alpha_{i_2+3-i_1} \alpha_{i_1+5-2k} + \\ &+ B^{k-1} \sum_{i_{k-2}=2k-1}^t \sum_{i_{k-3}=2k-1}^{i_{k-2}} \dots \sum_{i_1=2k-1}^{i_{k-2}} \alpha_{t+3-i_{k-2}} \cdot \alpha_{i_{k-2}+3-i_{k-3}} \cdot \dots \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+4-2k} \left. \right) \cdot \alpha'_k, \end{aligned}$$

$$\begin{aligned} \theta' &= \frac{1}{A^{n-1}} \left( A\theta + B\alpha_n - \sum_{k=3}^{n-1} (C_{k-1}^{k-2} A^{k-2} B \alpha_{n+2-k} + C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^n \alpha_{n+3-i_1} \alpha_{i_1+1-k} + \right. \\ &+ C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^n \sum_{i_1=k+3}^{i_2} \alpha_{n+3-i_2} \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1-k} + \dots + \\ &+ C_{k-1}^1 A B^{k-2} \sum_{i_{k-3}=2k-2}^n \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_{k-3}} \alpha_{n+3-i_{k-3}} \alpha_{i_{k-3}+3-i_{k-4}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+5-2k} + \\ &+ B^{k-1} \sum_{i_{k-2}=2k-1}^n \sum_{i_{k-3}=2k-1}^{i_{k-2}} \dots \sum_{i_1=2k-1}^{i_{k-2}} \alpha_{n+3-i_{k-2}} \cdot \alpha_{i_{k-2}+3-i_{k-3}} \cdot \dots \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+4-2k} \left. \right) \cdot \alpha'_k, \end{aligned}$$

where  $4 \leq t \leq n$ .

*b) Two algebras  $L(\beta_3, \beta_4, \dots, \beta_n, \gamma)$  and  $L'(\beta'_3, \beta'_4, \dots, \beta'_n, \gamma')$  are isomorphic if and only if there exist  $A, B \in \mathbb{C}$  such that  $A(A+B) \neq 0$  and the following conditions hold:*

$$\begin{aligned} \gamma' &= \frac{D^2}{A^n} \gamma, \quad \beta'_3 = \frac{D}{A^2} \beta_3, \\ \beta'_t &= \frac{1}{A^{t-1}} \left( D\beta_t - \sum_{k=3}^{t-1} (C_{k-1}^{k-2} A^{k-2} B \beta_{t+2-k} + C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^t \beta_{t+3-i_1} \cdot \beta_{i_1+1-k} + \right. \\ &+ C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^t \sum_{i_1=k+3}^{i_2} \beta_{t+3-i_2} \cdot \beta_{i_2+3-i_1} \cdot \beta_{i_1-k} + \dots + \\ &+ C_{k-1}^1 A B^{k-2} \sum_{i_{k-3}=2k-2}^t \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_{k-3}} \beta_{t+3-i_{k-3}} \beta_{i_{k-3}+3-i_{k-4}} \dots \beta_{i_2+3-i_1} \beta_{i_1+5-2k} + \\ &+ B^{k-1} \sum_{i_{k-2}=2k-1}^t \sum_{i_{k-3}=2k-1}^{i_{k-2}} \dots \sum_{i_1=2k-1}^{i_{k-2}} \beta_{t+3-i_{k-2}} \beta_{i_{k-2}+3-i_{k-3}} \dots \beta_{i_2+3-i_1} \beta_{i_1+4-2k} \left. \right) \beta'_k, \end{aligned}$$



where  $4 \leq t \leq n-1$ .

$$\begin{aligned} \beta'_n &= \frac{BD\gamma}{A^n} + \frac{1}{A^{n-1}} \left( D\beta_n - \sum_{k=3}^{n-1} (C_{k-1}^{k-2} A^{k-2} B\beta_{n+2-k} + C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^n \beta_{n+3-i_1} \cdot \beta_{i_1+1-k} + \right. \\ &+ C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^n \sum_{i_1=k+3}^{i_2} \beta_{n+3-i_2} \cdot \beta_{i_2+3-i_1} \cdot \beta_{i_1-k} + \dots + \\ &+ C_{k-1}^1 AB^{k-2} \sum_{i_{k-3}=2k-2}^n \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_2} \beta_{n+3-i_{k-3}} \beta_{i_{k-3}+3-i_{k-4}} \dots \beta_{i_2+3-i_1} \beta_{i_1+5-2k} + \\ &\left. + B^{k-1} \sum_{i_{k-2}=2k-1}^n \sum_{i_{k-3}=2k-1}^{i_{k-2}} \dots \sum_{i_1=2k-1}^{i_2} \beta_{n+3-i_{k-2}} \beta_{i_{k-2}+3-i_{k-3}} \dots \beta_{i_2+3-i_1} \beta_{i_1+4-2k} \right) \beta'_k, \end{aligned}$$

*Proof.* Consider the class  $\mu_1^{\bar{\alpha}, \theta}$ . Let  $\{e_0, e_1, \dots, e_n\}$  be a basis of algebra  $L(\alpha_3, \alpha_4, \dots, \alpha_n, \theta)$ , and  $\{e'_0, e'_1, \dots, e'_n\}$  be a basis of the algebra  $L'(\alpha'_3, \alpha'_4, \dots, \alpha'_n, \theta')$ .

It is easy to see that in algebra  $L(\alpha_3, \alpha_4, \dots, \alpha_n, \theta)$  the following is true:

$$[[e_0, e_1], e_1] = [[e_1, e_1], e_1].$$

We will consider a change of basis (4).

From Lemma 4.1 and equality (3) we obtain

$$R_{e_1}^m(e_{k-m}) = \sum_{i_m=k+m}^n \sum_{i_{m-1}=k+m}^{i_m} \dots \sum_{i_1=2m+1}^{i_2} \alpha_{i_m+3-i_{m-1}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+3-(k+m)} e_{i_m} \quad (6)$$

where  $m \leq n-k$   $m \leq k \leq n$ .

Now we substitute (6) in the equality (5) and using equalities (1), (3) we obtain the following:

$$\begin{aligned} e'_k &= (A+B) \left( \sum_{i=0}^{k-2} C_{k-1}^{k-1-i} A^{k-1-i} B^i R_{e_1}^i(e_{k-1}) + B^{k-1} R_{e_1}^{k-1}(e_0) \right) = (A+B) (A^{k-1} e_k + \\ &C_{k-1}^{k-2} A^{k-2} B \sum_{i=k+1}^n \alpha_{i+2-k} e_i + C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i=k+2}^n \sum_{i_1=k+2}^i \alpha_{i+3-i_1} \cdot \alpha_{i_1+1-k} e_i + \\ &C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i=k+3}^n \sum_{i_2=k+3}^i \sum_{i_1=k+3}^{i_2} \alpha_{i+3-i_2} \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1-k} e_i + \dots + \\ &C_{k-1}^1 AB^{k-2} \sum_{i=2k-2}^n \sum_{i_{k-3}=2k-2}^i \dots \sum_{i_1=2k-2}^{i_2} \alpha_{i+3-i_{k-3}} \cdot \dots \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+5-2k} e_i + \\ &B^{k-1} \sum_{i=2k-1}^n \sum_{i_{k-2}=2k-1}^i \dots \sum_{i_1=2k-1}^{i_2} \alpha_{i+3-i_{k-2}} \cdot \dots \cdot \alpha_{i_2+3-i_1} \alpha_{i_1+4-2k} e_i) = \\ &= (A+B) (A^{k-1} e_k + C_{k-1}^{k-2} A^{k-2} B \alpha_3 e_{k+1} + (C_{k-1}^{k-2} A^{k-2} B \alpha_4 + \\ &C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^{k+2} \alpha_{k+5-i_1} \cdot \alpha_{i_1+1-k}) e_{k+2} + \dots + (C_{k-1}^{k-2} A^{k-2} B \alpha_{t+2-k} + \\ &C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+1}^t \alpha_{t+3-i_1} \alpha_{i_1+1-k} + C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^t \sum_{i_1=k+3}^{i_2} \alpha_{t+3-i_2} \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1-k} + \\ &+ \dots + C_{k-1}^1 AB^{k-2} \sum_{i_{k-3}=2k-2}^t \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_2} \alpha_{t+3-i_{k-3}} \cdot \alpha_{i_{k-3}+3-i_{k-4}} \cdot \dots \cdot \alpha_{i_2+3-i_1} \alpha_{i_1+5-2k} + \\ &B^{k-1} \sum_{i_{k-2}=2k-1}^t \sum_{i_{k-3}=2k-1}^{i_{k-2}} \dots \sum_{i_1=2k-1}^{i_2} \alpha_{t+3-i_{k-2}} \cdot \alpha_{i_{k-2}+3-i_{k-3}} \cdot \dots \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+4-2k}) e_t + \\ &+ \dots + (C_{k-1}^{k-2} A^{k-2} B \alpha_{n+2-k} + C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+1}^n \alpha_{n+3-i_1} \alpha_{i_1+1-k} + \end{aligned}$$

$$\begin{aligned}
& C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^n \sum_{i_1=k+3}^{i_2} \alpha_{n+3-i_2} \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1-k} + \dots + \\
& C_{k-1}^1 A B^{k-2} \sum_{i_{k-3}=2k-2}^n \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_2} \alpha_{n+3-i_{k-3}} \cdot \alpha_{i_{k-3}+3-i_{k-4}} \cdot \dots \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+5-2k} + \\
& B^{k-1} \sum_{i_{k-2}=2k-1}^n \sum_{i_{k-3}=2k-1}^{i_{k-2}} \dots \sum_{i_1=2k-1}^{i_2} \alpha_{n+3-i_{k-2}} \cdot \alpha_{i_{k-2}+3-i_{k-3}} \cdot \dots \cdot \alpha_{i_2+3-i_1} \alpha_{i_1+4-2k} e_n) = \\
& = (A+B) \left( A^{k-1} e_k + \sum_{t=k+1}^n (C_{k-1}^{k-2} A^{k-2} B \alpha_{t+2-k} + C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^t \alpha_{t+3-i_1} \cdot \alpha_{i_1+1-k} + \right. \\
& C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^t \sum_{i_1=k+3}^{i_2} \alpha_{t+3-i_2} \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1-k} + \dots + \\
& C_{k-1}^1 A B^{k-2} \sum_{i_{k-3}=2k-2}^t \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_2} \alpha_{t+3-i_{k-3}} \cdot \alpha_{i_{k-3}+3-i_{k-4}} \cdot \dots \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+5-2k} + \\
& \left. + B^{k-1} \sum_{i_{k-2}=2k-1}^t \sum_{i_{k-3}=2k-1}^{i_{k-2}} \dots \sum_{i_1=2k-1}^{i_2} \alpha_{t+3-i_{k-2}} \cdot \alpha_{i_{k-2}+3-i_{k-3}} \cdot \dots \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+4-2k} e_t \right).
\end{aligned}$$

Consider the following products in the algebra  $L'(\alpha'_3, \alpha'_4, \dots, \alpha'_n, \theta')$ :

$$[e'_0, e'_1] = \sum_{k=3}^{n-1} \alpha'_k e'_k + \theta' e'_n, \quad [e'_1, e'_1] = \sum_{k=3}^n \alpha'_k e'_k.$$

Substituting expression  $e'_k$ , obtained above, into the product  $[e'_0, e'_1]$  and using the equality (1) with  $p=1$ , we derive the following equalities:

$$\begin{aligned}
[e'_0, e'_1] &= \sum_{k=3}^{n-1} \alpha'_k (A+B) \left( A^{k-1} e_k + \sum_{t=k+1}^n (C_{k-1}^{k-2} A^{k-2} B \alpha_{t+2-k} + C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^t \alpha_{t+3-i_1} \cdot \alpha_{i_1+1-k} + \right. \\
& C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^t \sum_{i_1=k+3}^{i_2} \alpha_{t+3-i_2} \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1-k} + \dots + \\
& + C_{k-1}^1 A B^{k-2} \sum_{i_{k-3}=2k-2}^t \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_2} \alpha_{t+3-i_{k-3}} \alpha_{i_{k-3}+3-i_{k-4}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+5-2k} + \\
& \left. B^{k-1} \sum_{i_{k-2}=2k-1}^t \sum_{i_{k-3}=2k-1}^{i_{k-2}} \dots \sum_{i_1=2k-1}^{i_2} \alpha_{t+3-i_{k-2}} \alpha_{i_{k-2}+3-i_{k-3}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+4-2k} e_t \right) + \\
& \theta' A^{n-1} (A+B) e_n = (A+B) \left( \sum_{k=3}^{n-1} A^{k-1} \alpha'_k e_k + \sum_{k=3}^{n-1} \sum_{t=k+1}^n (C_{k-1}^{k-2} A^{k-2} B \alpha_{t+2-k} + \right. \\
& C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^t \alpha_{t+3-i_1} \alpha_{i_1+1-k} + C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^t \sum_{i_1=k+3}^{i_2} \alpha_{t+3-i_2} \alpha_{i_2+3-i_1} \alpha_{i_1-k} + \\
& + \dots + C_{k-1}^1 A B^{k-2} \sum_{i_{k-3}=2k-2}^t \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_2} \alpha_{t+3-i_{k-3}} \alpha_{i_{k-3}+3-i_{k-4}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+5-2k} + \\
& \left. B^{k-1} \sum_{i_{k-2}=2k-1}^t \sum_{i_{k-3}=2k-1}^{i_{k-2}} \dots \sum_{i_1=2k-1}^{i_2} \alpha_{t+3-i_{k-2}} \alpha_{i_{k-2}+3-i_{k-3}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+4-2k} \right) \alpha'_k e_t + \\
& \theta' A^{n-1} e_n) = (A+B) \left( A^2 \alpha'_3 e_3 + \sum_{t=3}^{n-1} A^{t-1} \alpha'_t e_t + \sum_{t=3}^{n-1} \sum_{k=3}^{t-1} (C_{k-1}^{k-2} A^{k-2} B \alpha_{t+2-k} + \right.
\end{aligned}$$

$$\begin{aligned}
& C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^t \alpha_{t+3-i_1} \alpha_{i_1+1-k} + C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^t \sum_{i_1=k+3}^{i_2} \alpha_{t+3-i_2} \alpha_{i_2+3-i_1} \alpha_{i_1-k} + \\
& + \dots + C_{k-1}^1 A B^{k-2} \sum_{i_{k-3}=2k-2}^t \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_2} \alpha_{t+3-i_{k-3}} \alpha_{i_{k-3}+3-i_{k-4}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+5-2k} + \\
& B^{k-1} \sum_{i_{k-2}=2k-1}^t \sum_{i_{k-3}=2k-1}^{i_{k-2}} \dots \sum_{i_1=2k-1}^{i_2} \alpha_{t+3-i_{k-2}} \alpha_{i_{k-2}+3-i_{k-3}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+4-2k} \alpha'_k e_t \Big) + \\
& (A+B) \left( \theta' A^{n-1} + \sum_{k=3}^{n-1} (C_{k-1}^{k-2} A^{k-2} B \alpha_{n+2-k} + C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^n \alpha_{n+3-i_1} \alpha_{i_1+1-k} + \right. \\
& C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^n \sum_{i_1=k+3}^{i_2} \alpha_{n+3-i_2} \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1-k} + \dots + \\
& C_{k-1}^1 A B^{k-2} \sum_{i_{k-3}=2k-2}^n \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_2} \alpha_{n+3-i_{k-3}} \alpha_{i_{k-3}+3-i_{k-4}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+5-2k} + \\
& \left. B^{k-1} \sum_{i_{k-2}=2k-1}^n \sum_{i_{k-3}=2k-1}^{i_{k-2}} \dots \sum_{i_1=2k-1}^{i_2} \alpha_{n+3-i_{k-2}} \alpha_{i_{k-2}+3-i_{k-3}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+4-2k} \alpha'_k \right) e_n = \\
& (A+B) \left( A^2 \alpha'_3 e_3 + \sum_{t=3}^{n-1} (A^{t-1} \alpha'_t e_t + \sum_{k=3}^{t-1} (C_{k-1}^{k-2} A^{k-2} B \alpha_{t+2-k} + C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^t \alpha_{t+3-i_1} \alpha_{i_1+1-k} + \right. \\
& C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^t \sum_{i_1=k+3}^{i_2} \alpha_{t+3-i_2} \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1-k} + \dots + \\
& C_{k-1}^1 A B^{k-2} \sum_{i_{k-3}=2k-2}^t \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_2} \alpha_{t+3-i_{k-3}} \alpha_{i_{k-3}+3-i_{k-4}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+5-2k} + \\
& \left. B^{k-1} \sum_{i_{k-2}=2k-1}^t \sum_{i_{k-3}=2k-1}^{i_{k-2}} \dots \sum_{i_1=2k-1}^{i_2} \alpha_{t+3-i_{k-2}} \alpha_{i_{k-2}+3-i_{k-3}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+4-2k} \alpha'_k e_t \right) + \\
& (\theta' A^{n-1} + \sum_{k=3}^{n-1} (C_{k-1}^{k-2} A^{k-2} B \alpha_{n+2-k} + C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^n \alpha_{n+3-i_1} \alpha_{i_1+1-k} + \\
& C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^n \sum_{i_1=k+3}^{i_2} \alpha_{n+3-i_2} \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1-k} + \dots + \\
& C_{k-1}^1 A B^{k-2} \sum_{i_{k-3}=2k-2}^n \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_2} \alpha_{n+3-i_{k-3}} \alpha_{i_{k-3}+3-i_{k-4}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+5-2k} + \\
& \left. B^{k-1} \sum_{i_{k-2}=2k-1}^n \sum_{i_{k-3}=2k-1}^{i_{k-2}} \dots \sum_{i_1=2k-1}^{i_2} \alpha_{n+3-i_{k-2}} \alpha_{i_{k-2}+3-i_{k-3}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+4-2k} \alpha'_k \right) e_n \Big).
\end{aligned}$$

Similar expression for  $[e'_1, e'_1]$  can be easily obtained by substitution in the expression  $[e'_0, e'_1]$  instead  $\theta'$  the coefficient  $\alpha'_n$ , namely

$$\begin{aligned}
[e'_1, e'_1] &= (A+B) \left( A^2 \alpha'_3 e_3 + \sum_{t=4}^n (A^{t-1} \alpha'_t + \sum_{k=3}^{t-1} (C_{k-1}^{k-2} A^{k-2} B \alpha_{t+2-k} + \right. \\
& C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^t \alpha_{t+3-i_1} \alpha_{i_1+1-k} + C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^t \sum_{i_1=k+3}^{i_2} \alpha_{t+3-i_2} \alpha_{i_2+3-i_1} \alpha_{i_1-k} + \dots + \\
& \left. + C_{k-1}^1 A B^{k-2} \sum_{i_{k-3}=2k-2}^t \sum_{i_{k-4}=2k-2}^{i_{k-3}} \dots \sum_{i_1=2k-2}^{i_2} \alpha_{t+3-i_{k-3}} \alpha_{i_{k-3}+3-i_{k-4}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+5-2k} + \right.
\end{aligned}$$

$$+B^{k-1} \left( \sum_{i_{k-2}=2k-1}^t \sum_{i_{k-3}=2k-1}^{i_{k-2}} \cdots \sum_{i_1=2k-1}^{i_2} \alpha_{t+3-i_{k-2}} \alpha_{i_{k-2}+3-i_{k-3}} \cdots \alpha_{i_2+3-i_1} \alpha_{i_1+4-2k} \right) \alpha'_k e_t \Big).$$

On the other hand, we have

$$[e'_0, e'_1] = [Ae_0 + Be_1, (A+B)e_1 + B(\theta - \alpha_n)e_{n-1}] = (A+B)^2 \sum_{t=3}^{n-1} \alpha_t e_t + (A+B)(A\theta + B\alpha_n)e_n,$$

$$[e'_1, e'_1] = [(A+B)e_1 + B(\theta - \alpha_n)e_{n-1}, (A+B)e_1 + B(\theta - \alpha_n)e_{n-1}] = (A+B) \sum_{t=3}^n \alpha_t e_t.$$

Comparing the coefficients of the basis elements  $e_t$  and keeping in mind that the coefficient  $A+B$  is different from zero, we get the restrictions, that were outlined in the first assertion of the theorem.

Using Corollary 4.3, the assertion b) of the theorem is proved by applying similar arguments.  $\square$

**Remark 4.5.** From Theorem 4.4, we have that  $\alpha'_k$  is a polynomial of the form  $P_k(A, B, \alpha_1, \alpha_2, \dots, \alpha_k, \alpha'_1, \alpha'_2, \dots, \alpha'_{k-1})$ , where parameters  $\alpha_1, \alpha_2, \dots, \alpha_k, \alpha'_1, \alpha'_2, \dots, \alpha'_{k-1}$  are given and coefficients  $A, B$  are unknown, but satisfy the condition  $A(A+B) \neq 0$ . And  $\beta'_k$  is also a polynomial of the form  $Q_k(A, B, D, \beta_1, \beta_2, \dots, \beta_k, \beta'_1, \beta'_2, \dots, \beta'_{k-1})$ , where parameters  $\beta_1, \beta_2, \dots, \beta_k, \beta'_1, \beta'_2, \dots, \beta'_{k-1}$  are given and coefficients  $A, B, D$  are unknown, but satisfy the condition  $AD \neq 0$ . Therefore, the finding of parameters  $\alpha'_k$  and  $\beta'_k$  are recursive procedures. Consequently, we conclude that in any given dimension the problem of the classification (up to an isomorphism) of complex filiform Leibniz algebras, which are obtained from the naturally graded filiform non-Lie algebras, is algorithmically solvable task.

#### REFERENCES

1. Akbaraliev B. B. *Classification of six-dimensional complex filiform Leibniz algebras.* (Russian) Uzbek Mat. Zh., 2, 2004, p. 17 - 22.
2. Ayupov Sh.A., Omirov B.A. *On some classes of nilpotent Leibniz algebras,* (Russian) Sibirsk. Mat. Zh., 42 (1), 2001, p. 18 - 29; translation in Siberian Math. J., 42 (1), 2001, p. 15 - 24.
3. Barnes Donald W. *Some theorems on Leibniz algebras.* Comm. Algebra, 39(7), 2011, p. 2463 - 2472.
4. Boza L., Fedriani E. M., Núñez J. *A new method for classifying complex filiform Lie algebras.* Appl. Math. and Computation, 121(2-3), 2003, p. 611 - 630.
5. Boza L., Fedriani E. M., Núñez J. *Complex filiform Lie algebras of dimension 11.* Appl. Math. and Computation, 141(2-3), 2003, p. 611 - 630.
6. Casas J. M., Ladra M. *Non-abelian tensor product of Leibniz algebras and an exact sequence in Leibniz homology.* Comm. Algebra, 31(9), 2003, p. 4639 - 4646.
7. Fialowski A., Mandal A., Mukherjee G. *Versal deformations of Leibniz algebras.* J. K-Theory, 3(2), 2009, p. 327 - 358.
8. Frabetti A. *Leibniz homology of dialgebras of matrices.* J. Pure and Appl. Algebra, 129, 1998, p. 123 - 141.
9. Gómez J.R., Jiménez-Merchan A., Khakimjanov Y. *Low-dimensional filiform Lie algebras.* J. Pure and Appl. Algebra, 130, 1998, p. 133 - 158.
10. Goze M., Khakimjanov Yu. *Nilpotent Lie algebras.* Kluwer Academic Publishers, 361, 1996, 336 p.
11. Loday J.-L. *Une version non commutative des algèbres de Lie: les algèbres de Leibniz.* L'Ens. Math., 39, 1993, p. 269 - 293.
12. Loday J.-L., Pirashvili T. *Universal enveloping algebras of Leibniz algebras and (co)homology.* Math. Ann., 296, 1993, p. 139 - 158.
13. Omirov, B. A. *Conjugacy of Cartan subalgebras of complex finite-dimensional Leibniz algebras.* J. Algebra, 302(2), 2006, p. 887 - 896.
14. Omirov B. A., Rakhimov I. S. *On Lie-like complex filiform Leibniz algebras.* Bull. Aust. Math. Soc., 79(3), 2009, p. 391 - 404.
15. Rikhsiboev I. M. *Classification of seven-dimensional complex filiform Leibniz algebras.* (Russian) Uzbek Mat. Zh., 3, 2004, p. 57 - 61.
16. Vergne M. *Vergne M. Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes.* Bull. Soc. Math. France, 98, 1970, p. 81 - 116.