(7) F assoc. triple system $u \in F$

define $ab = \langle aub \rangle$

• Then (F, ab) is an associative algebra (of second kind)

Proof $(ab)c = \langle\langle aub\rangle uc\rangle = \langle au\langle buc\rangle\rangle = a(bc)$

• Suppose $\langle auu\rangle = a = \langle uua\rangle$ for all $a \in F$. Then

$ua = \langle uua\rangle = a$ and $au = \langle auu\rangle = a$

so u is a unit of (F, ab)

Finally $(ab)^\# = \langle u\langle aub\rangle u\rangle = \langle ubu\rangle au = \langle ub\langle uau\rangle\rangle$

$\langle x\langle uzy\rangle v\rangle \qquad \langle xyz\rangle uv = \langle xy\langle zuv\rangle\rangle$

(recall that $\langle xy\langle zuv\rangle\rangle = \langle\langle xyz\rangle uv\rangle = \langle x\langle uzy\rangle v\rangle$)

$$(ab)^\# = \langle b^\# a u\rangle = \langle u b a^\#\rangle$$

$b^\# a^\# = \langle\langle ubu\rangle u\langle uau\rangle\rangle = \langle ub\langle uu\langle uau\rangle\rangle\rangle$

$\langle\langle xyz\rangle u\ v\rangle \qquad = \langle xy\langle zuv\rangle\rangle = \langle x\langle uzy\rangle v\rangle$

$= \langle ub\langle uua^\#\rangle\rangle = \langle uba^\#\rangle = (ab)^\#$

(aub)

8 L is a Lie algebra with (abstract) product denoted $[a,b]$. Define a triple product $[abc] = [[a,b],c]$. Show $(L, [abc])$ is a Lie triple system.

Proof The axioms of a Lie triple system are

(i) $[aab] = 0$

(ii) $[abc] + [bca] + [cab] = 0$

(iii) $[de[abc]] = [[dea]bc] + [a[deb]c] + [ab[dec]]$

The axioms of a Lie algebra are

(i') $[a,a] = 0$

(ii') $[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$

(ii'') $[c,[a,b]] = [a,[c,b]] + [[c,a],b]$ ← KEY POINT

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proof of (i) $[aab] = [[a,a],b] = [0,b] = 0$ by (i')

proof of (ii)

$[abc] + [bca] + [cab]$

$= [[a,b],c] + [[b,c],a] + [[c,a],b] = 0$ by (ii')

Proof of (iii)

$$[de[abc]] = [\underbrace{[d,e]}_{x}, [\underbrace{[a,b]}_{y}, \underbrace{c}_{z}]]$$

$$[y,[x,z]] + [[x,y],z]$$

$$= \underbrace{[\underbrace{[a,b]}_{y}, [\underbrace{[d,e]}_{x}, \underbrace{c}_{z}]]}_{[ab[dec]]} + \underbrace{[[[d,e],[a,b]], c]}$$

(on the right track)

$$-[c,[[d,e],[a,b]]] = -[[c,[d,e]],[a,b]]$$

$$[x,[y\ \ z]]$$

$$-[[d,e],[c,[a,b]]]$$

$$= \quad [c,[d,e]] = [[c,d],e] + [d,[c,e]]$$

$$[c,[a,b]] = [[c,a],b] + [a,[c,b]]$$

$$-[[[c,d],e] + [d,[c,e]], [a,b]]$$

$$-[[d,e], [[c,a],b] + [a,[c,b]]]$$

NO GOOD go back to

look at $\big[\underbrace{[d,e]}_{x},\ [\underset{y}{a},\underset{z}{b}]\big] = \big[[[d,e],a],\ \cancel{}\ b\big] + \big[a,\ [[d,e],b]\big]$

then

$$-\Big[c,\ \underbrace{\big[[d,e],[a,b]\big]}\Big] = -\Big[c,\ \underbrace{\big[[[d,e],a],b\big]}\Big] \mp \Big[c,\ \big[a,\ [[d,e],b]\big]\Big]$$

$$= +\big[[dea]\,b\,\cancel{}c\big] + \underbrace{\Big[\big[a,\ \underbrace{[[d,e],b]}_{[deb]}\big],\ c\Big]}_{[a[deb]c]}$$

BULLSYE!
on the right track

BULLSEYE!

This proves (iii) $\blacksquare$