DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **4**, Number **6**, December **2011**

doi:10.3934/dcdss.2011.4.1359

pp. **1359–1370**

DERIVATIONS IN POWER-ASSOCIATIVE ALGEBRAS

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Abstract. In this paper we investigate derivations of a commutative power-associative algebra. Particular cases of stable and partially stable algebras are inspected. Some attention is paid to the Jordan case. Further results are given. Especially, we show that the core of a n^{th}-order Bernstein algebra which is power-associative is a Jordan algebra.

1. **Preliminaries.** Weighted algebras are non-associative algebras modeling concrete biological situations. Their origin lies in the work of I.M.H. Etherington (see [4]) who gave a mathematical formulation of the Mendelian laws in the language of algebras. Since then, many authors have contributed to the study of these algebras from various points of view, and there is at present a substantial bibliography on the subject. In a recent survey [18] a general introduction to algebras arising in genetics can be found. Different classes of weighted algebras have been defined through the literature according to their genetic relevance. Some considerable work has been done on the characterization of derivations of weighted algebras (see [1], [6], [7], [8], [9]), including a nice note by P. Holgate [10] on the genetic meaning of derivations.

Let K be an infinite commutative field of characteristic $\neq 2, 3, 5$, and let A be a commutative K-algebra, not necessarily associative, and without unit element. We define inductively the *principal powers* (resp. *plenary powers*) of an element x of A by:

$$x^1 = x, x^{k+1} = xx^k \text{ (resp. } x^{[1]} = x, x^{[k+1]} = x^{[k]}x^{[k]}) \ (k \geq 1).$$

We will say that the algebra A is power-associative if any subalgebra, generated by an element of A is associative. It is well known that if K has characteristic $\neq 2, 3, 5$, then A is power-associative if and only if $x^2x^2 = x^4$ for any x in A.

2000 *Mathematics Subject Classification.* Primary: 17A05, 17D92; Secondary: 17A36.

Key words and phrases. Derivations, Power-associative algebras, Peirce decomposition, Idempotent, Train algebras, Jordan Algebras, Stable algebras.

We will say that A is a *Jordan algebra* if the identity $x^2(yx) = (x^2y)x$ is satisfied for all elements x and y in A. It is known that Jordan algebras are power-associative.

If $\omega : A \to K$ is a nonzero homomorphism, then the pair (A, ω) will be called a weighted algebra over K and ω its weight function, or shortly its weight. A weighted algebra (A, ω) is said to be a *train algebra* of rank r if there exist $\gamma_1, \ldots, \gamma_{r-1} \in K$ such that

$$x^r + \gamma_1\omega(x)x^{r-1} + \cdots + \gamma_{r-1}\omega(x)^{r-1}x = 0$$

for all x in A, r being the smallest integer satisfying this identity. The rank of A determines then, in a unique way, the coefficients $\gamma_1, \ldots, \gamma_{r-1}$.

For any polynomial $Q(\lambda) = b_0\lambda^r + \cdots + b_{r-1}\lambda$, we define $Q(x) = b_0x^r + \cdots + b_{r-1}x$ for each $x \in \Delta = \{x \in A \mid \omega(x) = 1\}$. A polynomial $P(X)$, with zero constant term, is a *train polynomial* if $P(x) = 0$, $\forall x \in \Delta$. Such a polynomial admits 0 and 1 as roots. Moreover, in a convenient extension of the field K, the polynomial $P(X)$ is factorized in the form $P(X) = cX(X-1)(X-\lambda_1)\cdots(X-\lambda_{r-2})$, where c is a nonzero constant that can be chosen equal to 1. We then get a factorization of $P(X)$ as $P(X) = X(X-1)(X-\lambda_1)\cdots(X-\lambda_{r-2})$ and, formally, we can write $P(x) = x(x-1)(x-\lambda_1)\cdots(x-\lambda_{r-2})$ for all $x \in \Delta$, or equivalently, $P(x) = x(x-\omega(x))(x-\lambda_1\omega(x))\cdots(x-\lambda_{r-2}\omega(x))$ for all $x \in A$ ([23], page 35). The set of train polynomials is an ideal of the ring $K[X]$. The monic generator of this ideal is called the minimal train polynomial of the algebra A, and its degree is the rank of the algebra, while the corresponding equation is its *rank equation*.

Let A be a power-associative algebra which admits a nonzero idempotent e. It is well known that A has a Peirce decomposition given by $A = A_1 \oplus A_{1/2} \oplus A_0$, where $A_\lambda = \{x \in A \mid ex = \lambda x\}$. The Peirce components A_λ multiply according to

$$\begin{aligned} A_\lambda A_\lambda \subseteq A_\lambda,\ A_\lambda A_{1/2} \subseteq A_{1/2} \oplus A_{1-\lambda}\ (\lambda = 0, 1),& \\ A_{1/2}A_{1/2} \subseteq A_0 \oplus A_1,\ A_0A_1 = 0.& \end{aligned} \tag{1}$$

Following the notation of A.A. Albert [2], we define for any $x_1 \in A_1$ the following mappings: $S_{1/2}(x_1) : A_{1/2} \to A_{1/2},\ x_{1/2} \mapsto (x_1x_{1/2})_{1/2}$ and $S_0(x_1) : A_{1/2} \to A_0,\ x_{1/2} \mapsto (x_1x_{1/2})_0$. In the same way, for all $x_0 \in A_0$, the two following mappings are considered: $T_{1/2}(x_0) : A_{1/2} \to A_{1/2},\ x_{1/2} \mapsto (x_0x_{1/2})_{1/2}$ and $T_1(x_0) : A_{1/2} \to A_1,\ x_{1/2} \mapsto (x_0x_{1/2})_1$. We will make frequent use of some of the results of Albert and Kokoris on commutative power-associative algebras; namely, results (5), (6), (7), (38) of [2], ((2) and (3)) of [13]. We state them as

$$\begin{aligned} S_{1/2}(x_1y_1) &= S_{1/2}(x_1)S_{1/2}(y_1) + S_{1/2}(y_1)S_{1/2}(x_1), \\ 1/2S_0(x_1y_1) &= S_0(x_1)S_{1/2}(y_1) + S_0(y_1)S_{1/2}(x_1); \end{aligned} \tag{2}$$

$$\begin{aligned} T_{1/2}(x_0y_0) &= T_{1/2}(x_0)T_{1/2}(y_0) + T_{1/2}(y_0)T_{1/2}(x_0), \\ 1/2T_1(x_0y_0) &= T_1(x_0)T_{1/2}(y_0) + T_1(y_0)T_{1/2}(x_0); \end{aligned} \tag{3}$$

$$T_{1/2}(x_0)S_{1/2}(y_1) = S_{1/2}(y_1)T_{1/2}(x_0); \tag{4}$$

$$S_{1/2}(w_1)a_{1/2} = T_{1/2}(w_0)a_{1/2},\ \text{where } a_{1/2}^2 = w_1 + w_0; \tag{5}$$

$$\begin{aligned} x_\lambda(x_{1/2}y_{1/2}) = {}& [x_{1/2}(x_\lambda y_{1/2})_{1/2} + y_{1/2}(x_\lambda x_{1/2})_{1/2}]_\lambda \\ & + 1/2[x_{1/2}(x_\lambda y_{1/2})_{1-\lambda} + y_{1/2}(x_\lambda x_{1/2})_{1-\lambda}]_\lambda \quad (\lambda = 0, 1); \end{aligned} \tag{6}$$

where $x_0, y_0 \in A_0,\ x_1, y_1 \in A_1$ and $a_{1/2} \in A_{1/2}$.

We will say that A is *e-stable* if $A_\lambda A_{1/2} \subseteq A_{1/2}$ $(\lambda = 0, 1)$ and that A is *stable* if A is e-stable for every idempotent e. In particular, Jordan algebras are stable.

Let A be an algebra and let d be an endomorphism of A. We will say that d is *a derivation* of A if for any x, $y \in A$, $d(xy) = d(x)y + xd(y)$. The set $Der_K(A)$ of all derivations of A is a subalgebra of the Lie algebra $End_K(A)^-$, where $End_K(A)$ is the associative algebra of endomorphisms of A.

An ideal I of A is said to be *characteristic* if $d(I) \subseteq I$, for any derivation d of A.

The following lemma is just a translation of relations (2) to (7) when the algebra is e-stable.

Lemma 1.1. *Let $A = A_1 \oplus A_{1/2} \oplus A_0$. If A is e-stable, then for all $x_\lambda, y_\lambda \in A_\lambda$ ($\lambda = 0, 1$) and $a_{1/2} \in A_{1/2}$, we have:*

(*i*) $(x_1y_1)a_{1/2} = x_1(y_1a_{1/2}) + y_1(x_1a_{1/2})$;

(*ii*) $(x_0y_0)a_{1/2} = x_0(y_0a_{1/2}) + y_0(x_0a_{1/2})$;

(*iii*) $x_0(y_1a_{1/2}) = y_1(x_0a_{1/2})$;

(*iv*) $x_0(x_{1/2}y_{1/2}) = [x_{1/2}(y_{1/2}x_0)]_0 + [y_{1/2}(x_{1/2}x_0)]_0$,
$x_1(x_{1/2}y_{1/2}) = [x_{1/2}(y_{1/2}x_1)]_1 + [y_{1/2}(x_{1/2}x_1)]_1$;

(*v*) $(a^2_{1/2})_1a_{1/2} = (a^2_{1/2})_0a_{1/2} = \frac{1}{2}a^3_{1/2}$.

2. Characterization of derivations.

Theorem 2.1. *Let $A = A_1 \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of a commutative power-associative algebra and $d : A \longrightarrow A$ a derivation. Then d is determined by a unique quadruplet $(d(e), f_d, g_d, h_d)$ satisfying the following conditions:*

(*i*) $d(e) \in A_{1/2}$;

(*ii*) $d(x_1) = h_d(x_1) + (d(e)x_1)_0 + 2(d(e)x_1)_{1/2}$;

(*iii*) $d(x_{1/2}) = f_d(x_{1/2}) + 2(d(e)x_{1/2})_0 - 2(d(e)x_{1/2})_1$;

(*iv*) $d(x_0) = g_d(x_0) - (d(e)x_0)_1 - 2(d(e)x_0)_{1/2}$;

The endomorphisms $f_d \in End_K(A_{1/2})$, $g_d \in End_K(A_0)$ and $h_d \in End_K(A_1)$ verify:

(*v*) $h_d(x_1y_1) = h_d(x_1)y_1 + x_1h_d(y_1)$;

(*vi*) $g_d(x_0y_0) = g_d(x_0)y_0 + x_0g_d(y_0)$;

(*vii*)
$$f_d((x_1x_{1/2})_{1/2}) = [x_1f_d(x_{1/2}) + h_d(x_1)x_{1/2}]_{1/2} + [x_{1/2}(x_1d(e))_0]_{1/2} + 2[d(e)(x_1x_{1/2})_0]_{1/2},$$
$$g_d((x_1x_{1/2})_0) = [x_1f_d(x_{1/2}) + h_d(x_1)x_{1/2}]_0 + 2[x_{1/2}(x_1d(e))_{1/2} - d(e)(x_1x_{1/2})_{1/2}]_0;$$

(*viii*)
$$f_d((x_0x_{1/2})_{1/2}) = [x_0f_d(x_{1/2}) + g_d(x_0)x_{1/2}]_{1/2} - [x_{1/2}(x_0d(e))_1]_{1/2} - 2[d(e)(x_0x_{1/2})_1]_{1/2},$$
$$h_d((x_0x_{1/2})_1) = [x_0f_d(x_{1/2}) + g_d(x_0)x_{1/2}]_1 - 2[x_{1/2}(x_0d(e))_{1/2} - d(e)(x_0x_{1/2})_{1/2}]_1;$$

(*ix*)
$$g_d((x_{1/2}y_{1/2})_0) = [f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2})]_0 - [d(e)(x_{1/2}y_{1/2})_1]_0 - 2[x_{1/2}(d(e)y_{1/2})_1 + y_{1/2}(d(e)x_{1/2})_1]_0,$$
$$h_d((x_{1/2}y_{1/2})_1) = [f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2})]_1 + 2[x_{1/2}(d(e)y_{1/2})_0 + y_{1/2}(d(e)x_{1/2})_0]_1 + [d(e)(x_{1/2}y_{1/2})_0]_1.$$

Proof. Let d be a derivation of A. We have $d(e) = d(e^2) = 2ed(e)$, so $d(e) \in A_{1/2}$. By setting $d(x_1) = z_1 + z_{1/2} + z_0$ and deriving $ex_1 = x_1$, we have $d(x_1) = d(e)x_1 + ed(x_1) = z_1 + \frac{1}{2}z_{1/2} + d(e)x_1$, and therefore $z_{1/2} = 2(d(e)x_1)_{1/2}$ and $z_0 = (d(e)x_1)_0$. We check that the mapping $h_d : x_1 \mapsto z_1$ is linear, hence (*ii*) is valid. By setting $d(x_{1/2}) = r_1 + r_{1/2} + r_0$ and deriving $ex_{1/2} = \frac{1}{2}x_{1/2}$, we have $\frac{1}{2}d(x_{1/2}) = d(e)x_{1/2} + ed(x_{1/2}) = r_1 + \frac{1}{2}r_{1/2} + d(e)x_{1/2} = \frac{1}{2}r_1 + \frac{1}{2}r_{1/2} + \frac{1}{2}r_0$, hence $r_1 = -2(d(e)x_{1/2})_1$ and $r_0 = 2(d(e)x_{1/2})_0$. It is a simple matter to see that the mapping $f_d : x_{1/2} \mapsto r_{1/2}$ is linear, and so (*iii*) holds. Now, for $d(x_0) = t_1 + t_{1/2} + t_0$, since $ex_0 = 0$, we obtain $0 = d(e)x_0 + ed(x_0) = t_1 + \frac{1}{2}t_{1/2} + d(e)x_0$ and thus $t_{1/2} = -2(d(e)x_0)_{1/2}$ and $t_1 = -(d(e)x_0)_1$. In addition, the mapping $g_d : x_0 \mapsto t_0$ is linear, which proves (iv).

Using identity (2), we have

$$\begin{aligned} d(x_1y_1) &= x_1[h_d(y_1) + (y_1d(e))_0 + 2(y_1d(e))_{1/2}] \\ &\quad + [h_d(x_1) + (x_1d(e))_0 + 2(x_1d(e))_{1/2}]y_1 \\ &= x_1h_d(y_1) + h_d(x_1)y_1 + 2x_1(y_1d(e))_{1/2} + 2y_1(x_1d(e))_{1/2} \\ &= h_d(x_1y_1) + d(e)(x_1y_1)_0 + 2[d(e)(x_1y_1)]_{1/2}, \end{aligned}$$

and thus (v) follows. By symmetry, (vi) follows from identity (3). Now,

$$\begin{aligned} d(x_1x_{1/2}) &= x_1d(x_{1/2}) + d(x_1)x_{1/2} = x_1[f_d(x_{1/2}) + 2(d(e)x_{1/2})_0 - 2(d(e)x_{1/2})_1] \\ &\quad + [h_d(x_1) + d(e)(x_1d(e))_0 + 2d(e)(x_1d(e))_{1/2}]x_{1/2} \\ &= x_1f_d(x_{1/2}) + h_d(x_1)x_{1/2} - 2x_1(d(e)x_{1/2})_1 \\ &\quad + x_{1/2}(x_1d(e))_0 + 2x_{1/2}(x_1d(e))_{1/2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} d(x_1x_{1/2}) &= d((x_1x_{1/2})_{1/2}) + d((x_1x_{1/2})_0) \\ &= f_d((x_1x_{1/2})_{1/2}) + 2[d(e)(x_1x_{1/2})_{1/2}]_0 - 2[d(e)(x_1x_{1/2})_{1/2}]_1 \\ &\quad + g_d((x_1x_{1/2})_0) - [d(e)(x_1x_{1/2})_0)]_1 - 2[d(e)(x_1x_{1/2})_0]_{1/2}. \end{aligned}$$

By identifying the two expressions of $d(x_1x_{1/2})$, we obtain (vii), since

$$\begin{aligned} x_1(d(e)x_{1/2})_1 &= [x_{1/2}(x_1d(e))_{1/2} + d(e)(x_1x_{1/2})_{1/2}]_1 \\ &\quad + \frac{1}{2}[x_{1/2}(x_1d(e))_0 + d(e)(x_1x_{1/2})_0]_1 \end{aligned}$$

by (6). By symmetry, calculation of $d(x_0x_{1/2})$ lead us to $(viii)$. Finally, we have

$$\begin{aligned} d(x_{1/2}y_{1/2}) &= x_{1/2}[f_d(y_{1/2}) + 2(d(e)y_{1/2})_0 - 2(d(e)y_{1/2})_1] \\ &\quad + [f_d(x_{1/2}) + 2(d(e)x_{1/2})_0 - 2(d(e)x_{1/2})_1]y_{1/2} \\ &= [x_{1/2}f_d(y_{1/2}) + f_d(x_{1/2})y_{1/2}]_0 + [x_{1/2}f_d(y_{1/2}) + f_d(x_{1/2})y_{1/2}]_1 \\ &\quad + 2[x_{1/2}(d(e)y_{1/2})_0]_{1/2} + 2[y_{1/2}(d(e)x_{1/2})_0]_{1/2} + 2[x_{1/2}(d(e)y_{1/2})_0]_1 \\ &\quad - 2[x_{1/2}(d(e)y_{1/2})_1]_0 - 2[y_{1/2}(d(e)x_{1/2})_1]_0 + 2[y_{1/2}(d(e)x_{1/2})_0]_1 \\ &\quad - 2[y_{1/2}(d(e)x_{1/2})_1]_{1/2} - 2[x_{1/2}(d(e)y_{1/2})_1]_{1/2} \\ &= [x_{1/2}f_d(y_{1/2}) + f_d(x_{1/2})y_{1/2}]_0 + [x_{1/2}f_d(y_{1/2}) + f_d(x_{1/2})y_{1/2}]_1 \\ &\quad + 2[d(e)(x_{1/2}y_{1/2})_1]_{1/2} - 2[d(e)(x_{1/2}y_{1/2})_0]_{1/2} \\ &\quad + 2[x_{1/2}(d(e)y_{1/2})_0 + y_{1/2}(d(e)x_{1/2})_0]_1 \\ &\quad - 2[x_{1/2}(d(e)y_{1/2})_1 + y_{1/2}(d(e)x_{1/2})_1]_0. \end{aligned}$$

Since

$$\begin{aligned} [x_{1/2}(d(e)y_{1/2})_1]_{1/2} &+ [y_{1/2}(d(e)x_{1/2})_1]_{1/2} - [x_{1/2}(d(e)y_{1/2})_0]_{1/2} \\ &- [y_{1/2}(d(e)x_{1/2})_0]_{1/2} \\ &= [d(e)(x_{1/2}y_{1/2})_0]_{1/2} - [d(e)(x_{1/2}y_{1/2})_1]_{1/2} \end{aligned}$$

by linearizing the identity $S_{1/2}((a_{1/2}^2)_1)a_{1/2} = T_{1/2}((a_{1/2}^2)_0)a_{1/2}$ (see (5)).

Furthermore

$$\begin{aligned} d(x_{1/2}y_{1/2}) = g_d((x_{1/2}y_{1/2})_0) &- [d(e)(x_{1/2}y_{1/2})_0]_1 - 2[d(e)(x_{1/2}y_{1/2})_0]_{1/2} \\ &+ h_d((x_{1/2}y_{1/2})_1) + [d(e)(x_{1/2}y_{1/2})_1]_0 + 2[d(e)(x_{1/2}y_{1/2})_1]_{1/2}. \end{aligned}$$

By identifying the two expressions of $d(x_{1/2}y_{1/2})$, the identities (ix) follow. Conversely, let $d : A \longrightarrow A$ be a linear mapping obeying conditions (i) to (ix) of Theorem 2.1. For $x = x_1 + x_{1/2} + x_0$ and $y = y_1 + y_{1/2} + y_0$ in A, since $xy = x_1y_1 + x_1y_{1/2} + x_{1/2}y_1 + x_{1/2}y_{1/2} + x_{1/2}y_0 + x_0y_{1/2} + x_0y_0$, we check that $d(x_\lambda y_\mu) = x_\lambda d(y_\mu) + d(x_\lambda)y_\mu$ $(\lambda, \mu = 1, 1/2, 0)$. These equalities are satisfied because the conditions $(i) - (ix)$ rise from the above computations. □

Corollary 1. *The subspaces $J_\lambda = \{x_\lambda \in A_\lambda | x_\lambda A_{1/2} = 0\}$ and $J = J_1 + J_0$ are characteristic ideals satisfying $J_\lambda(A_{1/2}A_{1/2}) = J(A_{1/2}A_{1/2}) = 0$ $(\lambda = 1, 0)$. Moreover, if the algebra A is e-stable, then A_λ/J_λ and $(A_1 \oplus A_0)/J$ are Jordan algebras.*

Proof. The fact that J_λ, for $\lambda = 1, 0$ and J are ideals follow from Lemma 1 of [2] and the comments which follow its proof. By symmetry, it will be sufficient to prove the result for $\lambda = 1$. Let $x_1 \in J_1$, according to (ii) of Theorem 2.1, $d(x_1) = h_d(x_1)$ while (vii) gives $0 = f_d((x_1x_{1/2})_{1/2}) = (h_d(x_1)x_{1/2})_{1/2}$ and $0 = g_d((x_1x_{1/2})_0) = (h_d(x_1)x_{1/2})_0$, for all $x_{1/2} \in A_{1/2}$. Thus $h_d(x_1) \in J_1$, i.e. $d(J_1) \subseteq J_1$. Similarly, we show that $d(J_0) \subseteq J_0$. By (6), we have $x_\lambda(x_{1/2}y_{1/2}) = 0$. Hence $J_\lambda(x_{1/2}y_{1/2}) = 0$, for $\lambda = 1, 0$. Then $J(x_{1/2}y_{1/2}) = J_1(x_{1/2}y_{1/2}) + J_0(x_{1/2}y_{1/2}) = 0$. Now, if we suppose that A is e-stable, then J_0 and J_1 coincide respectively with $\ker(x_0 \mapsto 2T_{1/2}(x_0))$ and $\ker(x_1 \mapsto 2S_{1/2}(x_1))$. Lemma 1 of [2] says that A_λ/J_λ is a special Jordan algebra. Let us consider the homomorphism of algebras $A_1 \oplus A_0 \longrightarrow A_1/J_1 \times A_0/J_0$, $(x_1, x_0) \mapsto (\overline{x_1}, \overline{x_0})$, necessarily surjective, where $\overline{x_1}$ and $\overline{x_0}$ are respectively the coset of x_1 and x_0. Its kernel being J, we have the isomorphism $(A_1 \oplus A_0)/J \simeq A_1/J_1 \times A_0/J_0 \simeq A_1/J_1 \oplus A_0/J_0$. Thus, $A_1/J_1 \oplus A_0/J_0$ is a Jordan algebra. □

Let d and d' be two derivations. Since $[d, d']$ is a derivation, it is natural to look for the quadruplet associated to it.

Theorem 2.2. *Let d and d' be two derivations of A. Then, the quadruplet associated to $[d, d']$ is $([d, d'](e), f_{[d,d']}, g_{[d,d']}, h_{[d,d']})$ satisfying:*

(i) $[d, d'](e) = f_d(d'(e)) - f_{d'}(d(e))$;

(ii) $h_{[d,d']}(x_1) = [h_d, h_{d'}](x_1) - [d(e)(d'(e)x_1)_0 - d'(e)(d(e)x_1)_0]_1 - 4[d(e)(d'(e)x_1)_{1/2} - d'(e)(d(e)x_1)_{1/2}]_1$;

(iii) $f_{[d,d']}(x_{1/2}) = [f_d, f_{d'}](x_{1/2}) - 4\big([R_{d(e)}, R_{d'(e)}](x_{1/2})\big)_{1/2}$;

(iv) $g_{[d,d']}(x_0) = [g_d, g_{d'}](x_0) - [d(e)(d'(e)x_0)_1 - d'(e)(d(e)x_0)_1]_0 - 4[d(e)(d'(e)x_0)_{1/2} - d'(e)(d(e)x_0)_{1/2}]_0$.

Proof. First, the equality $dd'(e) = f_d(d'(e)) + 2(d(e)d'(e))_0 - 2(d(e)d'(e))_1$ gives (i). Using the identities of Theorem 2.1, we calculate $dd'(x_1)$ (resp. $d'd(x_1)$) by replacing $g_d((d'(e)x_1)_0)$ and $f_d((d'(e)x_1)_{1/2})$ (resp. $g_{d'}((d(e)x_1)_0)$ and $f_{d'}((d(e)x_1)_{1/2})$) by their expressions in (vii). After simplification, we get

$$\begin{aligned} [d, d'](x_1) = [h_d, h_{d'}](x_1) &+ \big([d, d'](e)x_1\big)_0 + 2\big([d, d'](e)x_1\big)_{1/2} \\ &- [d(e)(d'(e)x_1)_0 - d'(e)(d(e)x_1)_0]_1 \\ &- 4[d(e)(d'(e)x_1)_{1/2} - d'(e)(d(e)x_1)_{1/2}]_1. \end{aligned}$$

Hence (ii) is established, and by symmetry, we obtain (iv). Finally, the calculations of $dd'(x_{1/2})$ and $d'd(x_{1/2})$ give, after simplification,

$$\begin{aligned}[d,d'](x_{1/2}) = [f_d, f_{d'}](x_{1/2}) &+ 2\big([d,d'](e)x_{1/2}\big)_0 - 2\big([d,d'](e)x_{1/2}\big)_1\\ &- 4[d(e)(d'(e)x_{1/2})_0 - d'(e)(d(e)x_{1/2})_0]_{1/2}\\ &- 4[d(e)(d'(e)x_{1/2})_1 - d'(e)(d(e)x_{1/2})_1]_{1/2}.\end{aligned}$$

We deduce from above equality

$$f_{[d,d']}(x_{1/2}) = [f_d, f_{d'}](x_{1/2}) - 4\big([R_{d(e)}, R_{d'(e)}](x_{1/2})\big)_{1/2}.$$

Hence (iii) is satisfied. □

Example 1. ([2], p. 504) Let A be a commutative power-associative algebra whose multiplication table in the basis $\{e, e_1, e_2, e_3\}$ is given by $e^2 = e$, $e_1^2 = e_2^2 = e_3^2 = ee_3 = e_1e_3 = e_2e_3 = 0$, $ee_1 = e_1$, $ee_2 = \frac{1}{2}e_2$, $e_1e_2 = e_3$. We have $A_1 = Ke + Ke_1$, $A_{1/2} = Ke_2$, $A_0 = Ke_3$, $A_1A_{1/2} \subset A_0$. Let d be a derivation of A. For $d(e) = \alpha e_2$, $f_d(e_2) = \beta e_2$, $g_d(e_3) = \gamma e_3$, $h_d(e_1) = ae + be_1$, we have $0 = f_d((e_1e_2)_{1/2}) = (\beta e_3 + \frac{1}{2}ae_2 + be_3)_{1/2}$. Hence, $a = 0$. Now, the equality $g_d(e_3) = g_d(e_1e_2) = (\beta e_3 + be_3)_0 = (\beta + b)e_3$ gives $\gamma = \beta + b$. Consequently, $d(e) = \alpha e_2$, $d(e_1) = be_1$, $d(e_2) = \beta e_2$ and $d(e_3) = (\beta + b)e_3$. It is seen that the Lie algebra of derivations $Der_K(A) \simeq K \times K \times K$ with $[(\alpha, \beta, b), (\alpha', \beta', b')] = (\beta\alpha' - \beta'\alpha, 0, 0)$.

3. **Partially stable algebras.** Let $A = A_1 \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of a commutative power-associative algebra, which is partially stable. Then $A_\lambda A_{1/2} \subseteq A_{1/2}$ $(\lambda = 1, 0)$.

Proposition 1. *Every derivation d of A is determined by a unique quadruplet $(d(e), f_d, g_d, h_d)$ satisfying the following conditions:*

(i) $d(e) \in A_{1/2}$;

(ii) $d(x_1) = h_d(x_1) + 2d(e)x_1$;

(iii) $d(x_{1/2}) = f_d(x_{1/2}) + 2(d(e)x_{1/2})_0 - 2(d(e)x_{1/2})_1$;

(iv) $d(x_0) = g_d(x_0) - 2d(e)x_0$;

(v) *The endomorphisms g_d and h_d are respectively derivations of A_0 and A_1;*

(vi) $f_d(x_1x_{1/2}) = x_1f_d(x_{1/2}) + h_d(x_1)x_{1/2}$,
$f_d(x_0x_{1/2}) = x_0f_d(x_{1/2}) + g_d(x_0)x_{1/2}$;

(vii) $g_d((x_{1/2}y_{1/2})_0) = [f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2})]_0$,
$h_d((x_{1/2}y_{1/2})_1) = [f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2})]_1$;

$(viii)$ $[d(e)(x_\lambda x_{1/2})]_{1-\lambda} = [x_{1/2}(x_\lambda d(e))]_{1-\lambda}$ $(\lambda = 0, 1)$;

Moreover, if d and d' are two derivations of A, then the quadruplet $([d,d'](e), h_{[d,d']}, f_{[d,d']}, g_{[d,d']})$ is defined by :

(ix) $[d,d'](e) = f_d(d'(e)) - f_{d'}(d(e))$;

(x) $h_{[d,d']}(x_1) = [h_d, h_{d'}](x_1) - 4\big([R_{d(e)}, R_{d'(e)}](x_1)\big)_1$;

(xi) $f_{[d,d']}(x_{1/2}) = [f_d, f_{d'}](x_{1/2}) - 4[R_{d(e)}, R_{d'(e)}](x_{1/2})$;

(xii) $g_{[d,d']}(x_0) = [g_d, g_{d'}](x_0) - 4\big([R_{d(e)}, R_{d'(e)}](x_0)\big)_0$.

Proof. The result is deduced from Theorems 2.1 and 2.2 involving the fact that $A_\lambda A_{1/2} \subseteq A_{1/2}$. □

Example 2. Let $A =< e, u_1, u_2, u_3, u_4, w, v >$ be a commutative power-associative algebra of dimension 7 with multiplication table given by $e^2 = e$, $eu_i = \frac{1}{2}u_i$ $(i = 1, \dots, 4)$, $ew = w$, $u_1v = u_1w = u_3$, $u_2v = u_2w = u_4$, $u_2u_3 = -u_1u_4 = v + w$, other products are zero. Let d be a derivation of A. By using the equalities

$[d(e)(vu_1)]_1 = [u_1(d(e)v)]_1$ and $[d(e)(vu_2)]_1 = [u_2(d(e)v)]_1$, with $d(e) = \sum_{i=1}^4 a_i u_i$ and $f_d(u_j) = \sum_{i=1}^4 \alpha_{ij} u_i$, we have respectively $a_2 = 0$ and $a_1 = 0$. Since $0 = h_d(u_i^2) = 2[u_i f_d(u_i)]_1$, we obtain $\alpha_{41} = \alpha_{32} = \alpha_{23} = \alpha_{14} = 0$. The equalities $u_1u_2 = u_1u_3 = u_2u_4 = u_3u_4 = 0$ give $\alpha_{42} = \alpha_{31}$, $\alpha_{43} = \alpha_{21}$, $\alpha_{34} = \alpha_{12}$ and $\alpha_{24} = \alpha_{13}$. We have $h_d(w) = h_d((u_2u_3)_1) = (\alpha_{33} + \alpha_{22})w$, $g_d(v) = g_d((u_2u_3)_0) = (\alpha_{33} + \alpha_{22})v$, $f_d(u_3) = f_d(u_1v) = f_d(u_1)v + u_1 g_d(v) = (\alpha_{11} + \alpha_{22} + \alpha_{33})u_3 + \alpha_{21}u_4$ and $f_d(u_4) = f_d(u_2v) = \alpha_{12}u_3 + (2\alpha_{22} + \alpha_{33})u_4$. Hence $\alpha_{13} = 0$, $\alpha_{33} = \alpha_{11} + \alpha_{22} + \alpha_{33}$, and then, $\alpha_{22} = -\alpha_{11}$. Since $-h_d(w) = h_d((u_1u_4)_1) = -(\alpha_{44} + \alpha_{11})w$, i.e. $\alpha_{44} + \alpha_{11} = \alpha_{22} + \alpha_{33}$, then $\alpha_{44} = \alpha_{33} - 2\alpha_{11}$. Thus, $Der_K(A) \simeq K^2 \times K^5$, isomorphism of Lie algebras.

Corollary 2. *Assume that the algebra A satisfies*

$$[x_{1/2}(x_\lambda y_{1/2})]_{1-\lambda} = [y_{1/2}(x_\lambda x_{1/2})]_{1-\lambda},$$

for all $x_\lambda \in A_\lambda$ $(\lambda = 1, 0)$, $x_{1/2}$, $y_{1/2} \in A_{1/2}$. Then, each derivation d of A is determined by a unique quadruplet $(d(e), f_d, g_d, h_d)$ satisfying the following conditions:

(i) $d(e) \in A_{1/2}$;
(ii) $d(x_1) = h_d(x_1) + 2d(e)x_1$;
(iii) $d(x_{1/2}) = f_d(x_{1/2}) + 2(d(e)x_{1/2})_0 - 2(d(e)x_{1/2})_1$;
(iv) $d(x_0) = g_d(x_0) - 2d(e)x_0$;
(v) *The endomorphisms g_d and h_d are respectively derivations of A_0 and A_1;*
(vi) $f_d(x_1x_{1/2}) = x_1 f_d(x_{1/2}) + h_d(x_1)x_{1/2}$,
$f_d(x_0x_{1/2}) = x_0 f_d(x_{1/2}) + g_d(x_0)x_{1/2}$;
(vii) $g_d((x_{1/2}y_{1/2})_0) = [f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2})]_0$,
$h_d((x_{1/2}y_{1/2})_1) = [f_d(x_{1/2})y_{1/2} + x_{1/2}f_d(y_{1/2})]_1$.

Corollary 3. *Let A be a commutative Jordan algebra. Then, all derivations of A are determined by Corollary 2.*

Proof. A Jordan algebra is a power-associative algebra and is stable. It is sufficient to observe that any Jordan algebra verify the assumptions of Corollary 2 ([2], Lemma 7). □

4. **Train algebras which are power-associative.** In [3] it is shown that any nonzero idempotent of a train algebra (A, ω) of rank n, which is power associative, is at the same time principal and absolutely primitive and that the train equation is of the form $x^{n-t}(x - \omega(x))^t = 0$, or equivalently,

$$m(x) = x^n + \gamma_1\omega(x)x^{n-1} + \cdots + \gamma_t\omega(x)^t x^{n-t} = 0,$$

with $\gamma_k = (-1)^k \binom{t}{k}$, $k = 1, 2, \ldots, t$. The pair of integers $(n-t, t)$ is called *the presentation* of the algebra $(1 \le t \le n-1)$.

Let $A = A_1 \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of such an algebra, with respect to the idempotent $e \neq 0$. Then $A_1 = Ke \oplus \overline{A_1}$ and $\mathfrak{R} = \ker(\omega) = \overline{A_1} \oplus A_{1/2} \oplus A_0$ is the radical (maximal nilideal) of A, where $\overline{A_1} = A_1 \cap \mathfrak{R}$. Moreover, for any idempotent e, we have $x_0^{n-t} = 0$ and $x_1^t = 0$, for any $x_0 \in A_0$ and $x_1 \in \overline{A_1}$ ([3], Theorem 2.3).

Proposition 2. *Let $A = Ke \oplus \overline{A_1} \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of a power-associative train algebra relative to the idempotent $e \neq 0$. Suppose that $\mathfrak{R}^2 = 0$. Then the mapping $\varphi : Der_K(A) \longrightarrow L_K(A) = A_{1/2} \times End_K(A_{1/2}) \times$*

$End_K(\mathfrak{R}_0) \times End_K(\mathfrak{R}_1)$ such that $\varphi(d) = (d(e), f_d, g_d, h_d)$ is an isomorphism of Lie algebras, where the multiplication in $L_K(A)$ is defined by

$$[(u, f, g, h), (u', f', g', h')] = (f(u') - f'(u), [f, f'], [g, g'], [h, h'])$$

Proof. By Theorem 2.1, it is readily seen that the mapping φ is a monomorphism of vector spaces. Since $\mathfrak{R}^2 = 0$, Theorem 2.2 shows that φ is a homomorphism of Lie algebras. Let (u, f, g, h) be in $L_K(A)$. We define a linear mapping $d : A \to A$ by $d(x) = \alpha u + f(x_{1/2}) + g(x_0) + h(x_1)$, where $x = \alpha e + x_{1/2} + x_0 + x_1$ $(\alpha \in K)$. Then d is obviously a derivation of A, and therefore, φ is an epimorphism. □

An algebra is said to be *strictly train* if, for any extension $K \longrightarrow L$, where L is a commutative field, the algebra $A_L = A \otimes_K L$ over L is a train algebra. In this case, from the proprieties of the tensor product, it is clear that the minimal train polynomial of an element is unaltered on extension of the base field. It is the same thing for the minimal train polynomial of the algebra.

Following J. Tits [20] (see also [11], Theorem 1, p. 224), we can state the next result.

Theorem 4.1. *Let K be a commutative field of characteristic $\neq 2, 3, 5$, and let (A, ω) be a strictly train power-associative algebra of presentation $(n - t, t)$. If the characteristic of K is prime to the integer t, then $\omega \circ d = 0$ for every derivation d of A, i.e, the radical $\mathfrak{R}$ is a characteristic ideal.*

Proof. Let L be any extension of K. The extensions of the forms $\lambda_i(x) = \gamma_i \omega(x)^i$ and of the derivation d to $A_L = A \otimes_K L$ will be denoted by the same symbols. Let ξ be an indeterminate over K. The minimal train polynomial is unchanged on extension of K to L. For a and b in A_L, let us denote $\{a, b\}_i$ (resp. $\mu_i(a, b)$) the coefficient of ξ in $(a + \xi b)^i$ (resp. $\lambda_i(a + \xi b)$). As $m(a + \xi b)$ is identically zero, the coefficient of ξ in its expression must be zero, that is,

$$\{a, b\}_n + \sum_{i=1}^{t} \lambda_i(a)\{a, b\}_{n-i} + \sum_{i=1}^{t} \mu_i(a, b) a^{n-i} = 0. \tag{7}$$

By induction, we see that $\{a, d(a)\}_i = d(a^i)$. Then $d(m(a)) = d(a^n) + \sum_{i=1}^{t} \lambda_i(a) d(a^{n-i})$, and thus

$$\{a, d(a)\}_n + \sum_{i=1}^{t} \lambda_i(a)\{a, d(a)\}_{n-i} = 0, \quad \forall a \in A_L. \tag{8}$$

Setting $b = d(a)$ in (7) and making a substraction with (8), we obtain

$$\sum_{i=1}^{t} \mu_i(a, d(a)) a^{n-i} = 0.$$

By minimality of the rank of a, a being able to satisfy a polynomial identity of degree $n - 1$ of coefficient in K, so

$$\mu_i(a, b) = 0 \quad \text{for all } a \in A_L.$$

Then, since $\lambda_i(a + \xi b) = \gamma_i \omega(a + \xi b)^i = \gamma_i \sum_{j=0}^{i} \binom{i}{j} \omega(a)^{i-j} \omega(b)^j \xi^j$, we have $\mu_i(a, b) = \gamma_i \binom{i}{1} \omega(a)^{i-1} \omega(b)$. In particular, for $i = 1$, we get $0 = \mu_1(a, d(a)) = \gamma_1 \omega(d(a)) = -t\omega(d(a))$, because $\gamma_i = (-1)^i \binom{t}{i}$. So, if the characteristic of K is prime to t, then $\omega \circ d = 0$ for any derivation d of A. □

In [1], the authors have shown an analog result for uniquely weighted algebras over a field of characteristic zero (see Theorem 3.3).

The following example show that, if the characteristic of the field is not prime to t, it is possible to have $\omega \circ d \neq 0$.

Example 3. ([15], Exemple 2.5) Let K be a field of characteristic 7 and A_7 be the commutative and associative algebra of dimension 7 such that the multiplication table in the basis $\{e_0, e_1, \ldots, e_6\}$ is given by $e_i e_k = e_{i+k}$ if $i + k \leq 6$ and $e_i e_k = 0$ if not. The mapping $\omega : A_7 \longrightarrow K$ defined by $\omega(e_0) = 1$ and $\omega(e_i) = 0$ for $i \neq 0$ is a weight function of A_7. In addition, (A_7, ω) is uniquely weighted. Then A_7 is a power-associative train algebra with train equation $x(x - \omega(x))^7 = 0$. Let $d : A_7 \longrightarrow A_7$ be the linear mapping defined by $d(e_i) = ie_{i-1}$ if $i \neq 0$ and $d(e_0) = 0$. Then d is a derivation of A_7 with $\omega(d(e_1)) = 1$, and hence $\omega \circ d \neq 0$.

Let us suppose now that $\omega \circ d = 0$, for any derivation d of A.

Let $\mathfrak{A}$ be the commutative power-associative algebra obtained from A by adjoining a unit 1 in the usual fashion. Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_{1/2} \oplus \mathfrak{A}_0$ be the Peirce decomposition of $\mathfrak{A}$ relative to e. Then the element $v = 1 - e$ is an idempotent in $\mathfrak{A}$, which is orthogonal to e. The Peirce decomposition of $\mathfrak{A}$ relative to v is $\mathfrak{A} = \mathfrak{A}_v(1) \oplus \mathfrak{A}_v(1/2) \oplus \mathfrak{A}_v(0)$, where $\mathfrak{A}_1 = \mathfrak{A}_e(1) = \mathfrak{A}_v(0)$, $\mathfrak{A}_{1/2} = \mathfrak{A}_e(1/2) = \mathfrak{A}_v(1/2) = A_{1/2}$, $\mathfrak{A}_0 = \mathfrak{A}_e(0) = \mathfrak{A}_v(1)$. Moreover, $\mathfrak{A}_1 = Ke \oplus \mathfrak{R}_1$ and $\mathfrak{A}_0 = Kv \oplus \mathfrak{R}_0$ ([12]).

Theorem 4.2. *Let $\mathfrak{M} = \mathfrak{R}_1 + S_{1/2}(\mathfrak{R}_1)A_{1/2} + T_{1/2}(\mathfrak{R}_0)A_{1/2} + \mathfrak{R}_0$. Then $\mathfrak{M}$ is a characteristic ideal of A.*

Proof. Since $\mathfrak{R} = \mathfrak{R}_1 \oplus A_{1/2} \oplus \mathfrak{R}_0 = \ker(\omega)$ is the radical of A, $\mathfrak{R}$ is also the radical of $\mathfrak{A}$. Let $x_1 \in \mathfrak{R}_1$ and $y_{1/2} \in A_{1/2}$. We have $\omega(x_1 y_{1/2}) = 0$, so $x_1 y_{1/2} \in \mathfrak{R}$, hence $(x_1 y_{1/2})_0 \in \mathfrak{R} \cap A_0 = \mathfrak{R}_0$, i.e., $S_0(x_1)y_{1/2} \in \mathfrak{R}_0$, for all $x_1 \in \mathfrak{R}_1$ and $y_{1/2} \in A_{1/2}$. By symmetry, we show analogously that $T_1(x_0)y_{1/2} \in \mathfrak{R}_1$, for all $x_0 \in \mathfrak{R}_0$ and $y_{1/2} \in A_{1/2}$. So Lemma 1 of [12] is satisfied, without assuming that the characteristic of K is zero. Therefore, $\mathfrak{M}$ is an ideal of $\mathfrak{A}$ ([12], pages 544 and 549) and since $\mathfrak{M} \subseteq A$, then $\mathfrak{M}$ is an ideal of A. Finally, because the ideal $\mathfrak{R}$ is characteristic, we see by *(ii)* and *(iii)* of Theorem 2.1 that $g_d(\mathfrak{R}_0) \subseteq \mathfrak{R}_0$ and $h_d(\mathfrak{R}_1) \subseteq \mathfrak{R}_1$, while *(vii)* and *(viii)* of the same theorem show that $f_d(S_{1/2}(\mathfrak{R}_1)A_{1/2}) \subseteq S_{1/2}(\mathfrak{R}_1)A_{1/2}$ and $f_d(T_{1/2}(\mathfrak{R}_0)A_{1/2}) \subseteq T_{1/2}(\mathfrak{R}_0)A_{1/2}$, respectively. Consequently, $d(\mathfrak{M}) \subseteq \mathfrak{M}$, for all derivation d. □

It is known that if (A, ω) is a weighted algebra, then for each ideal $I \subset \ker(\omega)$, the mapping $\overline{\omega} : A/I \longrightarrow K$ defined by $\overline{\omega}(\pi(x)) = \omega(x)$ is a weight function of A/I, where $\pi : A \longrightarrow A/I$ is the canonical homomorphism. Furthermore, if (A, ω) is a train algebra, then also is $(A/I, \overline{\omega})$.

Theorem 4.3. *Let $A = A_1 \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of a power-associative train algebra. Then $A/\mathfrak{M}$ satisfies $\overline{x}^2 - \overline{\omega}(\overline{x})\overline{x} = 0$.*

Proof. Since $A = Ke \oplus \mathfrak{R}_1 \oplus A_{1/2} \oplus \mathfrak{R}_0$, the Peirce decomposition of $A/\mathfrak{M}$ relative to $\overline{e}$ is $\overline{A} = K\overline{e} \oplus \overline{A}_{1/2}$. Hence, the train equation of $\overline{A}$ is $\overline{x}^2 - \overline{\omega}(\overline{x})\overline{x} = 0$. □

Definition 4.4. [14] A n^{th}-order Bernstein algebra is a commutative weighted algebra (A, ω) satisfying

$$x^{[n+2]} = \omega(x)^{2^n} x^{[n+1]} \tag{9}$$

for all $x \in A$, where n is the smallest integer with such property.

It is known ([5],[16]) that any power-associative n^{th}-order Bernstein algebra is a train algebra satisfying $x^{2^n+1} - \omega(x)x^{2^n} = 0$, hence the train rank is $\leq 2^n + 1$. We have recently shown ([3], Theorem 5.5) that, if A is a power-associative train algebra of rank $m+1$, with train equation $x^{m+1} - \omega(x)x^m = 0$, i.e., of presentation $(m, 1)$, then A is a n^{th}-order Bernstein, where $n \geq 0$ is the unique integer satisfying $2^{n-1} < m \leq 2^n$.

Proposition 3. *Let $A = Ke \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of a power-associative Bernstein algebra, relative to the idempotent e. Every derivation d of A is determined by a unique a triplet $(d(e), f_d, g_d)$ satisfying the following conditions:*

(i) *$d(e) \in A_{1/2}$;*
(ii) *$d(x_{1/2}) = f_d(x_{1/2}) + 2(d(e)x_{1/2})_0$;*
(iii) *$d(x_0) = g_d(x_0) - 2d(e)x_0$;*
(iv) *The endomorphism g_d is a derivation of A_0;*
(v) *$f_d(x_0x_{1/2}) = x_0 f_d(x_{1/2}) + g_d(x_0)x_{1/2}$.*

Proof. Since in this case $A_1 = Ke$, it is enough to observe that $h_d = 0$ and to apply Proposition 1. □

The characterization of derivations of a power-associative train algebra of presentation $(1, m)$, i.e., with rank equation $x(x - \omega(x))^m = 0$, can be obtained by replacing x_0 by $x_1 \in \overline{A_1}$. In this case, h_d is a derivation of A_1, but not of $\overline{A_1}$. In other words, we have the following result.

Proposition 4. *Let $A = Ke \oplus \overline{A_1} \oplus A_{1/2}$ be the Peirce decomposition of a train algebra which is a power-associative algebra of presentation $(1, m)$, relative to the idempotent e. Then each derivation d of A is determined by a unique triplet $(d(e), f_d, h_d)$ satisfying the following conditions:*

(i) *$d(e) \in A_{1/2}$;*
(ii) *$d(x_{1/2}) = f_d(x_{1/2}) + 2(d(e)x_{1/2})_1$;*
(iii) *$d(x_1) = h_d(x_1) - 2d(e)x_1$;*
(iv) *The endomorphism h_d is a derivation of A_1;*
(v) *$f_d(x_1x_{1/2}) = x_1 f_d(x_{1/2}) + h_d(x_1)x_{1/2}$.*

Theorem 4.5 ([17], Proposition 6.7)**.** *Let $A = A_1 \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of a power-associative algebra relative to an idempotent $e \neq 1$. Then, the following conditions are equivalent:*

(i) *A is a Jordan algebra;*
(ii) *The algebra A satisfies the three conditions :*
  (a) *A is e-stable,*
  (b) *$[(x_\lambda y_{1/2})x_{1/2}]_{1-\lambda} = [(x_\lambda x_{1/2})y_{1/2}]_{1-\lambda}$ $(\lambda = 0, 1)$,*
  (c) *A_1 and A_0 are Jordan subalgebras.*

Theorem 4.6. *Let $A = A_1 \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of a power-associative algebra relative to an idempotent $e \neq 1$. If A satisfies conditions (a) and (b) of Theorem 4.5, the factor algebra A/J is a Jordan algebra.*

Proof. If $e \in J_1$, then $A_{1/2} = eA_{1/2} = 0$, i.e, $A = A_1 \oplus A_0$ and Corollary 1 gives the result. Assume that $e \notin J_1$ and consider the canonical homomorphism $\pi : A \to B = A/J$. Let $B = B_1 \oplus B_{1/2} \oplus B_0$ be the Peirce decomposition of the power-associative algebra B relative to the idempotent $\overline{e} = \pi(e)$. The algebra B satisfies also the conditions (a) and (b). Since $B_\lambda = \pi(A_\lambda)$, $\lambda = 1, 0$ and $B_\lambda \simeq A_\lambda/J_\lambda$ is a Jordan algebra, it follows that B satisfies again the condition (c). Theorem 4.5 completes the proof that $B = A/J$ is a Jordan algebra. □

Definition 4.7 ([21])**.** We call the *core* of a power-associative n^{th}-order Bernstein algebra, the ideal $C = Ke \oplus A_{1/2} \oplus A_{1/2}^2$ of A.

Theorem 4.8. *The core of a power-associative n^{th}-order Bernstein algebra is a Jordan algebra.*

Proof. Since $A_1 = Ke$, the algebra A is e-stable and satisfies condition (b) of Theorem 4.5. We must show that $Z = A_{1/2}^2$ is a Jordan algebra. By using the identities $x_0(x_{1/2}y_{1/2}) = (x_0x_{1/2})y_{1/2} + (x_0y_{1/2})x_{1/2}$ [(d) of Corollary 2] and $x_0^2(x_0x_{1/2}) = x_0(x_0^2x_{1/2})$, particular cases of $S_{1/2}(x_0)S_{1/2}(x_0^k) = S_{1/2}(x_0^k)S_{1/2}(x_0)$ $(k \geq 1)$ ([16], Lemme 1.4), we show that

$$\begin{aligned}
x_0^2(x_0(x_{1/2}y_{1/2})) &= x_0^2[(x_0x_{1/2})y_{1/2}] + x_0^2[(x_0y_{1/2})x_{1/2}] \\
&= [x_0^2(x_0x_{1/2})]y_{1/2} + (x_0x_{1/2})(x_0^2y_{1/2}) \\
&+ x_0^2(x_0y_{1/2}) + (x_0^2x_{1/2})(x_0y_{1/2}) \\
&= [x_0(x_0^2x_{1/2})]y_{1/2} + [x_0(x_0^2y_{1/2})]x_{1/2} \\
&+ (x_0x_{1/2})(x_0^2y_{1/2}) + (x_0^2x_{1/2})(x_0y_{1/2}) \\
&= x_0[(x_0^2y_{1/2})x_{1/2}] + x_0[(x_0^2x_{1/2})y_{1/2}] \\
&= x_0[x_0^2(x_{1/2}y_{1/2})].
\end{aligned}$$

Let $z \in Z$. One writes $z = \sum_{i=1}^m \alpha_i x_{1/2,i}y_{1/2,i}$ with $x_{1/2,i}$ and $y_{1/2,i}$ in $A_{1/2}$. We observe that

$$\begin{aligned}
x_0^2(zx_0) &= \sum_{i=1}^m \alpha_i x_0^2[(x_{1/2,i}y_{1/2,i})x_0] \\
&= \sum_{i=1}^m \alpha_i [x_0^2(x_{1/2,i}y_{1/2,i})]x_0 = (x_0^2z)x_0.
\end{aligned}$$

Consequently, $x_0^2(zx_0) = (x_0^2z)x_0$ for all x_0 and z in Z, and so Z is a Jordan algebra. $\square$

Corollary 4 ([21], Corollary 4.7)**.** *The ideal $A_{1/2} \oplus A_{1/2}^2$ is nilpotent in A.*

Acknowledgments. The authors thank the referees of the paper for the very helpful suggestions that shortened the paper.

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Received March 2009; revised November 2009.

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