

A assoc. alg A^+ special Jordan algebra ($= (A, \circ)$)

associative ideal : $B \subset A$ linear subsp. $AB + BA \subset B$

Jordan ideal $B \subset A^+$ linear subp. $A \circ B \subset B$

Lemma 1 If $B \subset A^+$ a Jordan ideal ~~then~~ then

$$\forall a, b \in B \quad \left. \begin{array}{c} \\ x \in A \end{array} \right\} \Rightarrow (ab + ba) \times \overline{} \times (ab + ba) \in B$$

Proof check first that

$$\begin{aligned} x(ab + ba) - (ab + ba)x &= ? \quad \text{⑤} \quad \text{③} \\ \text{①} \quad \text{②} &\quad \text{③} \quad \text{④} \quad a(xb - bx) + (xb - bx)a \\ &+ (xa - ax)b + b(xa - ax) \quad \text{⑥} \quad \text{④} \end{aligned}$$

$$\text{The right side} = a \circ (xb - bx) + b \circ (xa - ax)$$

$$\in B \circ A + B \circ A \subset B$$



def. $x \in A$ is trivial if $xAx = 0$

Theorem 1 Assume the assoc. alg A has no trivial non-zero elements. Then every NON-ZERO Jordan ideal of A^+ contains a non-zero associative ideal of A .

(2)

Proof let $B \neq 0$ be a Jordan ideal let $a, b \in B$

let $c := ab + ba$, By Lemma 1 $xc - cx \in B \forall x \in A$.

$c \in B$ so $xc + cx \in B$

$$\text{so } xc + cx + xc - cx \in B \\ 2xc \in B$$

so $xc \in B$ for all $x \in A$.

$$(xc)y + y(xc) \in B \quad \forall y \in A$$

~~$$xyc - cyx \in B$$~~

~~$$xcy + cyx \in B$$~~

~~$$xcy + yxc = xcy + cyx \in B$$~~

$$(yx)c - cyx \in B$$

~~$$ycc - ccy \in B$$~~

$$ycc + ccy \in B$$

$$yce + yec \in B$$

$$yec \in B$$

$$\text{so } xcy \in B$$

This says $\underbrace{AcA} \subset B$

this is an ideal, but it might be $\{0\}$ (NOT GOOD!)

Suppose $AcA = 0$

then $cAcAcAc = 0$

~~$$cAcAcAc = 0$$~~

$$cxcyczc = 0 \quad \forall x, y, z$$

$$\Rightarrow cAc = 0 \quad \forall x, z$$

$$cxcAczc = 0 \quad \forall x, z$$

$$\Rightarrow cxc = 0 \quad \forall x \in A$$

~~1~~

(3)

We now have $c = 0$. But $c = ab + ba$
 for any elements $a, b \in B$. Suppose $ab + ba = 0$

for all $a, b \in B$ In particular $a^2 = 0 \quad \forall a \in B$

and (2) $axa = a(\underbrace{ax+xa}_{\in B}) + (\underbrace{ax+xa}_{\in B})a$

so $aAa = 0 \quad \& \quad a = 0$ contradiction

Since a was arbitrary in B & $B \neq \{0\}$. \square

Corollary If A is a simple associative algebra,
 then A^+ is a simple Jordan algebra. $(A^2 \neq \{0\} \text{ & } \text{only } 0, A \text{ are the ideals})$

proof. If $x \in A$ satisfies $xA = 0$ then Ax is
 an assoc. ideal so $Ax = 0$ or $Ax = A$

If $Ax = A$, then $A^2 = \underbrace{Ax}_x A = 0$ so $Ax = 0$.
 $\cancel{xA} = 0$

Then Φx is an ideal $\left(\begin{array}{l} (2x)y = \lambda(xy) = 0 = 0 \cdot x \\ y(2x) = \lambda(xy) = 0 = 0 \cdot x \end{array} \right)$

& if $x \neq 0$ then $A = \Phi x \quad A^2 = 0$ contradiction

So $x = 0$

summary if $x \in A$ & $xA = 0$ then $x = 0$
 so

claim if $x \in A$ and $Ax = 0$ then $x = 0$.
 (same argument - in RED)

To complete the proof we show that A has no non-zero trivial elements.

For then by Theorem 1 any nonzero Jordan ideal $\stackrel{(\neq A)}{\text{would}}$ contain a non-zero associated ideal contradicting the simplicity of A .

So let c be a trivial element : $cAc = 0$. $\begin{matrix} NTP \\ c=0 \end{matrix}$

Aca is an ideal so $Aca = 0$ or $Aca = A$

suppose $Aca = A$ then $A^2 = AcA Aca A$

$\underbrace{ca}_{\substack{|| \\ 0}} ca A = 0$, contradicts A simple.

So $Aca = 0$ ~~so A simple~~

claim $Ac = 0$ let $x \in A$, then $xcA = 0$

which implies $xc = 0$ so $Ac = 0$

which implies $c = 0$ done.



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8.2

 V vector space over field $F (= \mathbb{R} \text{ or } \mathbb{C})$

let $g: V \rightarrow F$ be a quadratic map (also called a quadratic form)

$$g(\alpha x) = \alpha^2 g(x) \quad \alpha \in F$$

$$g(x+y) = \frac{g(x+y) - g(x) - g(y)}{2} \quad \text{is bilinear}$$

FACT $D := F \oplus V$ with

$$(\alpha, x)(\beta, y) = (\alpha\beta + g(x, y), \alpha y + \beta x)$$

is a Jordan algebra

Note:

$$\underline{g(x,x) = 4g(x) - 2g(x)}$$

$$\bullet = \# g(x)$$

$$\bullet g(x,y) = g(y,x) \\ (\text{symmetric})$$

~~$y z = (\alpha, x) \quad z^2 = (\alpha, x)(\alpha, x) = (\alpha^2 + g(x, x), 2\alpha x)$~~

~~$(1, 0)(\beta, y) = (\beta + g(0, y), y) = (\beta, y)$~~

~~$(\beta, y)(1, 0) = (\beta + g(y, 0), y) = (\beta, y)$~~

so $(1, 0)$ is a unit element. D is obviously commutative

~~$$\begin{aligned}
 & (\alpha, x) ((\beta, y))^2 (\gamma, z) = (\alpha, x) ((\beta^2 + g(y), 2\beta y) (\gamma, z)) \\
 & = (\alpha, x) (\beta^2 \gamma + g(y) \gamma, 2\beta y \gamma + (\beta^2 + g(y)) z) \\
 & = (\alpha (\beta^2 \gamma + g(y) \gamma) x + g(x, 2\beta y \gamma + (\beta^2 + g(y)) z, \\
 & \quad 2\alpha \beta y \gamma + \alpha (\beta^2 + g(y)) z \\
 & \quad + (\beta^2 \gamma + g(y) \gamma) x)
 \end{aligned}$$~~