

page 73 at the bottom

$$D = F \oplus V \quad F = \mathbb{R} \text{ or } \mathbb{C} \quad V = \text{vector space / } F$$

$$q(ax) = a^2 q(x) \quad q: V \rightarrow F \text{ a quadratic map}$$
$$q(x,y) = \frac{q(x+y) - q(x) - q(y)}{2} \text{ is bilinear}$$

D is a Jordan algebra

NOTE: $(0,x)(0,y) = (q(x,y), 0)$

$$(\alpha, x) + (\beta, y) = (\alpha + \beta, x + y)$$

$$(\alpha, x)(\beta, y) = (\alpha\beta + q(x,y), \alpha y + \beta x)$$

$$\alpha(\beta, y) = (\alpha\beta, \alpha y)$$

this says $V \times V \subset F$

Let U be an ideal of D (Jordan ideal !!)

$(\alpha, 0)(0, y) = (0, \alpha y)$

$$U = (\underbrace{U \cap F}_{\text{"}}) \oplus (\underbrace{U \cap V}_{\text{"}})$$

$F \times V \subset V$

$$\{(\alpha, 0) \in D : (\alpha, 0) \in U\}$$

$$= \{ (0, x) \in D : (0, x) \in U \}$$

$$u \in V, u \neq 0$$

Suppose $\exists z \neq 0 \quad z \in U \cap V$

$$z = (0, \frac{u}{z}) \in U$$

NOTE $F \times F \subset F$

q is non-degenerate means

If $x \in V$ & $q(x,y) = 0 \quad \forall y \in V$ then $x = 0$

$(\Leftrightarrow x \in V \text{ \& } q(y,x) = 0 \quad \forall y \in V \text{ then } x = 0)$ (q is symmetric)

~~if $\forall x \in V \quad q(x,u) \neq 0$ then $q(\frac{x}{q(x,u)}, u) = 1$~~

~~if $\forall x \in V \quad q(x,u) \neq 0$ then~~
so $\exists x \in V \quad q(x,u) = 1$

$$\begin{matrix} (0,x) & (0,u) & = & (g(x,u), 0) & \in & U \\ \uparrow & \uparrow & & & & \\ D & U & & & & \end{matrix}$$

so $(1,0) \in U$ so $U = D$ (since $(1,0)$ is the unit of D)

Conclusion 1 If U ideal in D satisfies $U \cap V \neq 0$ then $U = D$

Consider again an ideal U which is $\neq 0$

let $(p,v) \in U$ so $(p,v) \neq (0,0)$
 $p \neq 0$

If $\dim V \geq 2$ choose $y \neq 0$ orthogonal* to v (i.e. $g(y,v) = 0$)

Then $\underbrace{(p,v)}_{\in U} \underbrace{(0,y)}_{\in D} = (0, py) \in U \cap V$ so $U \cap V \neq 0$

so by the above conclusion 1 $U = D$.

Conclusion 2 If $\dim V \geq 2$ and $g(x,y)$ is non-degenerate then $D = F \oplus V$ is a simple Jordan algebra (with unit)

Exercise 3 for ch. 8 Is D a field? — see ~~the~~ ~~last~~ page (7) of these NOTES

(We are skipping the rest of chapter 8 for Now)

* If V is a vector space and $g(\cdot, \cdot)$ is a non-degenerate symmetric bilinear form ~~then~~ and $\dim V \geq 2$ then ~~there~~ for every vector $v \neq 0$ there is a vector $w \neq 0$ with $g(w,v) = 0$.

~~proof: if not $\exists v \neq 0$ s.t. $g(w,v) = 0$~~

Discussion for Exercise 2 (chapter 8)

Every $z = (\alpha, x) \in D = F \oplus V$ satisfies

$$z^2 = 2\alpha z + (d^2 + q(x, x)). \quad \text{see Mayberg p. 73}$$

For any such z let $f_z(t) = t^2 - 2\alpha t + (d^2 + q(x, x))$
 this is a polynomial of degree 2 with coefficients in F

Thus every element of D is a root of a quadratic polynomial (the polynomial depends on the element but it is always quadratic.)

If F is a field & if K is a field containing F as a subfield (think $F = \mathbb{Q}, K = \mathbb{R}$), an element $a \in K$ is "algebraic" ^{over F} if there is a polynomial $f(t) \neq 0$ with coefficients in F such that $f(a) = 0$.
 (example $\sqrt{2}$ is algebraic over \mathbb{Q} : $f(t) = t^2 - 2$)
 (K is an algebraic extension of F if all elts of K are algebraic over F)
 Thus D ^(low Jordan algebra) is algebraic over F (but D is not a field, or is it?)

In fact quadratic ^{over F} because all such polynomials $f_z(t) = t^2 - 2\alpha t + (d^2 + q(x, x))$ are of degree 2

Thus "quadratic extension" (which usually to field extensions)
 the term

More discussion of

④

Exercise 2 Assume $\dim V = 1$ & g non-degenerate

so $D = F \oplus Fv_0$ v_0 a fixed $\neq 0$ vector in V

~~$(\alpha, x)(\beta, y) = \alpha\beta + g(x, y)$~~

$(\alpha, x)(\beta, y) = (\alpha\beta + g(x, y), \alpha y + \beta x)$

x, β, x, y are all numbers (elements of F)

here $\alpha, \beta, x, y \in F$

$(\alpha y + \beta x)v_0$

$\vec{x} = x v_0$
 $\vec{y} = y v_0$

$g: F \times F \rightarrow F$

For a fixed $y \in F$

$x \rightarrow g(x, y)$ is a linear transf $F \rightarrow F$

so $g(x, y) = \lambda(y)x$ where $\lambda: F \rightarrow F$
 $\lambda(y) \in F$

$g(x, y_1 + y_2) = \lambda(y_1 + y_2)x$ so $\lambda: F \rightarrow F$
" " λ is a linear transf

$g(x, y_1) + g(x, y_2)$
" "

so $\lambda(y) = \lambda(y \cdot 1) = y \lambda(1)$

$\lambda(y_1)x + \lambda(y_2)x$
" "
 $(\lambda(y_1) + \lambda(y_2))x$

Thus $g(x, y) = \lambda(1)xy$

g is non-degenerate $\Leftrightarrow \lambda_0 \neq 0$

Conclusion A bilinear form $g: F \times F \rightarrow F$ has the

form $g(x, y) = \lambda_0 xy$ for a fixed $\lambda_0 \in F$

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More on

Quadratic & bilinear forms

$$f: \mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}$$

$\vec{x} \rightarrow f(\vec{x}, \vec{y})$ is a linear functional on \mathbb{F}^n

so $\exists \lambda(\vec{y}) \in \mathbb{F}^n$

"
 $(\lambda_1(\vec{y}), \dots, \lambda_n(\vec{y}))$

$$f(\vec{x}, \vec{y}) = \sum_{j=1}^n x_j \lambda_j(\vec{y})$$

~~$\lambda(x) = \lambda$~~
 ~~$\lambda(y) = \lambda$~~
 ~~$y \lambda$~~

$$\lambda: \mathbb{F}^n \rightarrow \mathbb{F}^n$$

$$f(\vec{x}, \vec{y} + \vec{z}) = \sum_{j=1}^n x_j \lambda_j(\vec{y} + \vec{z})$$

"
 $f(\vec{x}, \vec{y}) + f(\vec{x}, \vec{z})$

"
 $\sum_j x_j \lambda_j(\vec{y}) + \sum_j x_j \lambda_j(\vec{z})$

(6)

$$\sum_j x_j \left(\lambda_j(\vec{y} + \vec{z}) - \lambda_j(\vec{y}) - \lambda_j(\vec{z}) \right) = 0$$

↑
this vector in \mathbb{F}^n
↓

$$\left(\lambda_1(\vec{y} + \vec{z}) - \lambda_1(\vec{y}) - \lambda_1(\vec{z}), \dots, \lambda_n(\vec{y} + \vec{z}) - \lambda_n(\vec{y}) - \lambda_n(\vec{z}) \right)$$

is orthogonal to \mathbb{F}^n , so it is zero.

So each $\lambda_j(\vec{y} + \vec{z}) = \lambda_j(\vec{y}) + \lambda_j(\vec{z})$
 λ_j is
 a

linear functional on \mathbb{F}^n so $\exists \vec{z}_j \in \mathbb{F}^n$

so $\exists \lambda_j(\vec{y}) = \sum_{k=1}^n y_k z_{jk}$ $\vec{z}_j = (z_{j1}, \dots, z_{jn})$

so $q(\vec{x}, \vec{y}) = \sum_j x_j \lambda_j(\vec{y})$
 $= \sum_{j,k} x_j y_k z_{jk}$

q is the ~~quadratic~~ bilinear form of a matrix $(z_{jk})_{j,k=1}^n$
 To continue

(See the file by Yafaev)

Exercise 3 chapter 8 Is D a field?

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To be a field, the multiplication in $D = F \oplus V$ must be associative i.e.

$$((\alpha, x)(\beta, y))(\gamma, z) = (\alpha, x)((\beta, y)(\gamma, z))$$

AND

every non-zero element must have an inverse i.e.

if $(\alpha, x) \neq (0, 0)$ need to find (β, y)

such that $(\alpha, x)(\beta, y) = (1, 0)$

i.e. $(\alpha\beta + \gamma(x, y), \alpha y + \beta x) = (1, 0)$

You may assume $\dim V = 1$