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Bernstein Algebras: Lattice Isomorphisms and Isomorphisms*

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Abstract: We present here some cases of Bernstein algebras which are determined up to isomorphisms by their lattices of subalgebras.

0. Introduction

The origin of Bernstein algebras lies in genetics and in the study of the stationary evolution operators, see Lyubich [1]. Holgate [2] was the first to give a formulation of Bernstein's problem into the language of nonassociative algebras. For a summary of known results see Wörz-Busekros [3], Ch 9. Further investigations on Bernstein algebras have been taken up by Alcalde, Burgueño, Labra, Micali, (see [5]).

On the other hand, the study of the relationship between lattice isomorphisms and isomorphisms has been done by Barnes ([7] and [8]) for associative and Lie algebras and by J. A. Laliena for alternative algebras [9]. For Jordan algebras similar studies has been done by J. A. Laliena (presented in "Jornadas sobre Modelos Algebraicos no Asociativos y sus Aplicaciones", celebrated in Zaragoza, April 1989) and completed by J. A. Anquela (personal communication, still not submitted).

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1. Preliminaries

A finite-dimensional commutative algebra A over a field K is called *baric* if there exists a nontrivial homomorphism $\omega : A \to K$, called *weight homomorphism*.

A baric algebra is called a *Bernstein algebra* if:

$$(x^2)^2 = \omega(x)^2 x^2 \quad \text{for all } x \text{ in } A.$$

In the following, let K be a commutative infinite field of characteristic different from 2.

Let us list several results on Bernstein algebras which can be found in [3].

For every Bernstein algebra the nontrivial homomorphism $\omega : A \to K$ is uniquely determined.

Every Bernstein algebra A possesses at least one non-zero idempotent.

Every Bernstein algebra A with non-zero idempotent e can be decomposed into the internal direct sum of subspaces:

$$A = Ke \oplus \operatorname{Ker} \omega,$$

with:

$$\operatorname{Ker} \omega = U_e \oplus V_e,$$

where:

$$U_e = \{ex \mid x \in \operatorname{Ker} \omega\} = \{x \in A \mid ex = \tfrac{1}{2}x\},$$

$$V_e = \{x \in A \mid ex = 0\}.$$

The subspaces U_e and V_e of A satisfy:

$$U_e V_e \subseteq U_e, \quad V_e^2 \subseteq U_e, \quad U_e^2 \subseteq V_e, \quad U_e V_e^2 = \langle 0 \rangle.$$

The set of idempotent elements of A is given by:

$$\mathcal{J} = \{e + u + u^2 \mid u \in U_e\} \quad \text{for any idempotent } e \text{ in } A.$$

If $e_1 = e + \sigma + \sigma^2$ with σ in U_e is another idempotent of A we have the following relations between the corresponding subspaces:

$$U_{e_1} = \{ u + 2\sigma u \mid u \in U_e \},$$

$$V_{e_1} = \{ v - 2(\sigma + \sigma^2) v \mid v \in V_e \}.$$

It follows that, although the decomposition of a Bernstein algebra depends on the choice of the idempotent e, the dimension of the subspace U_e of A is an invariant of A. If $\dim_K A = n+1$, then one can associate to $A = Ke \oplus U_e \oplus V_e$ a pair of integers $(r+1, s)$, called *the type* of A, where:

$$r = \dim_K U_e, \qquad s = \dim_K V_e, \qquad \text{hence } r + s = n.$$

In the same way Wörz-Busekros shows in [3] that $\dim_K U_e^2$ and $\dim_K (U_eV_e + V_e^2)$ are also invariants of the algebra A.

Other useful identities can be found in [3] and [5] and will be cited if necessary.

A classification of all the Bernstein algebras of dimensions 2 and 3 is given by Wörz-Busekros in [3]. We reproduce it here to get a classification of these algebras up to isomorphisms.

In dimension 2 there exist up to isomorphisms exactly 2 Bernstein algebras, one algebra of type (1,1): $Ke \oplus Kv$, with $e^2 = e$, $ev = 0$, $v^2 = 0$, and one algebra of type (2,0) : $Ke \oplus Ku$, with $e^2 = e$, $eu = \frac{1}{2}u$, $u^2 = 0$.

In dimension 3 there exists up to isomorphisms exactly one Bernstein algebra of type (1,2): $Ke \oplus Kv_1 \oplus Kv_2$, whose multiplication table is $e^2 = e$, $ev_i = 0$, $v_i v_k = 0$, and one algebra of type (3,0): $Ke \oplus Ku_1 \oplus Ku_2$ whose multiplication table is $e^2 = e$, $eu_i = \frac{1}{2}u_i$, $u_i u_k = 0$. We will denote these algebras by $A_{(1)}$ and $A_{(2)}$ respectively. The 3-dimensional Bernstein algebras of type (2,1) are:

$Ke \oplus Ku \oplus Kv$ with $e^2 = e$, $eu = \frac{1}{2}u$, $ev = 0$ and the remaining products given by the table 1, depending on $\dim_K U_e^2$ and $\dim_K (U_eV_e + V_e^2)$.

TABLE 1.

$\dim_K(U_eV_e+V_e^2)$	$\dim_K U_e^2$	u^2	uv	v^2	
0	0	0	0	0	(a)
0	1	αv	0	0	(b)
1	0	0	βu	γu	(c)

with $\alpha \neq 0$ and $(\beta, \gamma) \neq (0,0)$.

The algebra (a) is the trivial algebra of type (2,1), see [6], and will be called $A_{(3)}$. The algebra (b) is isomorphic to another algebra with the same multiplication table and $\alpha = 1$ and will be called $A_{(4)}$. In (c), if $\beta \neq 0$ then we can put $\beta=1$ and $\gamma=0$ to obtain an isomorphic algebra which will be called $A_{(5)}$. If $\beta=0$, the resulting algebra is isomorphic to another algebra of (c) with $\beta=0$ and $\gamma=1$ which will be called $A_{(6)}$. On the other hand one can see that the algebras $A_{(1)} \ldots A_{(6)}$ are nonisomorphic.

In the same way Wörz-Busekros shows in [6] that for every decomposition $n=r+s$, there exists, up to isomorphism, exactly one $(n+1)$-dimensional, trivial (with $(\mathrm{Ker}\,\omega)^2 = 0$), Bernstein algebra of type $(r+1, s)$.

2. On the Length of a Bernstein Algebra

Let A be an algebra over a commutative field K. We denote by $\mathcal{L}(A)$ the lattice of all subalgebras of A. By an $\mathcal{L}$-isomorphism (lattice isomorphism) of the algebra A onto an algebra B over the same field, we mean an isomorphism:

$$\mathcal{L}(A) \to \mathcal{L}(B) \quad \text{of } \mathcal{L}(A) \text{ onto } \mathcal{L}(B).$$

We put $\ell(A)$, the *length* of A, for the supremum of the lengths of all the chains in $\mathcal{L}(A)$ (by the length of a chain we mean its cardinality minus one). Clearly we have $\dim_K A \geq \ell(A)$ and if the algebra A is finite-dimensional then $\ell(A)$ is the maximum, not only the supremum. We remark that, for a solvable algebra A, we have $\ell(A) = \dim_K A$.

THEOREM 1: *Let A be a Bernstein algebra over a commutative field K of characteristic different from 2. Then $\ell(A) = \dim_K A$.*

PROOF: Let $\omega : A \to K$ be the weight homomorphism of the algebra A, $e \in A$ be an idempotent and $A = Ke \oplus U_e \oplus V_e$ be the decomposition of A with respect to e. In [3] we can see $A^2 = Ke \oplus U_e \oplus U_e^{\,2}$, and this algebra is a Bernstein algebra with weight homomorphism the restriction $\tilde{\omega}: A^2 \to K$ of ω to the algebra A^2. Then we have that the corresponding subspaces of the decomposition of the algebra A^2 with respect to e are:

$$\tilde{U}_e = \{ x \in A^2 \mid ex = \tfrac{1}{2}x \} = U_e,$$

$$\tilde{V}_e = \{ x \in A^2 \mid ex = 0 \} = U_e^{\,2} = \tilde{U}_e^{\,2}.$$

Then $(A^2)^2 = A^2$ and, in this situation, we can apply the theorem 2 of [4] and conclude that $\operatorname{Ker}\tilde{\omega} = U_e \oplus U_e^{\,2} \supseteq (\operatorname{Ker}\omega)^2$ is right nilpotent, hence nilpotent, (see proposition 4.1. on page 82 of [10]), and then solvable. But now $\operatorname{Ker}\omega$ is solvable, too, and hence $\ell(\operatorname{Ker}\omega) = \dim_K \operatorname{Ker}\omega$.

So we have:

$\ell(A) \geq \ell(\operatorname{Ker}\omega) + 1 = \dim_K \operatorname{Ker}\omega + 1 = \dim_K A$ and this completes the proof.•

In what follows, by "subalgebra" we mean a proper subalgebra.

COROLLARY: *The dimensions of the subalgebras of a Bernstein algebra are invariant by any $\mathfrak{L}$-isomorphism.*

PROOF: If A_1 is a subalgebra of a Bernstein algebra A such that $A_1 \subseteq \operatorname{Ker}\omega$, then A_1 is solvable and hence $\ell(A_1) = \dim_K A_1$. In the other case the algebra A_1 is a Bernstein algebra and we have $\ell(A_1) = \dim_K A_1$. Clearly the length of an algebra is invariant by any $\mathfrak{L}$-isomorphism.•

REMARK: If K is a field of characteristic 2 the proofs of the previous results are not valid because $\operatorname{Ker}\omega$ is not, in general, solvable. We can consider, for example, the commutative algebra $A = Ke \oplus Kx \oplus Ky \oplus Kz$ whose multiplication table is given by $e^2 = e$, $xy = z$, $yz = x$, $xz = y$, where the missing products are zero, over any field K of characteristic 2. Clearly A is a Bernstein algebra over K with weight homomorphism

$\omega : A \rightarrow K$ given by $\omega(e) = 1$, $\omega(x) = \omega(y) = \omega(z) = 0$ linearly extended, but $(\mathrm{Ker}\,\omega)^2 = \mathrm{Ker}\,\omega$.

3. Subalgebras of Dimension Two

We are going to study the 2-dimensional algebras which can be found in a Bernstein algebra.

If A_1 is a 2-dimensional subalgebra of a Bernstein algebra A such that $A_1 \not\subset \mathrm{Ker}\,\omega$, then A_1 is a 2-dimensional Bernstein algebra and hence it is either $Ke \oplus Ku$ or $Ke \oplus Kv$, with the notation of §1 .

In the other case we have $A_1 \subseteq \mathrm{Ker}\,\omega$ and then A_1 is a commutative solvable algebra such that $(x^2)^2 = 0$ holds in A_1. If A_1 is not trivial we have $A_1{}^2 = Kz$ with $z \neq 0$ and one can see that A_1 is isomorphic to another algebra $B = Kz \oplus Kw$ whose multiplication table is given by:

$z^2 = 0$, $zw = 0$, $w^2 = z$, which will be called $B_{(2)}$, or:

$z^2 = 0$, $zw = z$, $w^2 = 0$, which will be called $B_{(3)}$.

By $B_{(1)}$ we mean A_1 when is trivial. Clearly these algebras are nonisomorphic.If we look at the lattice of subalgebras of these algebras we can conclude the following table 2:

TABLE 2.

ALGEBRA	Multiplication table	Subalgebras	Number of subalgebras
$Ke \oplus Ku$	$e^2 = e$ $eu = \frac{1}{2}u$ $u^2 = 0$	Ku, $K(e + \alpha u)$	$\lvert K \rvert + 1$
$Ke \oplus Kv$	$e^2 = e$ $ev = 0$ $v^2 = 0$	Ke, Kv	2
$B_{(1)} = Kz \oplus Kw$	$z^2 = 0$ $zw = 0$ $w^2 = 0$	Kw, $K(z + \alpha w)$	$\lvert K \rvert + 1$
$B_{(2)} = Kz \oplus Kw$	$z^2 = 0$ $zw = 0$ $w^2 = z$	Kz	1
$B_{(3)} = Kz \oplus Kw$	$z^2 = 0$ $zw = z$ $w^2 = 0$	Kz, Kw	2

where α is in K and we write $|K|$ for the cardinality of the field K, which is infinite, in this case.

4. Lattice Isomorphism and Isomorphism

We begin with the low-dimensional cases.

THEOREM 2: *Two Bernstein algebras of dimension less than or equal to 3 over an infinite field of characteristic different from 2 are $\mathcal{L}$-isomorphic if and only if they are isomorphic.*

PROOF: The "if" is obvious. Let us prove the "only if".

Clearly they have the same dimension (see §2), and the result is valid in dimension 2 (see table 2). Let (A, ω) and $(\tilde{A}, \tilde{\omega})$ be two 3-dimensional Bernstein algebras over K, and $\phi: \mathcal{L}(A) \to \mathcal{L}(\tilde{A})$ be an $\mathcal{L}$-isomorphism.

(I) Let us suppose $A \cong A_{(1)}$. Then there exits only one idempotent e and $A = Ke \oplus Kv_1 \oplus Kv_2$ with $e^2=e$, $ev_i=0$, $v_iv_k=0$, $V_e = \mathrm{Ker}\,\omega = Kv_1 \oplus Kv_2 \cong B_{(1)}$.

The 1-dimensional subalgebras of A are Kx with $x^2=x$ or $x^2=0$. So it is easy to see that, in this case, they are Ke or Kv with v in V_e.

We consider the 2-dimensional subalgebras of A: if A_1 is a 2-dimensional subalgebra of A and $A_1 \subseteq \mathrm{Ker}\,\omega$ then $A_1 = \mathrm{Ker}\,\omega \cong B_{(1)}$ which has $|K| + 1$ subalgebras. In the other case A_1 possesses an idempotent, i.e., $Ke \subseteq A_1$, and $A_1 = Ke \oplus Kv$ with v in V_e, which has 2 subalgebras.

Hence Ke is the only 1-dimensional subalgebra of A not contained in 2-dimensional subalgebras with $|K| + 1$ subalgebras. On the other hand, Ke is contained in an infinite number of 2-dimensional subalgebras with only 2 subalgebras.

If $\tilde{A}$ is a Bernstein algebra of type (1, 2), then clearly $\tilde{A} \cong A$. Let us suppose this is false and we will get a contradiction.

We denote $Kx = \phi(Ke)$ with $0 \neq x$ in $\widetilde{A}$ and let $\widetilde{A}_1$ be a 2-dimensional sulalgebra of $\widetilde{A}$ such that $Kx \subseteq \widetilde{A}_1$, then $\widetilde{A}_1$ has exactly 2 subalgebras. It follows from table 2:

$\widetilde{A}_1 = K\tilde{e} \oplus K\tilde{v}$ with $\tilde{e}^2 = \tilde{e}$, $\tilde{e}\tilde{v} = 0$, $\tilde{v}^2 = 0$, or:

$\widetilde{A}_1 = Kz \oplus Kw$ with $z^2 = 0$, $zw = z$, $w^2 = 0$.

In the second case we have $\widetilde{A}_1 = \operatorname{Ker}\tilde{\omega}$; we take $\widetilde{A}_2$ and $\widetilde{A}_3$, two different subalgebras of $\widetilde{A}$ satisfying the same conditions as $\widetilde{A}_1$. Then, for the same reasons we have:

$\widetilde{A}_2 = K\tilde{e} \oplus K\tilde{v}$ with $\tilde{e}^2 = \tilde{e}$, $\tilde{e}\tilde{v} = 0$, $\tilde{v}^2 = 0$, and $\tilde{\omega}(\tilde{e}) = 1$

$\widetilde{A}_3 = K\tilde{e}_1 \oplus K\tilde{v}_1$ with $\tilde{e}_1{}^2 = \tilde{e}_1$, $\tilde{e}_1\tilde{v}_1 = 0$, $\tilde{v}_1{}^2 = 0$, and $\tilde{\omega}(\tilde{e}_1) = 1$, since $\widetilde{A}_2$ and $\widetilde{A}_3$ cannot be $\operatorname{Ker}\tilde{\omega}$.

But now $\tilde{v}$ is in $\widetilde{V}_{\tilde{e}}$, hence $\dim_K \widetilde{V}_{\tilde{e}} \geq 1$ and $\dim_K \widetilde{V}_{\tilde{e}} \leq 1$, because the type of $\widetilde{A}$ is not (1, 2). It follows $\widetilde{V}_{\tilde{e}} = K\tilde{v}$. In the same way $\widetilde{V}_{\tilde{e}_1} = K\tilde{v}_1$. Besides that, $\widetilde{A}$ is a Bernstein algebra of type (2, 1).

We have: $0 \neq x \in \operatorname{Ker}\tilde{\omega} \cap \widetilde{A}_2 \cap \widetilde{A}_3$, hence $Kx = K\tilde{v} = K\tilde{v}_1$.

On the other hand, the idempotents $\tilde{e}$ and $\tilde{e}_1$ are connected by:

$\tilde{e}_1 = \tilde{e} + \sigma + \sigma^2$ with σ in $\widetilde{U}_{\tilde{e}}$, and $\sigma \neq 0$, since $\widetilde{A}_2$ and $\widetilde{A}_3$ are different subalgebras. Therefore $\widetilde{U}_{\tilde{e}} = K\sigma$.

Since $0 \neq \tilde{v} \in K\tilde{v} = K\tilde{v}_1 = \widetilde{V}_{\tilde{e}_1}$, we have $\tilde{v} = \lambda\{\tilde{v} - 2(\sigma+\sigma^2)\tilde{v}\}$, with λ in K. Since $(\sigma+\sigma^2)\tilde{v}$ is in $\widetilde{U}_{\tilde{e}}$, and the sum $\widetilde{U}_{\tilde{e}} \oplus \widetilde{V}_{\tilde{e}}$ is direct, we can conclude:

$$\lambda = 1 \quad \text{and} \quad (\sigma+\sigma^2)\tilde{v} = 0.$$

On the other hand σ^2 is in $\widetilde{V}_{\tilde{e}} = K\tilde{v}$, and $\tilde{v}^2 = 0$, hence $\sigma^2\tilde{v} = 0$, and thus $\sigma\tilde{v} = 0$, i.e., $\widetilde{U}_{\tilde{e}}\widetilde{V}_{\tilde{e}} = 0$.

If $\sigma^2 \neq 0$, we can consider $K\sigma \oplus K\sigma^2 = \tilde{U}_{\tilde{e}} \oplus \tilde{V}_{\tilde{e}} \cong B_{(2)}$, which has only one subalgebra. As A has not any subalgebras of this type, we get a contradiction. Then $\sigma^2 = 0$. Therefore we take $K\sigma \oplus K\tilde{v} = \tilde{U}_{\tilde{e}} \oplus \tilde{V}_{\tilde{e}} \cong B_{(1)}$, which has $|K|+1$ subalgebras and contains $Kx = K\tilde{v}$ and this is a contradiction.

Then, the second posibility for $\tilde{A}_1$ is not valid, i.e., $x \notin \operatorname{Ker} \tilde{\omega}$. Hence $Kx = K\tilde{e}_0$, where $\tilde{e}_0$ is an idempotent of $\tilde{A}$. As above, we take $\tilde{A}_2$ and $\tilde{A}_3$, which are Bernstein algebras with only one idempotent. Thus $\tilde{e}_0 = \tilde{e} = \tilde{e}_1$. Now, $\tilde{v}$ and $\tilde{v}_1$ cannot be linearly independent, because $\dim_K \tilde{V}_{\tilde{e}} \leq 1$, and cannot be linearly dependent, since $\tilde{A}_2$ and $\tilde{A}_3$ are different subalgebras, which is a contradiction.

(II) Let us suppose $A \cong A_{(2)}$. Then $A = Ke \oplus Ku_1 \oplus Ku_2$ with $e^2 = e$, any idempotent of A, $eu_i = \frac{1}{2}u_i$, $u_i u_k = 0$, $U_e = \operatorname{Ker}\omega = Ku_1 \oplus Ku_2 \cong B_{(1)}$.

The 1-dimensional subalgebras of A are exactly the 1-dimensional subspaces of A, Kx, since $x^2 = \omega(x)\, x$ holds for all x in A.

Concerning the 2-dimensional subalgebras of A, if A_1 is a 2-dimensional subalgebra of A and $A_1 \subseteq \operatorname{Ker}\omega$ then $A_1 = \operatorname{Ker}\omega \cong B_{(1)}$ which has $|K| + 1$ subalgebras. In the other case A_1 possesses an idempotent e, and $A_1 = Ke \oplus Ku$ with u in U_e, which has $|K| + 1$ subalgebras. Thus, none of the 2-dimensional subalgebras of A has a finite number of subalgebras.

From (I), $\tilde{A}$ cannot be isomorphic to $A_{(1)}$. If it were isomorphic to $A_{(3)}$, $A_{(4)}$, or $A_{(5)}$, it would have a 2-dimensional subalgebra with exactly 2 subalgebras ($Ke \oplus Kv$, with the notation of Table 1). If it were isomorphic to $A_{(6)}$, it would have a 2-dimensional subalgebra with only one subalgebra ($Ku \oplus Kv$, with the same notation). Both situations lead us to a contradiction. Then it must be that $\tilde{A} \cong A_{(2)} \cong A$.

(III) Finally, let us suppose A is a Bernstein algebra of type (2, 1). Then we can write $A = Ke \oplus Ku \oplus Kv$, with $e^2 = e$, $eu = \frac{1}{2}u$, $ev = 0$, for any idempotent e of A. From (I) and (II) we can conclude that $\tilde{A}$ is a Bernstein algebra of the same type.

If $A \cong A_{(3)}$, we can write $u^2=0$, $uv=0$, $v^2=0$ for any decomposition of A. Besides that, either a 2-dimensional subalgebra A_1 of A is $A_1 = \operatorname{Ker}\omega \cong B_{(1)}$, which has $|K|+1$

subalgebras, or A_1 possesses an idempotent. In the second case $A_1 = Ke \oplus Kx$ with $0 \neq x$ in $\operatorname{Ker}\omega = Ku \oplus Kv$. We put $x = \alpha u + \beta v$, with α, β in K. If $\alpha\beta \neq 0$, we have $u = (2\alpha)^{-1}ex \in A_1$, hence $v \in A_1$ and then $A_1 = A$, which is a contradiction.Therefore we have $Ke \oplus Kx = Ke \oplus Ku$, with an infinite number of subalgebras, or $Ke \oplus Kx = Ke \oplus Kv$, which has 2 subalgebras. Either way A has only 2-dimensional subalgebras with 2 or more subalgebras.

If $A \cong A_{(5)}$, we can put $u^2=0$, $uv=u$, $v^2=0$. Then, a 2-dimensional subalgebra A_1 of A is $A_1 = \operatorname{Ker}\omega \cong B_{(3)}$, which has 2 subalgebras, or A_1 possesses an idempotent $e + \lambda u$, λ in K. In the last case we put $A_1 = K(e+\lambda u) \oplus Kx$ with $0 \neq x$ in $\operatorname{Ker}\omega = Ku \oplus Kv$. As above, $x = \alpha u + \beta v$, and, if $\alpha\beta \neq 0$, $x^2 = 2\alpha\beta u \in A_1$ and hence $A_1 = A$, which is absurd. Thus we conclude $\alpha=0$ or $\beta=0$. If we had $\alpha=0$, then we would have $\beta \neq 0$ and $\lambda\beta u = (e+\lambda u)\beta v \in A_1$; hence $\lambda=0$ and $A_1 = Ke \oplus Kv$, which has 2 subalgebras. If we had $\beta=0$, then we would have $\alpha \neq 0$ and $A_1 = Ke \oplus Ku$, which has $|K|+1$ subalgebras. Again, A has only 2-dimensional subalgebras with 2 or more subalgebras.

If $A \cong A_{(4)}$ or $A \cong A_{(6)}$, it has a 2-dimensional subalgebra, $\operatorname{Ker}\omega \cong B_{(2)}$ with a unique proper subalgebra.

Moreover, we have shown that the algebras $A_{(3)}$ and $A_{(5)}$ are determined by their lattices. They can be $\mathfrak{L}$-isomorphic neither to $A_{(4)}$ nor to $A_{(6)}$ because of what we said above. And the algebra $A_{(3)}$ cannot be $\mathfrak{L}$-isomorphic to $A_{(5)}$ because $A_{(5)}$ has only one 2-dimensional subalgebra with an infinite number of subalgebras, $Ke \oplus Ku$, and $A_{(3)}$ has two of them, $\operatorname{Ker}\omega$ and $Ke \oplus Ku$, using the previous notation.

It remains to show why the algebras $A_{(4)}$ and $A_{(6)}$ cannot be $\mathfrak{L}$-isomorphic.

As above, if we put $A_{(6)} = Ke \oplus Ku \oplus Kv$, with $e^2 = e$, $eu = \frac{1}{2}u$, $ev = 0$, $u^2 = 0$, $uv = 0$, $v^2 = u$, a 2-dimensional subalgebra A_1 of $A_{(6)}$ such that $A_1 \not\subset \operatorname{Ker}\omega$ possesses an idempotent $e + \lambda u$, λ in K. We write $A_1 = K(e+\lambda u) \oplus Kx$, $x = \alpha u + \beta v$, with α, β in K. If $\beta \neq 0$, we have $x^2 = \beta^2 u$ is in A_1, and hence $A_1 = A$, which is a contradiction. Thus it must be $\beta=0, \alpha \neq 0$, and $A_1 = Ke \oplus Ku$, which has an infinite number of subalgebras. Then we can conclude that the 2-dimensional subalgebras of $A_{(6)}$ have only one subalgebra or an infinite number of subalgebras. Now if we put $A_{(4)} = Ke \oplus Ku \oplus Kv$, with $e^2 = e$, $eu = \frac{1}{2}u$, $ev = 0$, $u^2 = v$, $uv = 0$, $v^2 = 0$, we can consider $Ke \oplus Kv$, which has exactly 2 subalgebras. It follows that $A_{(4)}$ and $A_{(6)}$ cannot be $\mathfrak{L}$-isomorphic.

In our situation this means that the algebra $\widetilde{A}$ must be isomorphic to the algebra A.•

We are going to generalize the ideas of the previous theorem to get similar results for some extreme cases of Bernstein algebras in any dimension.

THEOREM 3: *If two Bernstein algebras are $\mathfrak{L}$-isomorphic and one of them is an $(n+1)$-dimensional Bernstein algebra of type $(n+1, 0)$, then they are isomorphic.*

PROOF: Let $(\widetilde{A}, \widetilde{\omega})$ be an $(n+1)$-dimensional Bernstein algebra of type $(n+1, 0)$. Let (A, ω) be a Bernstein algebra and $\phi : \mathfrak{L}(\widetilde{A}) \to \mathfrak{L}(A)$ be an $\mathfrak{L}$-isomorphism.

We can write: $\widetilde{A} = K\widetilde{e} \oplus \operatorname{Ker} \widetilde{\omega}$ with $\widetilde{U}_{\widetilde{e}} = \operatorname{Ker} \widetilde{\omega}$, $\widetilde{V}_{\widetilde{e}} = 0$, for any idempotent $\widetilde{e}$ in $\widetilde{A}$ and suppose $n \geq 2$.

If $\widetilde{A}_1$ is a 2-dimensional subalgebra of $\widetilde{A}$, either $\widetilde{A}_1 \subset \operatorname{Ker} \widetilde{\omega}$ or $\widetilde{A}_1$ possesses an idempotent $\widetilde{e}$.In the first case, $\widetilde{A}_1$ is trivial and it has $|K| + 1$ subalgebras. In the second case we have $\widetilde{A}_1 = K\widetilde{e} \oplus K\widetilde{u}$, with $\widetilde{u}$ in $\widetilde{U}_{\widetilde{e}}$, which also has $|K| + 1$ subalgebras.

Besides that, we can see that:

-Every pair of 1-dimensional subalgebras of $\widetilde{A}$ generates a 2-dimensional subalgebra of $\widetilde{A}$.

-The n-dimensional subalgebras of $\widetilde{A}$ are $\operatorname{Ker} \widetilde{\omega}$ or $K\widetilde{e} \oplus W$ where W is (n-1)-dimensional and $W \subseteq \widetilde{U}_{\widetilde{e}}$. In both cases we can see that a n-dimensional subalgebra of $\widetilde{A}$ contains n 1-dimensional subalgebras generated by elements linearly independent, let $\widetilde{x}_1, \dots, \widetilde{x}_n$ be those elements. Moreover any subset of m elements of $\{\widetilde{x}_1, \dots, \widetilde{x}_n\}$ generates a m-dimensional subalgebra of $\widetilde{A}$.

Now, let us write $A = Ke \oplus U_e \oplus V_e$ for any idempotent e in A. Then, since $\operatorname{Ker} \omega = U_e \oplus V_e$ is a n-dimensional subalgebra of A, we have $\operatorname{Ker} \omega = \phi(\widetilde{A}_2)$, where

$\widetilde{A}_2$ is a n-dimensional subalgebra of $\widetilde{A}$. Therefore we can write $\widetilde{A}_2 = K\widetilde{x}_1 \oplus \dots \oplus K\widetilde{x}_n$ for the preceding elements.

We take $Kx_i = \phi(K\widetilde{x}_i)$, $0 \neq x_i \in A$, for i=1, ..., n. If the elements $x_1, \dots, x_n$ were not linearly independent, there would exist j such that:

$$x_j \in Kx_1 + \overset{\wedge}{\underset{\dots}{j}} + Kx_n$$

Then we would have:

$$\phi(K\widetilde{x}_j) = Kx_j \subseteq Kx_1 \vee \overset{\wedge}{\underset{\dots}{j}} \vee Kx_n = \phi(K\widetilde{x}_1) \vee \overset{\wedge}{\underset{\dots}{j}} \vee \phi(K\widetilde{x}_n) =$$

$$= \phi(K\widetilde{x}_1 \vee \overset{\wedge}{\underset{\dots}{j}} \vee K\widetilde{x}_n) = \phi(K\widetilde{x}_1 \oplus \overset{\wedge}{\underset{\dots}{j}} \oplus K\widetilde{x}_n)$$

Thus: $K\widetilde{x}_j \subseteq K\widetilde{x}_1 \oplus \overset{\wedge}{\underset{\dots}{j}} \oplus K\widetilde{x}_n$, which is a contradiction. Hence the elements $x_1, \dots, x_n$ are linearly independent and we have:

$$\operatorname{Ker} \omega = U_e \oplus V_e = Kx_1 \oplus \dots \oplus Kx_n .$$

If $x_i \in U_e$ for all i = 1, ..., n we have $\operatorname{Ker} \omega = U_e$ and A is a Bernstein algebra of type (n+1, 0) isomorphic to $\widetilde{A}$, as we wanted to prove.

In the other case there would exist j such that:

$$x_j = u + v \text{ with } u \text{ in } U_e \text{ and } 0 \neq v \text{ in } V_e.$$

If $u^2 = w \neq 0$ we would have $w^2 = 0$ and we could consider the subalgebra $Ke \oplus Kw$ which has only 2 subalgebras, and this is impossible because A and $\widetilde{A}$ are $\mathfrak{L}$-isomorphic. Hence, $u^2 = 0$ and Ku is a subalgebra of A. Now we take the subalgebras $Kx_j = K(u+v)$ and Ku; they generate a 2-dimensional subalgebra which contains u and v, and thus it is $Ku \oplus Kv$. This algebra must have $|K| + 1$ subalgebras and, since it is contained in $\operatorname{Ker} \omega$, it must be trivial. But then $v^2 = 0$ and we get a contradiction (as above with w).•

THEOREM 4: *If two Bernstein algebras are $\mathfrak{L}$-isomorphic and one of them is a Bernstein algebra (n+1)-dimensional of type (1, n), then they are isomorphic.*

PROOF: We will carry out an induction on n.

For $n = 0,1$ the result is obvious. Let us suppose the result valid for all $0 \leq k \leq n-1$, with $n \geq 2$ and we will prove it for n.

Let $(\widetilde{A}, \widetilde{\omega})$ be a (n+1)-dimensional Bernstein algebra of type (1, n), (A, ω) be a Bernstein algebra and $\phi : \mathcal{L}(\widetilde{A}) \rightarrow \mathcal{L}(A)$ be an $\mathcal{L}$-isomorphism.

We can write: $\widetilde{A} = K\widetilde{e} \oplus \mathrm{Ker}\, \widetilde{\omega}$ with $\widetilde{V}_{\widetilde{e}} = \mathrm{Ker}\, \widetilde{\omega}$, $\widetilde{U}_{\widetilde{e}} = 0$, for the unique idempotent $\widetilde{e}$ in $\widetilde{A}$.

If $\widetilde{A}_1$ is a 2-dimensional subalgebra of $\widetilde{A}$, either $\widetilde{A}_1 \subseteq \mathrm{Ker}\, \widetilde{\omega}$ or $\widetilde{A}_1$ possesses an idempotent $\widetilde{e}$. In the first case, $\widetilde{A}_1$ is trivial and it has $|K| + 1$ subalgebras. In the second case we have $\widetilde{A}_1 = K\widetilde{e} \oplus K\widetilde{v}$, with $\widetilde{v}$ in $\widetilde{V}_{\widetilde{e}}$, which has 2 subalgebras.

Besides that, $K\widetilde{e}$ is the only 1-dimensional subalgebra of $\widetilde{A}$ not contained in 2-dimensional subalgebras with an infinite number of subalgebras. Moreover, a 2-dimensional subalgebra of $\widetilde{A}$ contains $K\widetilde{e}$ if and only if it has exactly 2 subalgebras.

On the other hand, $\widetilde{A}$ has at least four different n-dimensional subalgebras: $\mathrm{Ker}\, \widetilde{\omega}$ and at least three Bernstein algebras with decomposition $K\widetilde{e} \oplus \widetilde{W}$ where $\widetilde{W}$ is a (n-1)-dimensional subspace of $\widetilde{V}_{\widetilde{e}}$.

Therefore A has at least four different n-dimensional subalgebras, and we can take two different n-dimensional subalgebras which are different from $\mathrm{Ker}\, \omega$ and $\phi(\mathrm{Ker}\, \widetilde{\omega})$. Then, each one of them is a Bernstein algebra $\mathcal{L}$-isomorphic to a Bernstein algebra of type (1, n-1), hence isomorphic to a Bernstein algebra of that type, by the induction assumption.

Let us write these algebras:

$$Ke \oplus W, \text{ with } W \subseteq V_e,\ e^2 = e,$$

$$Ke_1 \oplus W_1, \text{ with } W_1 \subseteq V_{e_1},\ e_1^2 = e_1.$$

If V_e were n-dimensional, we would have the type of A is (1, n) and hence it is isomorphic to $\widetilde{A}$.

If V_e is (n-1)-dimensional, we have $V_e=W$, $V_{e_1}=W_1$, and A is of type (2, n-1) with $V_e^2=0$. We put $U_e=Ku$. If $0\neq v=u^2$, we would have $v^2=0$, $uv=0$, since $u^3=0$ (see [3]), and $Ku\oplus Kv$ would be a 2-dimensional subalgebra of A with only one subalgebra, but that is not possible in A. Thus, it must be $u^2=0$.

Hence the idempotents are connected by $e_1=e+\lambda u$, $0\neq\lambda$ in K, since these algebras are different.

Now, we consider their sum as vectorial subspaces:

$$(Ke\oplus V_e)+(Ke_1\oplus V_{e_1})=A,$$

hence the subalgebra $A_1=(Ke\oplus V_e)\cap(Ke_1\oplus V_{e_1})$ is (n-1)-dimensional.

If $A_1\not\subset \operatorname{Ker}\omega$, A_1 possesses an idempotent, but since the algebras $Ke\oplus V_e$ and $Ke_1\oplus V_{e_1}$ have only one idempotent, it follows $e=e_1$ and we have a contradiction. Hence $A_1\subseteq \operatorname{Ker}\omega$, and then $A_1\subseteq V_e\cap V_{e_1}$. Because of the dimensions of the subspaces, we have shown:

$$V_e\cap V_{e_1}=V_e=V_{e_1}$$

Now, any w in V_e is in $V_{e_1}=\{\,v-2\lambda uv \mid v\in V_e\,\}$ and we can write:

$$w=v-2\lambda uv \text{ with } w, v \text{ in } V_e.$$

Since $U_e\oplus V_e$ is direct we conclude $uv=0$, then $uw=0$, hence $U_eV_e=0$.

Therefore A is a trivial Bernstein algebra of type (2, n-1).

If we take $Kx=\phi(K\tilde{e})$, it is contained in a 2-dimensional subalgebra of A with 2 subalgebras, say A_2. If A_2 were contained in $\operatorname{Ker}\omega$, it would be isomorphic to $B_{(3)}$, but this impossible because the latter is not trivial. Then $A_2=Ke\oplus Kv$ with v in V_e for an idempotent e of A and $Kx=Ke$ or $Kx=Kv$. In both cases Kx would be contained in a 2-dimensional subalgebra of A with an infinite number of subalgebras, which is a contradiction.●

THEOREM 5: *If two Bernstein algebras are* $\mathfrak{L}$*-isomorphic and one of them is a trivial Bernstein algebra which is (n+1)-dimensional of type (r+1, s), then they are isomorphic.*

PROOF: We will carry out an induction on n. For $n = 0,1$ the result is obvious, and we know it for $n = 2$. Let us suppose the result valid for all $0 \leq k \leq n-1$, with $n \geq 3$ and we will prove it for n. The cases $r = 0$ and $s = 0$ are the theorems 3 and 4 and we will suppose $r\, s \neq 0$.

Let $(\tilde{A}, \tilde{\omega})$ be an (n+1)-dimensional Bernstein algebra of type (r+1, s), (A, ω) be a Bernstein algebra and $\phi : \mathfrak{L}(\tilde{A}) \to \mathfrak{L}(A)$ be an $\mathfrak{L}$-isomorphism.

We can write: $\tilde{A} = K\tilde{e} \oplus \operatorname{Ker} \tilde{\omega}$ with $\operatorname{Ker} \tilde{\omega} = \tilde{U}_{\tilde{e}} \oplus \tilde{V}_{\tilde{e}}$, for any idempotent $\tilde{e}$ in $\tilde{A}$ and it is easy to see that any 2-dimensional subalgebra of $\tilde{A}$ must have at least 2 subalgebras.

(I) We will show here that if $A \not\cong \tilde{A}$, then A has two n-dimensional subalgebras which are trivial Bernstein algebras, one of type (r, s), and the other of type (r+1, s-1).For any idempotent $\tilde{e}$ in $\tilde{A}$, let

$\tilde{U}_{\tilde{e}} = K\langle \tilde{u}_1, \ldots, \tilde{u}_r \rangle$, with $\{\tilde{u}_1, \ldots, \tilde{u}_r\}$ a basis of $\tilde{U}_{\tilde{e}}$ over K

$\tilde{V}_{\tilde{e}} = K\langle \tilde{v}_1, \ldots, \tilde{v}_s \rangle$, with $\{\tilde{v}_1, \ldots, \tilde{v}_s\}$ a basis of $\tilde{V}_{\tilde{e}}$ over K.

(a) If $r \geq 2$ and $s \geq 2$, we take for example the following subalgebras of A:

$A_1 = \phi(K\tilde{e} \oplus K\langle \tilde{u}_2, \ldots, \tilde{u}_r \rangle \oplus \tilde{V}_{\tilde{e}})$, $\quad A_2 = \phi(K\tilde{e} \oplus \tilde{U}_{\tilde{e}} \oplus K\langle \tilde{v}_2, \ldots, \tilde{v}_s \rangle)$,

$A_3 = \phi(K\tilde{e} \oplus K\langle \tilde{u}_1, \ldots, \tilde{u}_{r-1} \rangle \oplus \tilde{V}_{\tilde{e}})$, $\quad A_4 = \phi(K\tilde{e} \oplus \tilde{U}_{\tilde{e}} \oplus K\langle \tilde{v}_1, \ldots, \tilde{v}_{s-1} \rangle)$.

At most one of them can be contained in $\operatorname{Ker}\omega$. Hence among them there are at least 3 Bernstein algebras and therefore, at least one $\mathfrak{L}$-isomorphic to a trivial Bernstein algebra of type (r, s) and one $\mathfrak{L}$-isomorphic to a trivial Bernstein algebra of type (r+1, s-1).We apply the induction assumption for these algebras.

(b) If $r = 1$, we have $s = n - r = n - 1 \geq 2$ and we can take the algebras:

$A_1=\phi(K\tilde{e}\oplus\tilde{V}_{\tilde{e}})$, $\qquad A_2=\phi(K\tilde{e}\oplus\tilde{U}_{\tilde{e}}\oplus K\langle\tilde{v}_2,\dots,\tilde{v}_s\rangle)$,

$A_3=\phi(K(\tilde{e}+\tilde{u}_1)\oplus\tilde{V}_{\tilde{e}})$, $\qquad A_4=\phi(K\tilde{e}\oplus\tilde{U}_{\tilde{e}}\oplus K\langle\tilde{v}_1,\dots,\tilde{v}_{s-1}\rangle)$,

and work as in (a).

(c) Finally, if $s=1$, we have $r=n-s=n-1\geq 2$. Now we take, for example:

$$\phi(K\tilde{e}\oplus K\langle\tilde{u}_2,\dots,\tilde{u}_r\rangle\oplus\tilde{V}_{\tilde{e}}),$$

$$\phi(K\tilde{e}\oplus K\langle\tilde{u}_1,\dots,\tilde{u}_{r-1}\rangle\oplus\tilde{V}_{\tilde{e}}),$$

$$\phi(K(\tilde{e}+\tilde{u}_1)\oplus K\langle\tilde{u}_2,\dots,\tilde{u}_r\rangle\oplus\tilde{V}_{\tilde{e}}),$$

and we conclude that there are in A two trivial Bernstein algebras of type (r, s).

Put $A_1=\phi(K\tilde{e}\oplus\tilde{U}_{\tilde{e}})$. If it is not contained in $\mathrm{Ker}\,\omega$, it is a Bernstein algebra, and hence it is a trivial Bernstein algebra of type (r+1, s-1) by the induction assumption. Otherwise $A_1=\mathrm{Ker}\,\omega$ and each of its 2-dimensional subalgebras has an infinite number of subalgebras.

We consider: $Kx_0=\phi(K\tilde{u}_0)$, where we put $\tilde{u}_0=\tilde{e}$, $Kx_i=\phi(K\tilde{u}_i)$, for $i=1,\dots,r$. If $x_0,\dots,x_r$ are not linearly independent, there must exist i such that:

$$0\neq Kx_i\cap(Kx_0\vee\overset{\wedge}{\underset{\dots}{i}}\vee Kx_r)=\phi(K\tilde{u}_i)\cap(\phi(K\tilde{u}_0)\vee\overset{\wedge}{\underset{\dots}{i}}\vee\phi(K\tilde{u}_r))=$$

$$=\phi(K\tilde{u}_i\cap(K\tilde{u}_0\vee\overset{\wedge}{\underset{\dots}{i}}\vee K\tilde{u}_r))=\phi(K\tilde{u}_i\cap(K\tilde{u}_0+\overset{\wedge}{\underset{\dots}{i}}+K\tilde{u}_r)).$$

Then $0\neq K\tilde{u}_i\cap(K\tilde{u}_0+\overset{\wedge}{\underset{\dots}{i}}+K\tilde{u}_r)$, which is a contradiction. Hence $\{x_0,\dots,x_r\}$ is a basis of $A_1=\mathrm{Ker}\,\omega$ over K.

Besides that, for all $i\neq j$ the algebra $Kx_i\vee Kx_j$ is a 2-dimensional subalgebra of $\mathrm{Ker}\,\omega$ and it has an infinite number of subalgebras, hence it is trivial. Thus we have $\mathrm{Ker}\,\omega$ is also trivial.

On the other hand, we can take the two trivial Bernstein algebras of type (r, s=1) which are contained in A:

$B = Ke \oplus W \oplus Kv$, with $0 \neq v$ in V_e, $W \subsetneq U_e$

$B_1 = Ke_1 \oplus W_1 \oplus Kv_1$, with $0 \neq v_1$ in V_{e_1}, $W_1 \subsetneq U_{e_1}$.

If U_e is r-dimensional, A is of type (r+1, s) and trivial, hence isomorphic to $\widetilde{A}$, but we are supposing this is false. Therefore U_e is (r-1)-dimensional and $W = U_e$, $W_1 = U_{e_1}$, V_e and V_{e_1} are 2-dimensional. But since A is a trivial Bernstein algebra, we can write $V_e = V_{e_1}$, $U_e = U_{e_1}$ and so we can put:

$B = Ke \oplus U_e \oplus Kv$, $B_1 = Ke_1 \oplus U_e \oplus Kv_1 = Ke \oplus U_e \oplus Kv_1$, with v, v_1 in V_e.

Since $B \neq B_1$, the elements v and v_1 must be linearly independent and besides that, we know the multiplication table of A. If we consider U, an (r - 2)-dimensional subspace of U_e (we include the case r - 2 = 0 !), and the subalgebra $A_2 = Ke \oplus U \oplus V_e$ of A, which is n-dimensional, we can take $\phi^{-1}(A_2)$ and it cannot be Ker $\tilde{\omega}$ since it has a 2-dimensional subalgebra, $\phi^{-1}(Ke \oplus Kv)$, with only 2 subalgebras, and that is not possible in a trivial algebra. Hence $\phi^{-1}(A_2)$ is a trivial Bernstein algebra of type (r -1, 2) by the induction assumption, which is impossible since $\widetilde{A}$ has type (r+1, 1).

(II) For any e, idempotent in A, it is easy to see that $U_e^2 = 0$:

For if there exists u in U_e such that $u^2 \neq 0$, then the subalgebra $Ku \oplus Ku^2$ would have only one subalgebra, which is not possible in A.

(III) Finally, we will show that A must be isomorphic to $\widetilde{A}$.

We consider in A the two subalgebras of (I):

$B = Ke \oplus W \oplus H$, a trivial Bernstein algebra with $e^2 = e$, W an (r -1)-dimensional subspace of U_e, H a s-dimensional subspace of V_e, and:

$B_1 = Ke_1 \oplus W_1 \oplus H_1$, a trivial Bernstein algebra with $e_1^2 = e_1$, W_1 an r-dimensional subspace of U_{e_1}, H_1 a (s -1)-dimensional subspace of V_{e_1}.

We can conclude from these decompositions that the type of A is (r+1, s) and as a consequence of (II) $U_{e_1} = U_e$ and then we have:

$B = Ke \oplus W \oplus H = Ke \oplus W \oplus V_e$, $B_1 = Ke_1 \oplus W_1 \oplus H_1 = Ke_1 \oplus U_e \oplus H_1$

$U_e^2 = 0$, $V_e^2 = 0$, $W\,V_e = 0$, $U_e\,H_1 = 0$.

Now, we must show $U_eV_e = 0$ and this will complete the proof.

It is easy to see that two different 1-dimensional subalgebras of $\tilde{A}$ generate a subalgebra of dimension 2 or 3, hence the same must happen in A. Besides that, if the algebra is 3-dimensional, it has a 2-dimensional subalgebra with only 2 subalgebras.

Let us suppose there exist u in U_e and v in V_e with $0 \neq uv$ in U_e. Then $u \notin W$ and $v \notin H_1$ and we can write $U_e = Ku \oplus W$, $V_e = Kv \oplus H_1$.

Now if $Ku \vee Kv$ is 2-dimensional, it is equal to $Ku \oplus Kv$, hence $uv \in Ku$ and $Ke \oplus Ku \oplus Kv \cong A_{(5)}$. As in the last part of the proof of (I), $\phi^{-1}(Ke \oplus Ku \oplus Kv)$ cannot be in Ker $\tilde{\omega}$, and hence it is a Bernstein algebra isomorphic to $A_{(5)}$, which cannot be in $\tilde{A}$.

If $Ku \vee Kv$ is 3-dimensional, the work is a bit harder.

Clearly $uv \notin Ku$; we put $uv = \lambda u + w$, w in W. If we had $\lambda \neq 0$, by taking $u_1 = uv$, we would have $u_1 v = \lambda u_1$, and we could work as above. Therefore, we can suppose $uv \in W$ and then $Ku \vee Kv = Ku \oplus Kv \oplus K(uv)$ has a 2-dimensional subalgebra which has exactly 2 subalgebras. But since we are working in Ker ω, this latter subalgebra is isomorphic to $B_{(3)}$, say $Ka \oplus Kb$, $a^2 = b^2 = 0$, $ab = b$. Then if one writes a and b as a linear combination of u, v, and uv, from $ab = b$, one obtains $b = 0$, which is a contradiction.•

Finally we include here another extreme case of Bernstein algebra of any dimension which is also determined by its lattice of subalgebras.

LEMMA: *Any non trivial (n+1)-dimensional Bernstein algebra of type (2, n-1) has the property that for any idempotent e, either* $U_eV_e + V_e^2 = 0$ *or* $U_e^2 = 0$.

PROOF: We write $A = Ke \oplus Ku \oplus V_e$, the decomposition of A with respect to an idempotent e of A, and let us suppose $u^2 \neq 0$. We define the bilinear form:

$$F:\ V_e \times V_e \rightarrow K$$

by $F(v, w) \in K$ such that $vw = F(v, w)\,u$.

From the elementary theory of bilinear forms, we can get a basis of V_e, $\{v_1, \ldots, v_{n-1}\}$ such that $v_i v_j = 0$ for $i \neq j$. We put $v_i^2 = a_i u$, $u v_i = b_i u$.

As one can see in [3], $u(u v_i) = 0$ and $(v_i^2)^2 = 0$. From this we have $a_i = b_i = 0$, which concludes the proof.•

If a Bernstein algebra satisfies the condition $U_e V_e + V_e^2 = 0$ for any idempotent e it is called in [3] a *normal Bernstein algebra.*

As a consequence of this lemma, we have:

COROLLARY: *Over a commutative field of characteristic different from 2 there exist, up to isomorphism, exactly one non trivial, normal Bernstein algebra of type (2, n-1) :* $A = Ke \oplus Ku \oplus V$, *where* $e^2 = 0$, $eu = \frac{1}{2}u$, $eV = 0$, $uV = 0$, $V^2 = 0$, $0 \neq u^2$ *is in V, and* $V = V_e$ *does not depend on the choice of e.* •

Now we can prove the following theorem.

THEOREM 6: *If two Bernstein algebras are* $\mathfrak{L}$*-isomorphic and one of them is a normal Bernstein algebra (n+1)-dimensional of type (2, n-1), then they are isomorphic.*

PROOF: Let (A, ω) be a normal Bernstein algebra (n+1)-dimensional of type (2, n-1); we can suppose it is not trivial by theorem 5. We write as above $A = Ke \oplus Ku \oplus V$ and put $\{u^2, v_2, \ldots, v_{n-1}\}$, a basis of V. Let $(\tilde{A}, \tilde{\omega})$ be a Bernstein algebra $\mathfrak{L}$-isomorphic to A. As we have already proved the result for $n = 1, 2$, we can take $n \geq 3$.

It is easy to see some properties of the lattice of subalgebras of A:

-the 1-dimensional subalgebras of A are Ke_1, with $e_1^2 = e_1$, and Kv for any v in V.

-the 2-dimensional subalgebras of A are:

$Kv \oplus Kw$, v, w in V, a trivial algebra isomorphic to $B_{(1)}$, with an infinite number of subalgebras

$Ku^2 \oplus K(u + v)$, v in V, which is isomorphic to $B_{(2)}$, with Ku^2 as its only subalgebra.

$Ke_1 \oplus Kv$, $e_1^2 = e_1$, v in V, i.e., the Bernstein algebra of type (1, 1).

Hence the subalgebras of the form Ke_1 are contained only in these latter subalgebras, and as we are supposing $n \geq 3$, the dimension of V is at least 2 and there are an infinite number of subalgebras containing Ke_1.

We can also take $Ke \oplus V$ and $K(e+u) \oplus V$, two different subalgebras which are trivial Bernstein algebras of type (1, n-1). In this way, if we consider their images under ϕ, at most one of them coincides with Ker $\tilde{\omega}$ and so, at least one of them is a trivial Bernstein algebra of type (1, n-1) by using theorem 4. We put for this algebra $K\tilde{e} \oplus W$, W a (n-1)-dimensional subspace of $\tilde{V}_{\tilde{e}}$. If $\tilde{V}_{\tilde{e}}$ were n-dimensional, $\tilde{A}$ should be trivial and so should be A, again by theorem 4, but it is not. Then $W = \tilde{V}_{\tilde{e}}$ and $\tilde{A}$ is of type (2, n-1), with $\tilde{V}_{\tilde{e}}^2 = 0$.

If we proof $\tilde{U}_{\tilde{e}} \tilde{V}_{\tilde{e}} = 0$, $\tilde{A}$ will be normal, and since it cannot be trivial by theorem 5, we would conclude the result by using the preceding corollary.

Let us suppose the contrary and we will get a contradiction. By the previous lemma $\tilde{U}_{\tilde{e}}^2$ must be 0.

We can take $\tilde{v}$ in $\tilde{V}_{\tilde{e}}$ such that, putting $\tilde{U}_{\tilde{e}} = K\tilde{u}$, we have $\tilde{u}\tilde{v} = \tilde{u}$. Hence $K\tilde{u} \oplus K\tilde{v} \cong B_{(3)}$ and $K\tilde{u}$, $K\tilde{v}$ are its subalgebras. Hence $K\tilde{u} \oplus K\tilde{v} = \phi(Ke_1 \oplus Kv)$, the image by ϕ of one 2-dimensional subalgebra of A with only 2 subalgebras, and $\phi(Ke_1) = K\tilde{u}$ or $\phi(Ke_1) = K\tilde{v}$. In either case this means that $\phi(Ke_1)$ is in a 2-dimensional subalgebra with an infinite number of subalgebras, and that is not possible.•

REMARK: The results we have obtained are also valid if the field K is finite but has at least 5 elements, in order to have the identities and properties of the decomposition respect an idempotent (see [3]).

This paper is just a first approach to lattice theory applied to the study of Bernstein algebras. The "high degree of nilpotency" of these algebras makes one think that the lattice of subalgebras determines the structure of the algebra to a "high degree" since the techniques of this paper are simple in general.

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