LOW DIMENSIONAL BERNSTEIN–JORDAN ALGEBRAS

TERESA CORTÉS AND FERNANDO MONTANER

0. *Introduction*

Bernstein algebras were introduced by Holgate [**5**] as an algebraic formulation of the problem of classifying the stationary evolution operators in genetics (see [**7**]). Since then, many authors have contributed to the study of these algebras, and there is a fairly extensive bibliography on the subject. Known results include classification theorems for some types of Bernstein algebras (for instance, Bernstein algebras of dimension less than or equal to four, see [**10**, **8**, **2**]) as well as some other structural results (see for instance [**1**, **5**, **9**, **4**]).

In [**3**], the authors suggested an approach to the study of the structure of Bernstein algebras through two main ideas: direct products and the related concepts of decomposability and indecomposability, and the reduction of the general problem to the study of Bernstein–Jordan algebras. The procedure depended on getting what was called a reduced Bernstein algebra (which is, in particular, a Jordan algebra), and on the description of the indecomposable factors of the algebra so obtained. In this paper we use these ideas to classify Bernstein–Jordan algebras of low dimension. Thus, in Section 3 we describe reduced Bernstein algebras of dimension less than or equal to 5 through their indecomposable factors. We use this information in Section 4 to recover all Bernstein–Jordan algebras of dimension less than or equal to 5 by gluing together reduced algebras and the trivial ideal that, when factored out, creates reduced algebras. After a Section 1 of preliminaries, where we collect several known facts about Bernstein algebras which will be used later, we devote Section 2 to proving some technical lemmas which will ease the computations of Section 3.

1. *Preliminaries*

Throughout this paper Φ will denote an infinite field of characteristic not two.

A finite dimensional commutative algebra A over the field Φ together with a homomorphism of algebras (weight homomorphism) $\omega: A \to \Phi$ is called a *Bernstein algebra* if every $x \in A$ satisfies $(x^2)^2 = \omega(x)^2 x^2$.

Next we recall some known results that can be found in [**10**].

For any Bernstein algebra A, the weight homomorphism is uniquely determined.

Each Bernstein algebra possesses at least one idempotent, and to each idempotent $e \in A$ is associated a Peirce decomposition $A = \Phi e + U_e + V_e$, where

$$U_e = \{x \in A \mid 2ex = x\} \quad \text{and} \quad V_e = \{x \in A \mid ex = 0\},$$

and $\operatorname{Ker}\omega = U_e + U_e$. The Peirce subspaces multiply according to

$$U_e V_e \subseteq U_e, \quad V_e^2 \subseteq U_e, \quad U_e^2 \subseteq V_e, \quad U_e V_e^2 = 0.$$

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The set of idempotent elements of A is given by $\mathrm{I}(A) = \{e+\sigma+\sigma^2 \mid \sigma\in U_e\}$, where e is any given idempotent of A. For two idempotents e and $f = e+\sigma+\sigma^2$, we have the following relations between the corresponding Peirce spaces $U_f = \{u+2\sigma u \mid u\in U_e\}$ and $V_f = \{v-2v(\sigma+\sigma^2) \mid v\in V_e\}$. This shows that the numbers $\dim U_e$ and $\dim V_e$ do not depend on the idempotent e. The pair $(\dim U_e+1, \dim V_e)$ is called the *type* of A.

Also we have the following identities which hold in any Bernstein algebra:

(1) $(xy)(zt)+(xz)(yt)+(xt)(yz) = 0$ for any $x, y, z, t\in \operatorname{Ker}\omega$,
(2) $u_1(u_2 v)+u_2(u_1 v) = 0$,
(3) $u^3 = 0$, and its linearization $u_1(u_2 u_3)+u_2(u_3 u_1)+u_3(u_1 u_2) = 0$,
(4) $u^2(uv) = 0$, for any $u, u_1, u_2, u_3\in U_e$, and any $v\in V_e$.

If we have that the algebra A is in fact a Jordan algebra, we also know that $V_e^2 = 0$ for any idempotent e and as a consequence [1]:

(5) $(uv_1)v_2+(uv_2)v_1 = 0$ for any $u\in U_e$ and any $v_1, v_2\in V_e$.

In this case, the Peirce subspaces with respect to idempotents e, f, related by $f = e+\sigma+\sigma^2$ with $\sigma\in U_e$, satisfy

$$U_f = \{u+2\sigma u \mid u\in U_e\} \quad\text{and}\quad V_f = \{v-2\sigma v \mid v\in V_e\}.$$

Finally we recall the concepts on which will be based the study we shall carry out in the following sections; they can be found in [3].

Let $(A_\iota, \omega_\iota)_{\iota\in I}$ be a family of Bernstein algebras. We consider in $\prod_{\iota\in I} A_\iota$ the set

$$\times_{\iota\in I}(A_\iota, \omega_\iota) = \{(x_\iota)\in\textstyle\prod_{\iota\in I} A_\iota \mid \omega_i(x_i) = \omega_j(x_j) \text{ for all } i,j\in I\}.$$

It is possible to define a weight homomorphism on $\times_{\iota\in I}(A_\iota, \omega_\iota)$ by setting $\omega((x_\iota)) = \omega_i(x_i)$ which clearly is independent of the index i. Then it is straightforward to prove that $(\times_{\iota\in I}(A_\iota, \omega_\iota), \omega)$ is a Bernstein algebra which is the direct product of the family under consideration.

A Bernstein algebra is called *decomposable* if it is isomorphic to the direct product of at least two Bernstein algebras none of them isomorphic to the ground field. Otherwise, it is called *indecomposable*. Any finite-dimensional Bernstein algebra is a direct product of indecomposable Bernstein algebras. On the other hand it is known [3, Lemma 2.3] that a Bernstein algebra is decomposable if and only if $\operatorname{Ker}\omega$ is a direct sum of two nonzero ideals of the algebra.

In any Bernstein algebra A the intersection of all the subspaces U_e, with $e\in \mathrm{I}(A)$, is an ideal of A denoted by $U_0(A)$. For any $e\in\mathrm{I}(A)$ it is readily shown that $U_0(A) = \{u\in U_e \mid uU_e = 0\}$. A Bernstein algebra A for which $U_0(A) = 0$ is called *reduced*. A reduced algebra is always a Jordan algebra. Reduced algebras are easily obtained because for any algebra A, the quotient algebra $A/U_0(A)$ turns out to be reduced.

2. *Technical lemmas*

In this section we prove some facts which will allow us to simplify the description of Bernstein–Jordan algebras. The central result Lemma 2.2 imposes restrictions on the dimension of a Bernstein algebra for which some strings of products do not quickly reach zero. In particular, for low dimensional algebras this forces orthogonality of the subspaces U_e and V_e of the Peirce decomposition attached to the idempotent e.

Define the subspaces $H_e^1 = U_e V_e$, $H_e^k = H_e^{k-1} V_e$ for $k \geqslant 2$. We first show that the least index k for which $H_e^k = 0$ does not depend on the choice of the idempotent e.

LEMMA 2.1. *Let A be a Bernstein algebra which is a Jordan algebra and let $e, f \in \mathrm{I}(A)$ be related by $f = e + \sigma + \sigma^2$ with $\sigma \in U_e$. Then $H_f^k \subseteq H_e^k + H_e^k \sigma + H_e^k \sigma^2$ and in particular, $H_e^k = 0$ if and only if $H_f^k = 0$.*

Proof. We will carry out an induction on k.

The subspace $H_f^1 = U_f V_f$ is generated as a vector space by the products $u'v'$ with $u' \in U_f$ and $v' \in V_f$. Put $u' = u + 2\sigma u$ with $u \in U_e$ and $v' = v - 2\sigma v$ with $v \in V_e$. We have

$$u'v' = uv + 2(\sigma u) v - 2(\sigma v) u - 4(\sigma u)(\sigma v) = uv + 2(uv)\sigma + 2(uv)\sigma^2 \in H_e^1 + H_e^1 \sigma + H_e^1 \sigma^2,$$

since $(\sigma u) v \in U_e^2 V_e \subseteq V_e^2 = 0$, $-(\sigma v) u = (uv)\sigma$ by (2), and $-2(\sigma u)(\sigma v) = (uv)\sigma^2$ by (1).

Now if the result holds up to the exponent k, then

$$H_f^{k+1} = H_f^k V_f \subseteq (H_e^k + H_e^k \sigma + H_e^k \sigma^2)(V_e + V_e \sigma).$$

Performing the products we obtain the following:

$$\begin{array}{ll}
H_e^k V_e = H_e^{k+1}, \quad (H_e^k \sigma) V_e \subseteq U_e^2 V_e \subseteq V_e^2 = 0, & \\
(H_e^k \sigma^2) V_e = (H_e^k V_e)\sigma^2 = H_e^{k+1} \sigma^2 & \text{by (5)}, \\
H_e^k (V_e \sigma) = \sigma(V_e H_e^k) = H_e^{k+1} \sigma & \text{by (2)}, \\
(H_e^k \sigma)(V_e \sigma) \subseteq (H_e^k V_e)\sigma^2 = H_e^{k+1} \sigma^2 & \text{by (1)}, \\
(H_e^k \sigma^2)(V_e \sigma) \subseteq ((\sigma V_e)\sigma^2) H_e^k = 0 & \text{by (2) and (4)}.
\end{array}$$

In the next result we show that the characteristic number k so attached to the algebra A provides a lower bound for the dimension of the subspaces U_e.

LEMMA 2.2. *Let A be a reduced Bernstein algebra. If $H_e^k \neq 0$ for some idempotent e and some $k \equiv 1$ or $2 \pmod 4$, then $\dim U_e \geqslant 2^{k+1}$.*

Proof. Fix an idempotent e in A and consider elements $u, z \in U_e$ and $v_1, \ldots, v_p \in V_e$. We shall make use of the following properties of the expression

$$h(u, v_1, \ldots, v_p) = (\ldots((uv_1)v_2)\ldots)v_p.$$

(a) We have $h(u, v_1, \ldots, v_p) z = (-1)^p h(z, v_p, \ldots, v_1) u$. Indeed, for $p = 1$ this is just (2) and by induction:

$$\begin{aligned}
h(u, v_1, \ldots, v_p) z &= h((\ldots((uv_1)v_2)\ldots)v_{p-1}, v_p) z \\
&= -h(z, v_p)((\ldots((uv_1)v_2)\ldots)v_{p-1}) = -(zv_p) h(u, v_1, \ldots, v_{p-1}) \\
&= -(-1)^{p-1} h(zv_p, v_{p-1}, \ldots, v_1) u = (-1)^p h(z, v_p, \ldots, v_1) u.
\end{aligned}$$

(b) We have $h(u, v_1, \ldots, v_p) = (-1)^{\operatorname{sign}\pi} h(u, v_{\pi(1)}, \ldots, v_{\pi(p)})$ for any permutation $\pi \in S_p$, since $H_e^t \subseteq U_e$, and any t, so we can apply (5) successively. In particular, $h(u, v_1, \ldots, v_p) v_i = 0$, for any $1 \leqslant i \leqslant p$.

(c) From (a) and (b) we obtain $h(u, v_1, \ldots, v_p) z = (-1)^{p(p+1)/2} h(z, v_1, \ldots, v_p) u$, so that in particular when $k \equiv 1$ or $2 \pmod 4$ this makes $h(u, v_1, \ldots, v_k) u = 0$.

Now, since we have $H_e^k \neq 0$, we can take elements $u \in U_e$, $v_1, \ldots, v_k \in V_e$ such that $h(u, v_1, \ldots, v_k) \neq 0$. Since A is reduced there exists some $z \in U_e$ such that $h(u, v_1, \ldots, v_k) z \neq 0$ and so $h(z, v_1, \ldots, v_k) u \neq 0$ by the former considerations. We set

$$u_0 = u, \quad z_0 = z, \quad u_{i_1 \ldots i_t} = h(u, v_{i_1}, \ldots, v_{i_t}), \quad z_{i_1 \ldots i_t} = h(z, v_{i_1}, \ldots, v_{i_t})$$

for $1 \leqslant i_1 < \ldots < i_t \leqslant k$. Next we show that these are linearly independent elements in U_e. Assume that there exists $\lambda_0, \mu_0, \lambda_{i_1 \ldots i_t}, \mu_{i_1 \ldots i_t} \in \Phi$ for $1 \leqslant i_1 < \ldots < i_t \leqslant k$, such that

$$\sum \lambda_{i_1 \ldots i_t} u_{i_1 \ldots i_t} + \sum \mu_{i_1 \ldots i_t} z_{i_1 \ldots i_t} = 0.$$

Successively multiplying this expression by $v_1, \ldots, v_k$ and applying (b) we obtain

$$\lambda_0 h(u, v_1, \ldots, v_k) + \mu_0 h(z, v_1, \ldots, v_k) = 0.$$

Next multiplying this expression by u we get, using (c), $\mu_0 h(z, v_1, \ldots, v_k) u = 0$, which implies that $\mu_0 = 0$ and $\lambda_0 h(u, v_1, \ldots, v_k) = 0$, hence $\lambda_0 = 0$.

We give a partial order to the set of indexes $\{i_1 \ldots i_t \mid 1 \leqslant i_1 < \ldots < i_t \leqslant k\}$ setting $i_1 \ldots i_t \leqslant i_1 \ldots i_l$ if either $t < l$ or $t = l$ and the inequality holds in the lexicographical order. Now suppose that $i_1 \ldots i_m$ is the least index for which either $\lambda_{i_1 \ldots i_m} \neq 0$ or $\mu_{i_1 \ldots i_m} \neq 0$. Multiplying the expression $\sum \lambda_{i_1 \ldots i_t} u_{i_1 \ldots i_t} + \sum \mu_{i_1 \ldots i_t} z_{i_1 \ldots i_t} = 0$ by $v_{j_1} \ldots v_{j_{k-m}}$, where $\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_{k-m}\} = \{1, \ldots, k\}$ and applying (b) we obtain

$$\lambda_{i_1 \ldots i_m} h(u, v_1, \ldots, v_k) + \mu_{i_1 \ldots i_m} h(z, v_1, \ldots, v_k) = 0.$$

Next multiplying this expression by u we get by (c) $\mu_{i_1 \ldots i_m} h(z, v_1, \ldots, v_k) u = 0$, which implies $\mu_{i_1 \ldots i_m} = 0$ and $\lambda_{i_1 \ldots i_m} h(u, v_1, \ldots, v_k) = 0$, hence $\lambda_{i_1 \ldots i_m} = 0$, thus giving a contradiction and proving that the elements $u_{i_1 \ldots i_t}, z_{i_1 \ldots i_t}$ are linearly independent. Since there are 2^{k+1} of them, the result holds.

Finally we include here two more results which will be used in the next section when we study reduced algebras.

LEMMA 2.3. *Let A be a reduced indecomposable Bernstein algebra. If for some idempotent e of A we have $U_e \neq 0$ and $U_e V_e = 0$, then $U_e^2 = V_e$.*

Proof. Set $V_e = U_e^2 + S$ a direct sum of vector spaces. Then clearly S and $U_e + U_e^2$ are ideals of A and their sum composes $\operatorname{Ker} \omega$. Since A is indecomposable, that means that $S = 0$ or $U_e = 0$.

LEMMA 2.4. *Let A_1, A_2 be two Bernstein algebras. Then, $A_1 \times A_2$ is reduced if and only if A_1 and A_2 are reduced.*

3. *Low dimensional reduced Bernstein algebras*

In what follows Φ will be an algebraically closed field of characteristic not 2.

This section is devoted to the classification of reduced algebras of dimension less than or equal to 5. Following the approach of [3] we start with a description of indecomposable reduced algebras.

Bernstein algebras of dimension less than or equal to 4 which satisfy these conditions are easily obtained from the known classifications [**10**, **2**] together with [**3**]. These algebras are

$$A_0 = \Phi e, \qquad A_2 = \Phi\langle e, u, v\rangle \quad \text{with } u^2 = v,$$
$$A_1 = \Phi\langle e, v\rangle, \quad A_3 = \Phi\langle e, u_1, u_2, v\rangle \quad \text{with } u_1 u_2 = v,$$

reduced.

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where e is an idempotent and $u, u_1, u_2 \in U_e$, $v \in V_e$, and all other products are equal to zero.

With respect to 5-dimensional algebras, note first that if $U_e V_e \neq 0$, by Lemma 2.2 we have $\dim U_e \geqslant 4$. Hence $U_e = \operatorname{Ker}\omega$ and $U_e^2 = 0$, contradicting the indecomposability of A. Thus $U_e V_e = 0$ and since A is reduced and indecomposable, Lemma 2.3 implies that $U_e = 0$ or $U_e^2 = V_e$. Now we consider separately each of the possible types of a 5-dimensional algebra.

Type (1, 4): the only possibility is $A = \Phi e + V_e$ with $V_e^2 = 0$, which is clearly decomposable.

Type (2, 3): in any Peirce decomposition we have $\dim U_e = 1$, hence $\dim U_e^2 \leqslant 1$, and since $\dim V_e = 3$, we cannot have $U_e^2 = V_e$. Thus there are no indecomposable reduced algebras of this type.

Type (3, 2): write $A = \Phi e + U_e + V_e$. Since $U_e^2 = V_e$, there exists some $u \in U_e$ with $u^2 \neq 0$. Pick u_1, an element satisfying these conditions and put $v_1 = u_1^2$ and $U_e = \Phi\langle u_1, u_2\rangle$. We consider two cases.

Case 1: $\{u_1^2, u_2^2\}$ are linearly dependent so that $u_2^2 = \lambda v_1$ with $\lambda \in \Phi$ and $V_e = U_e^2 = \Phi\langle v_1, v_2\rangle$ with $v_2 = u_1 u_2$.

If $\lambda = 0$, this algebra is obviously reduced. Now let us show that it is also indecomposable. First notice that any non-zero ideal I of A contained in $\operatorname{Ker}\omega$ has nonzero intersection with V_e. Indeed, $I = I \cap U_e + I \cap V_e$, hence if $I \cap V_e = 0$, then $I \cap U_e \neq 0$. But if $0 \neq \alpha u_1 + \beta u_2 \in I$, then $0 \neq (\alpha u_1 + \beta u_2) u_1 = \alpha v_1 + \beta v_2 \in I$. On the other hand, if $\operatorname{Ker}\omega = I_1 + I_2$ is a direct sum of such ideals, since $U_e = I_1 \cap U_e + I_2 \cap U_e$, either I_1 or I_2 contains an element of the form $\alpha u_1 + \beta u_2$ with $\alpha \neq 0$. Thus it contains the elements $(\alpha u_1 + \beta u_2) u_2 = \alpha v_2$ and $(\alpha u_1 + \beta u_2) u_1 = \alpha v_1 + \beta v_2$ and hence is contains V_e. Since we have just shown that any of these ideals intersects V_e nontrivially, $I_1 \cap I_2 \neq 0$. We shall denote this algebra by B_4.

If $\lambda \neq 0$, since Φ is algebraically closed we can first multiply by suitable scalars to obtain $u_1^2 = u_2^2 = v_1$ and $u_1 u_2 = v_2$. Then the elements $z_1 = u_1 + u_2$, $z_2 = u_1 - u_2$, $w_1 = 2(v_1 + v_2)$, $w_2 = 2(v_1 - v_2)$ satisfy $z_1^2 = w_1$, $z_2^2 = w_2$, $z_1 z_2 = 0$, which gives the following decomposition of $\operatorname{Ker}\omega$ as a direct sum of ideals of A:

$$\operatorname{Ker}\omega = \Phi\langle z_1, w_1\rangle + \Phi\langle z_2, w_2\rangle.$$

Thus A is decomposable.

Case 2: $\{u_1^2, u_2^2\}$ are linearly independent so that $u_2^2 = v_2$ and $u_1 u_2 = a v_1 + b v_2$, with $a, b \in \Phi$.

If $a = 0$, setting $z_1 = u_1 - b u_2$, $z_2 = u_2$, $w_1 = v_1 - b^2 v_2$, $w_2 = v_2$, we obtain that A is decomposable as in the last part of Case 1. The same argument applies to the case $b = 0$. Hence we can assume that $a \neq 0 \neq b$. On the other hand, imposing $(u_1 + \alpha u_2)(u_1 + \beta u_2) = 0$, we obtain that α and β should be the solutions of the equation $a x^2 + x + b = 0$. If this equation had two different solutions, we would get two linearly independent elements in U_e having zero product and this would imply, as before, the decomposability of A. Thus $4ab = 1$ and taking $z_1 = u_1$, $z_2 = u_2 - 2a u_1$, $w_1 = v_1$, $w_2 = (1/4a) v_2 - a v_1$, we obtain the multiplication table of B_4.

Type (4, 1): write $A = \Phi e + U_e + V_e$ with $V_e = \Phi v$. The product in U_e can be described by a bilinear form F: $U_e \times U_e \to \Phi$ given by $F(u, u') = \lambda$ if $uu' = \lambda v$. As A is reduced, it follows that F is nondegenerate and we can find a basis $\{u_1, u_2, u_3, u_4\}$ of U_e such that $u_i^2 = v$, $u_i u_j = 0$ for $i \neq j$ since Φ is algebraically closed. Reciprocally, this implies that A is reduced. On the other hand, it is also indecomposable. For if I is a nonzero ideal of A contained in $\operatorname{Ker}\omega$, it is easily seen that $I \cap V_e \neq 0$, hence $V_e \subseteq I$. We shall denote this algebra by A_4.

Type (5, 0): the only possibility is $A = \Phi e + U_e$, with $U_e^2 = 0$, which is obviously decomposable.

We summarize the previous discussion in the following.

PROPOSITION 3.1. *A Bernstein algebra of dimension less than or equal to* 5 *over an algebraically closed field of characteristic not* 2 *is reduced and indecomposable if and only if it is isomorphic to one of the following*: A_0, A_1, A_2, A_3, A_4, B_4.

From Lemma 2.4 we know that all the reduced Bernstein algebras of dimension less than or equal to 5 can be obtained as direct products of the algebras listed in Proposition 3.1. So we get the following theorem.

THEOREM 3.2. *A Bernstein algebra of dimension less than or equal to* 5 *over an algebraically closed field of characteristic not* 2 *is reduced if and only if it is isomorphic to one of the following*:

(i) A_0 *of dimension* 1,
(ii) A_1 *of dimension* 2,
(iii) A_2, $A_1 \times A_1$ *of dimension* 3,
(iv) A_3, $A_1 \times A_2$, $A_1 \times A_1 \times A_1$ *of dimension* 4,
(v) A_4, B_4, $A_1 \times A_3$, $A_2 \times A_2$, $A_1 \times A_1 \times A_2$, $A_1 \times A_1 \times A_1 \times A_1$ *of dimension* 5.

4. *Low dimensional Bernstein–Jordan algebras*

In this section we prove our main result, constructing all Bernstein–Jordan algebras of dimension less than or equal to 5 over an algebraically closed field of characteristic not 2. Bernstein algebras of dimension less than 5 over any infinite field of characteristic not 2 have already been classified in [**10**, **2**]. Thus, to get all Bernstein–Jordan algebras of these dimensions we only have to look at those classifications. The algebras that appear with the additional hypothesis of the field being algebraically closed are the following:

of dimension 1: A_0,
of dimension 2: A_1, $C_1 = \Phi\langle e, u\rangle$ with $u^2 = 0$,
of dimension 3: $C_1 \times C_1$, $A_1 \times A_1$, $C_1 \times A_1$, A_2,
of dimension 4: $C_1 \times C_1 \times C_1$, $A_1 \times A_1 \times A_1$, $C_1 \times A_1 \times A_1$, $A_1 \times A_2$,
$C_1 \times C_1 \times A_1$, $C_1 \times A_2$, A_3, $C_3 = \langle e, u_1, u_2, v\rangle$ with $u_1 v = u_2$.

With respect to 5-dimensional algebras the procedure we shall use is based on the description of reduced algebras given in Section 3. Let A be a Bernstein–Jordan algebra of dimension 5 over Φ. The algebra $A/U_0(A)$ is reduced, hence it is one of the algebras listed in Theorem 3.2. Therefore, to obtain the original algebra A it is enough to solve the extension problem associated to the short exact sequence

$$U_0(A) \longrightarrow A \xrightarrow{\pi} A/U_0(A).$$

Here we know the multiplication in $A/U_0(A)$ and the multiplication in $U_0(A)$, which is trivial. Now, to get the multiplication in A we first take any idempotent in $A/U_0(A)$. Any of its preimages e in A is an idempotent of A, by [3, (3.7)]. Moreover, $U_0(A) \subseteq U_e$ by [3, (3.4)]. Then if $A = \Phi e + U_e + V_e$ is the Peirce decomposition of A with respect to e, we have $A/U_0(A) = \Phi(e+U_0(A)) + U_e/U_0(A) + (V_e + U_0(A))/U_0(A)$, the Peirce decomposition of $A/U_0(A)$ with respect to $e+U_0(A)$. Notice also that if u_1, $u_2 \in U_e$, $v \in V_e$, and $\pi(u_1)\pi(u_2) = \pi(v)$, then $u_1 u_2 - v \in U_0(A) \cap V_e = 0$, hence $u_1 u_2 = v$. On the other hand, if we want A to be a Jordan algebra, we have to impose $V_e^2 = 0$, and $(uv)v = 0$, for any $u \in U_e$, and any $v \in V_e$. So, we have only to determine the products uv with $u \in U_e$, $v \in V_e$. Since all reduced algebras appearing in Theorem 3.2 satisfy $U_e V_e = 0$, the algebra A must satisfy $U_e V_e \subseteq U_0(A)$. We shall next use these considerations to solve the extension problem for each of the algebras in Theorem 3.2.

If $A/U_0(A) = A_0$, then $A = A_0 + U_0(A) = \Phi e + U_e$, with $U_e^2 = 0$, hence A is $C_1 \times C_1 \times C_1 \times C_1$.

If $A/U_0(A) = A_1$, then $A = A_1 + U_0(A) = \Phi\langle e, u_1, u_2, u_3, v\rangle$, where the only possibly nonzero products in $\operatorname{Ker}\omega$ are the $u_i v$. Since the homomorphism $R_v: U_e \to U_e$ given by $R_v(u) = uv$ satisfies $R_v^2 = 0$, its minimal polynomial has to be x or x^2. In the first case A satisfies $(\operatorname{Ker}\omega)^2 = 0$ and hence it is isomorphic to $C_1 \times C_1 \times C_1 \times A_1$. In the second case, we can find a basis $\{u_1, u_2, u_3\}$ of U_e such that $u_1 v = u_2$, $u_2 v = u_3 v = 0$. This makes the algebra isomorphic to $C_1 \times C_3$.

If $A/U_0(A) = A_2$, then $A = A_2 + U_0(A) = \Phi\langle e, u_1, u_2, u_3, v\rangle$ with $u_1^2 = v$ and all other products in U_e equal to zero. Now $u_i v = u_i u_1^2 = -2u_1(u_i u_1) = 0$, by (3). Hence A is isomorphic to $C_1 \times C_1 \times A_2$.

If $A/U_0(A) = A_1 \times A_1$, then $A = A_1 \times A_1 + U_0(A) = \Phi\langle e, u_1, u_2, v_1, v_2\rangle$, where the only possibly nonzero products in $\operatorname{Ker}\omega$ are the $u_i v_j$. If $U_e V_e = 0$, then A is the algebra $C_1 \times C_1 \times A_1 \times A_1$. Next if $\dim U_e V_e = 1$, put $U_e V_e = \Phi z_2$ and $U_e = \Phi\langle z_1, z_2\rangle$. Notice first that $z_2 v_i = \lambda_i z_2$, hence from $(z_2 v_i)v_i = 0$, it follows that $z_2 v_i = 0$, for $i = 1, 2$. Thus, $z_1 v_i = \alpha_i z_2$, with, for example, $\alpha_1 \neq 0$. Set $w_1 = \alpha_1^{-1} v_1$, and $w_2 = \alpha_1 v_2 - \alpha_2 v_1$. This gives $z_2 w_i = 0$, $z_1 w_1 = z_2$ and $z_1 w_2 = 0$ and so A is the algebra $A_1 \times C_3$. Finally, we consider the case $U_e V_e = U_e$. The expression $((uw_1)w_2)w_3$ for $u \in U_e$, $w_i \in V_e$ is skew-symmetric in the w_i by (5). Since V_e has dimension 2 this implies that $((U_e V_e)V_e)V_e = 0$, which contradicts $U_e V_e = U_e$.

If $A/U_0(A) = A_3$, then $A = A_3 + U_0(A) = \Phi\langle e, u_1, u_2, u_3, v\rangle$ with $u_1 u_2 = v$ and all other products in U_e equal to zero. Now $u_1 v = u_1(u_1 u_2) = -\frac{1}{2}u_2 u_1^2 = 0$, and similarly $u_2 v = 0$. Next, $u_3 v = u_3(u_1 u_2) = -u_1(u_2 u_3) - u_2(u_3 u_1) = 0$. Hence $U_e V_e = 0$ and A is isomorphic to $C_1 \times A_3$.

If $A/U_0(A) = A_1 \times A_2$, then $A = A_1 \times A_2 + U_0(A) = \Phi\langle e, u_1, u_2, v_1, v_2\rangle$ with $u_1^2 = v_1$ and all other products in U_e equal to zero. Now $u_1 v_1 = u_1^3 = 0$, and $u_2 v_1 = u_2 u_1^2 = -2u_1(u_2 u_1) = 0$. Next $u_i v_2 = \alpha_i u_2 \in U_0(A)$. We have $0 = (u_2 v_2)v_2 = \alpha_2^2 u_2$, hence $u_2 v_2 = 0$. In case $\alpha_1 = 0$, we get the algebra $C_1 \times A_1 \times A_2$. Otherwise we can assume that $\alpha_1 = 1$. We next show that this algebra is indecomposable. Suppose that $\operatorname{Ker}\omega = I_1 + I_2$ is a direct sum of ideals of A. Since $U_e = U_e \cap I_1 + U_e \cap I_2$, one of these ideals, say I_1, contains an element of the form $\alpha u_1 + \beta u_2$ with $\alpha \neq 0$. Hence

$\alpha v_1 = (\alpha u_1 + \beta u_2) u_1$ and $\alpha u_2 = (\alpha u_1 + \beta u_2) v_2$ are in I_1. Hence, $\Phi\langle u_1, u_2, v_1\rangle$ is contained in I_1. On the other hand $V_e = V_e \cap I_1 + V_e \cap I_2$, so one of these ideals must contain an element of the form $\gamma v_1 + \delta v_2$ with $\delta \neq 0$. If that ideal is I_1 then $I_1 = \operatorname{Ker}\omega$ and $I_2 = 0$. If on the contrary $\gamma v_1 + \delta v_2 \in I_2$, then $\delta u_2 = (\gamma v_1 + \delta v_2) u_1 \in I_2$, hence $I_1 \cap I_2 \neq 0$, giving a contradiction. We shall denote this algebra by D_4.

If $A/U_0(A) = A_1 \times A_1 \times A_1$, then $A = A_1 \times A_1 \times A_1 + U_0(A) = \Phi\langle e, u, v_1, v_2, v_3\rangle$, where the only possibly nonzero products in $\operatorname{Ker}\omega$ are uv_i. From $(uv_i)v_i = 0$, we obtain as before $U_e V_e = 0$, so that $(\operatorname{Ker}\omega)^2 = 0$ and A is isomorphic to

$$C_1 \times A_1 \times A_1 \times A_1.$$

Finally we summarize all the preceding discussion in the following main theorem of this paper. Notice that any two of the algebras listed below are nonisomorphic. This follows from [3, Theorem 2.6] and the fact that the indecomposable algebras which appear are obviously nonisomorphic to each other.

THEOREM 4.1. *Any Bernstein–Jordan algebra of dimension less than or equal to* 5 *over an algebraically closed field of characteristic not* 2 *is isomorphic to exactly one of the following algebras*:

of dimension 1: A_0,

of dimension 2: C_1, A_1,

of dimension 3: $C_1 \times C_1$, $C_1 \times A_1$, $A_1 \times A_1$, A_2,

of dimension 4: $C_1 \times C_1 \times C_1$, $C_1 \times C_1 \times A_1$, $C_1 \times A_1 \times A_1$, $A_1 \times A_1 \times A_1$, $C_1 \times A_2$, $A_1 \times A_2$, C_3, A_3.

of dimension 5: $C_1 \times C_1 \times C_1 \times C_1$, $C_1 \times C_1 \times C_1 \times A_1$, $C_1 \times C_1 \times A_1 \times A_1$, $C_1 \times A_1 \times A_1 \times A_1$, $A_1 \times A_1 \times A_1 \times A_1$, $C_1 \times C_1 \times A_2$, $C_1 \times A_1 \times A_2$, $A_1 \times A_1 \times A_2$, $C_1 \times C_3$, $C_1 \times A_3$, $A_1 \times C_3$, $A_1 \times A_3$, $A_2 \times A_2$, D_4, B_4, A_4.

As a final remark, it can be pointed out that the methods used here also provide a way of facing the problem of classification of all Bernstein–Jordan algebras. However, as shown in [3], when the dimension rises, there appear infinite families of indecomposable reduced Bernstein–Jordan algebras of the same dimension. This suggests that the structure of indecomposable reduced algebras may be rather intricate, and shows the difficulties of a direct approach to the classification of these algebras. Therefore, unless new techniques for handling indecomposable reduced algebras are introduced, the general classification problem will remain intractable.

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Departamento de Matemáticas
Facultad de Ciencias
Universidad de Oviedo
Calvo Sotelo s/n
33007 Oviedo
Spain

Departamento de Matemáticas
Facultad de Ciencias
Universidad de Zaragoza
50009 Zaragoza
Spain