BOL. SOC. BRAS. MAT., VOL. 14 N.º 1 (1983), 63-80

On the derivation algebra of zygotic algebras for polyploidy with multiple alleles

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1. Introduction

The terminology and notations of this paper are those of [1] of which this one is a natural continuation. In that one, we have calculated the derivation algebra of $G(n+1,2m)$, the gametic algebra of a $2m$-ploid and $n+1$-allelic population. In particular, it was shown that the dimension of this derivation algebra depends only on n. The integer m is related to the nilpotence degree of certain nilpotent derivations of a basis ([1], th. 3 and 4), as it is easily seen.

The problem now is the determination of the derivations of $Z(n+1,2m)$, the zygotic algebra of the same $2m$-ploid and $n+1$-allelic population. As $Z(n+1,2m)$ is the commutative duplicate for $G(n+1,2m)$ ([10], Ch. 6C), the first idea to obtain derivations in $Z(n+1,2m)$ is to try to duplicate derivations of $G(n+1,2m)$. We recall briefly that given a genetic algebra A with a canonical basis $C_0, C_1, \dots, C_n$ then the set of symbols $C_i{*}C_j$ $(0 \leqq i \leqq j \leqq n)$ is a basis of the duplicate $A{*}A$ of A ([10], Ch. 6C). In particular if $\dim A = n+1$ then $\dim(A{*}A) = \dfrac{(n+1)(n+2)}{2}$. The multiplication in $A{*}A$ is given by

$$(C_i{*}C_j)(C_k{*}C_\ell) = (C_iC_j){*}(C_kC_\ell)$$

where C_iC_j (resp. C_kC_ℓ) is the product, in A, of C_i and C_j (resp. C_k and C_ℓ). An intrinsic construction of $A{*}A$ is the following: take the tensor product vector space $A \otimes A$ and define a multiplication by $(a \otimes b)(c \otimes d) = (ab) \otimes (cd)$. Then let J be the two-sided ideal generated by the elements $a \otimes b - b \otimes a$, $a, b \in A$ and take $A{*}A = (A \otimes A)/J$ ([10]).

Lemma 1. *Let $\delta : A \to A$ be a derivation. There exists one and only one derivation $\delta^*: A{*}A \to A{*}A$ such that $\delta^*(a{*}b) = \delta(a){*}b + a{*}\delta(b)$ for all a, b in A.*

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Proof. Let $\theta : A \times A \to A \otimes A$ be the the canonical bilinear mapping given by $\theta(a, b) = a \otimes b$. Then $\theta \circ (\delta \times 1_A) : A \times A \to A \otimes A$ is bilinear. The same holds for $\theta \circ (1_A \times \delta) : A \times A \to A \otimes A$. Hence $\theta \circ (\delta \times 1_A) + \theta \circ (1_A \times \delta)$ is again bilinear and induces $\bar{\delta} : A \otimes A \to A \otimes A$, linear and satisfying $\bar{\delta}(a \otimes b) = \delta(a) \otimes b + a \otimes \delta(b)$ for all a, b in A. This mapping $\bar{\delta}$ satisfies $\bar{\delta}(J) \subset J$. In fact, take one generator $a \otimes b - b \otimes a$ of J. We have

$$\begin{aligned} \bar{\delta}(a \otimes b - b \otimes a) &= \bar{\delta}(a \otimes b) - \bar{\delta}(b \otimes a) = \\ \delta(a) \otimes b + a \otimes \delta(b) &- \delta(b) \otimes a - b \otimes \delta(a) = \\ &= (\bar{\delta}(a) \otimes b - b \otimes \delta(a)) + (a \otimes \delta(b) - \delta(b) \otimes a), \end{aligned}$$

which is an element of J. By the well known lemma on quotients, $\bar{\delta}$ induces $\delta^* : A^*A \to A^*A$ such that $\delta^*(a^*b) = a^*\delta(b) + \delta(a)^*b$ for all a, b in A. It rests to prove that δ^* is a derivation of A^*A. As A^*A is generated by the elements a^*b, a, b in A, it is enough to prove the following equality:

$$\delta^*((a^*b)(c^*d)) = \delta^*(a^*b)(c^*d) + (a^*b)\delta^*(c^*d)$$

for all a, b, c, d in A. In fact, we have:

$$\begin{aligned} \delta^*((a^*b)(c^*d)) &= \delta^*((ab)^*(cd)) = \delta(ab)^*(cd) + (ab)^*\delta(cd) = \\ &= (\delta(a)b + a\delta(b))^*(cd) + (ab)^*(\delta(c)d + c\delta(d)) = \\ &= \delta(a)b^*(cd) + a\delta(b)^*(cd) + (ab)^*\delta(c)d + (ab)^*c\delta(d) = \\ &= (\delta(a)^*b)(c^*d) + (a^*\delta(b))(c^*d) + (a^*b)(\delta(c)^*d) + (a^*b)(c^*\delta(d)) = \\ &= [\delta(a)^*b + a^*\delta(b)](c^*d) + (a^*b)[\delta(c)^*d + c^*\delta(d)] = \\ &= \delta^*(a^*b)(c^*d) + (a^*b)\delta^*(c^*d). \end{aligned}$$

The unicity of δ^* is clear.

We shall call δ^* the duplicate of δ and the correspondence $\delta \to \delta^*$ the duplication mapping.

Proposition 1. *The correspondence $\delta \to \delta^*$ is an injective homomorphism of Lie algebras.*

Proof. Let δ_1 and δ_2 be derivations of A, $a, b \in A$. We have:

$$\begin{aligned} (\delta_1 + \delta_2)^*(a^*b) &= (\delta_1 + \delta_2)(a)^*b + a^*(\delta_1 + \delta_2)(b) = \\ &= \delta_1(a)^*b + \delta_2(a)^*b + a^*\delta_1(b) + a^*\delta_2(b) = \\ &= \delta_1(a)^*b + a^*\delta_1(b) + \delta_2(a)^*b + a^*\delta_2(b) = \\ &= \delta_1^*(a^*b) + \delta_2^*(a^*b) = (\delta_1^* + \delta_2^*)(a^*b). \end{aligned}$$

As a^*b, with $a, b \in A$, is a generating set of A^*A, we have $(\delta_1+\delta_2)^* = = \delta_1^* + \delta_2^*$. In a similar way, we prove that $(\lambda\delta)^* = \lambda\delta^*$ for all $\lambda \in R$.

Now

$$\begin{aligned}
(\delta_1 \circ \delta_2 - \delta_2 \circ \delta_1)^*(a^*b) &= [(\delta_1 \circ \delta_2)^* - (\delta_2 \circ \delta_1)^*](a^*b) = \\
&= \delta_1(\delta_2(a))^*b + a^*\delta_1(\delta_2(b)) - \delta_2(\delta_1(a))^*b - a^*\delta_2(\delta_1(b)) = \\
&= \delta_1(\delta_2(a))^*b + \delta_2(a)^*\delta_1(b) + a^*\delta_1(\delta_2(b)) + \delta_1(a)^*\delta_2(b) - \\
&- \delta_2(\delta_1(a))^*b - \delta_1(a)^*\delta_2(b) - a^*\delta_2(\delta_1(b)) - \delta_2(a)^*\delta_1(b) = \\
&= \delta_1^*(\delta_2(a)^*b) + \delta_1^*(a^*\delta_2(b)) - \delta_2^*(\delta_1(a)^*b) - \delta_2^*(a^*\delta_1(b)) =
\end{aligned}$$

$= (\delta_1^* \circ \delta_2^*)(a^*b) - (\delta_2^* \circ \delta_1^*)(a^*b) = (\delta_1^* \circ \delta_2^* - \delta_2^* \circ \delta_1^*)(a^*b)$ and so

$$(\delta_1 \circ \delta_2 - \delta_2 \circ \delta_1)^* = \delta_1^* \circ \delta_2^* - \delta_2^* \circ \delta_1^*.$$

We show now that $\delta^* = 0$ implies $\delta = 0$. Take a basis $C_0, C_1, \ldots, C_n$ of A. If $\delta(C_i) = \sum_{k=0}^{n} \alpha_{ki} C_k$ $(i = 0, 1, \ldots, n)$ then:

$$\begin{aligned}
0 &= \delta^*(C_i{}^*C_i) = \delta(C_i)^*C_i + C_i{}^*\delta(C_i) = 2C_i{}^*\delta(C_i) = \\
&= 2C_i{}^*\Big(\sum_{k=0}^{n} \alpha_{ki}C_k\Big) = \sum_{k=0}^{i} 2\alpha_{ki}C_k{}^*C_i + \sum_{k=i+1}^{n} 2\alpha_{ki}C_i{}^*C_k,
\end{aligned}$$

for all $i = 0, 1, \ldots, n$. As $C_0{}^*C_i, \ldots, C_i{}^*C_i, C_i{}^*C_{i+1}, \ldots, C_i{}^*C_n$ are part of a basis of A^*A we have $\alpha_{ki} = 0$ for all $k = 0, 1, \ldots, n$ and so $\delta = 0$.

Remark. In general the correspondence $\delta \to \delta^*$ is not an isomorphism of Lie algebras. We give an example of a class of genetic algebras where $\delta \to \delta^*$ is not an isomorphism. But, in contrast to this, we will have an isomorphism for every one of the gametic algebras $G(n+1, 2m)$.

For each $n \geqq 1$, we call K_n the trivial genetic algebra of dimension $n+1$ having a basis $C_0, C_1, \ldots, C_n$ such that $C_0^2 = C_0$ and all other products are zero. The weight function $\omega : K_n \to R$ is given by $\omega(C_0) = 1$ and $\omega(C_i) = 0$ $(i = 1, \ldots, n)$. Given $x = \omega(x)C_0 + \sum_{i=1}^{n} \alpha_i C_i \in K_n$ and $y = \omega(y)C_0 + \sum_{j=1}^{n} \beta_j C_j$ we have $xy = \omega(x)\omega(y)C_0$. The algebra K_n is the Bernstein algebra of dimension $n+1$ and type $(1, n)$ ([10], Chap. 9B, th. 9.10).

Lemma 2. *The derivations of K_n are the linear mappings $\delta : K_n \to K_n$ such that $\omega \circ \delta = 0$ and $\delta(C_0) = 0$. Hence the derivation algebra of K_n has dimension n^2.*

Proof. Suppose δ is a derivation. Then $\omega \circ \delta = 0$ ([1], th. 1). If $\delta(C_0) = u \in \operatorname{Ker} \omega$ then $C_0^2 = C_0$ implies

$$u = \delta(C_0) = 2C_0\delta(C_0) = 2C_0u = 0.$$

Suppose now $\delta : K_n \to K_n$ satisfies $\omega \circ \delta = 0$ and $\delta(C_0) = 0$. Then

$$\delta(xy) = \delta(\omega(x)\omega(y)C_0) = \omega(x)\omega(y)\delta(C_0) = 0.$$

On the other hand,

$$\delta(x)y + x\delta(y) = \omega(\delta(x))\omega(y)C_0 + \omega(x)\omega(\delta(y))C_0 = 0$$

and so δ is a derivation of K_n.

We have shown that δ is completely determined by $\delta(C_1), \ldots, \delta(C_n)$ with $\delta(C_i) \in \operatorname{Ker} \omega$, $(i = 1, \ldots, n)$ and so there is a one-to-one correspondence between derivations and sequences $A_1, \ldots, A_n$ of elements of $\operatorname{Ker} \omega$. This completes the proof.

Lemma 3. *For each* $n \geqq 1$, $K_n{*}K_n$ *is isomorphic to* $\dfrac{K_{n(n+3)}}{2}$.

Proof. It is enough to prove that $(K{*}K)^2$ is a one dimensional algebra spanned by $C_0{*}C_0$. In fact, if $C_0, C_1, \ldots, C_n$ is a basis of K_n, then $C_i{*}C_j$, with $0 \leqq i \leqq j \leqq n$ is a canonical basis of $K_n{*}K_n$.

Now

$$(C_0{*}C_0)^2 = C_0{*}C_0 \text{ and } (C_i{*}C_j)(C_k{*}C_\ell) = C_iC_j{*}C_kC_\ell = 0,$$

if i, j, k or $\ell \neq 0$.

Corollary. *For each* K_n, $n \geqq 1$, *the duplication mapping is not an isomorphism.*

Proof. By lemmas 2 and 3, the derivation algebra of $K_n{*}K_n$ has dimension $\frac{1}{4}(n^2(n+3)^2)$ which is greater than n^2.

2. Multiallelism only

It is well known that $G(n+1,2)$ has a canonical basis $C_0, C_1 \ldots, C_n$ such that $C_0^2 = C_0$, $C_0C_i = \frac{1}{2}C_i$ $(i = 1, \ldots, n)$ and $C_iC_j = 0$ $(1 \leqq i, j \leqq n)$. By duplication, we obtain a canonical basis $C_i{*}C_j$ $(0 \leqq i \leqq j \leqq n)$ of

$Z(n+1,2)$, the zygotic algebra of the same diploid and $(n+1)$-allelic population. The multiplication is given by

$$(C_0{}^*C_0)^2 = C_0{}^*C_0,\ (C_0{}^*C_0)(C_0{}^*C_i) = \frac{1}{2}C_0{}^*C_i \quad (i=1,\ldots,n),$$
$$(C_0{}^*C_i)(C_0{}^*C_j) = \frac{1}{4}C_i{}^*C_j (1 \leqq i,j \leqq n) \text{ and } (C_i{}^*C_j)(C_k{}^*C_\ell) = 0$$

when $1 \leqq i \leqq j \leqq n$ or $1 \leqq k \leqq \ell \leqq n$. Let us decompose $Z(n+1,2)$ as $Z(n+1,2) = V_0 \oplus V_1 \oplus V_2$ where $V_0 = \langle C_0{}^*C_0\rangle$, $V_1 = \langle C_0{}^*C_i : i=1,\ldots,n\rangle$ and $V_2 = \langle C_i{}^*C_j : 1 \leqq i \leqq j \leqq n\rangle$ ($\langle\ldots\rangle$ indicates the subspace generated by ...). Observe that $V_1 \oplus V_2$ is the kernel of the weight function, which is 1 for $C_0{}^*C_0$ and 0 otherwise.

Theorem 1. *Suppose $\delta : Z(n+1,2) \to Z(n+1,2)$ is a derivation. Then there exist $A, B_1, \ldots, B_n$ in V_1 such that*

(i) $\delta(C_0{}^*C_0) = A$;
(ii) For each $1 \leqq i \leqq n$, $\delta(C_0{}^*C_i) = B_i + 2A(C_0{}^*C_i)$;
(iii) *For each* $1 \leqq i \leqq j \leqq n$, $\delta(C_i{}^*C_j) = 4(C_0{}^*C_i)B_j + 4(C_0{}^*C_j)B_i$.

Conversely, given $A, B_1, \ldots, B_n$ in V_1, there exists one and only one derivation δ of $Z(n+1,2)$ such that (i), (ii) and (iii) hold.

Proof. (i): By ([1], th. 1) we have $\omega \circ \delta = 0$. Call $\delta(C_0{}^*C_0) = A + z_2$ with $A \in V_1$ and $z_2 \in V_2$. As $(C_0{}^*C_0)^2 = C_0{}^*C_0$ we have

$$A + z_2 = \delta(C_0{}^*C_0) = 2(C_0{}^*C_0)\delta(C_0{}^*C_0) = 2(C_0{}^*C_0)(A+z_2) =$$
$$= 2(C_0{}^*C_0)A + 2(C_0{}^*C_0)z_2 = A.$$

Equating components we have $z_2 = 0$. It rests $\delta(C_0{}^*C_0) = A$.

(ii): Call $\delta(C_0{}^*C_i) = B_i + D_i$ with $B_i \in V_1$, $D_i \in V_2$. From $(C_0{}^*C_0)(C_0{}^*C_i) = \frac{1}{2}C_0{}^*C_i$ we obtain

$$A(C_0{}^*C_i) + (C_0{}^*C_0)(B_i + D_i) = \frac{1}{2}(B_i + D_i) \text{ or } A(C_0{}^*C_i) + \frac{1}{2}B_i = \frac{1}{2}(B_i + D_i).$$

But $A(C_0{}^*C_i) \in V_2$, so $A(C_0{}^*C_i) = \frac{1}{2}D_i$, which means $\delta(C_0{}^*C_i) = B_i + 2A(C_0{}^*C_i)$.

(iii): From $C_i{}^*C_j = 4(C_0{}^*C_i)(C_0{}^*C_j)$ $(1 \leqq i \leqq j \leqq n)$ we obtain

$$\delta(C_i{}^*C_j) = 4[\delta(C_0{}^*C_i)(C_0{}^*C_j) + (C_0{}^*C_i)\delta(C_0{}^*C_j)] =$$
$$= 4[[B_i + 2A(C_0{}^*C_i)](C_0{}^*C_j) + (C_0{}^*C_i)[B_j + 2A(C_0{}^*C_j)]] =$$
$$= 4[B_i(C_0{}^*C_j) + (C_0{}^*C_i)B_j].$$

Conversely, given $A, B_1, \ldots, B_n$ in V_1 define $\delta : Z(n+1,2) \to Z(n+1,2)$ by the formulae above. It is routine to verify that δ is indeed a derivation. Also the unicity of δ is clear.

Corollary. *The derivation algebra of $Z(n+1,2)$ has dimension $n(n+1)$ and so every derivation of $Z(n+1,2)$ is the duplicate of one and only one derivation of $G(n+1,2)$.*

3. Polyploidy only

The gametic algebra $G(2,2m)$ has a canonical basis $C_0, C_1, \ldots, C_m$ such that $C_iC_j = 0$ when $i+j > m$ and $C_iC_j = t_{i+j}C_{i+j}$ when $i+j \leqq m$, where $t_k (k=0, 1, \ldots, m)$ are the t-roots of $G(2,2m)$. Hence $Z(2,2m)$ has a canonical basis $C_i{}^*C_j$ $(0 \leqq i \leqq j \leqq m)$ where the multiplication is given by

$$(C_i{}^*C_j)(C_k{}^*C_\ell) = \begin{cases} 0 \text{ when } i+j>m \text{ or } k+\ell>m \\ t_{i+j}t_{k+\ell}C_{i+j}{}^*C_{k+\ell} \text{ when } i+j \leqq m \text{ and } k+\ell \leqq m \end{cases}$$

The t-roots of $Z(2,2m)$ are $t_0, t_1, \ldots, t_m$ (where $t_k = \binom{2m}{k}^{-1}\binom{m}{k}$) and 0, this one with multiplicity $m(m+1)/2$. The weight function ω of $Z(2,2m)$ is given by $\omega(C_0{}^*C_0) = 1$ and $\omega(C_i{}^*C_j) = 0$ for all $(i,j) \neq (0,0)$.

We have also a direct sum decomposition $Z(2,2m) = V_0 \oplus V_1 \oplus \ldots \oplus V_{2m}$ where $V_k (0 \leqq k \leqq 2m)$ is the subspace of $Z(2,2m)$ generated by the vectors $C_i{}^*C_j$, $0 \leqq i \leqq j \leqq n$, such that $i+j=k$. In particular $V_0 = \langle C_0{}^*C_0 \rangle$, $V_1 = \langle C_0{}^*C_1 \rangle$, $V_2 = \langle C_0{}^*C_2,\ C_1{}^*C_1 \rangle$ and so on. The dimension of V_k is $\dfrac{k}{2}+1$ when k is even and $\dfrac{k+1}{2}$ when k is odd. From the multiplication table of $Z(2,2m)$ we see that every element of V_k is an absolute divisor of zero if $m+1 \leqq k \leqq 2m$. This means that if $v_k \in V_k$ and $m+1 \leqq k \leqq 2m$, for every $x \in Z(2,2m)$ we have $xv_k = 0$. Also we have the following relation for $v_k \in V_k$ and $0 \leqq k \leqq m$: $(C_0{}^*C_0)v_k$ is a scalar multiple of $C_0{}^*C_k$. In fact, if $v_k = \alpha_0 C_0{}^*C_k + \alpha_1 C_1{}^*C_{k-1} + \ldots$ we have

$$\begin{aligned}(C_0{}^*C_0)v_k &= \alpha_0(C_0{}^*C_0)(C_0{}^*C_k) + \alpha_1(C_0{}^*C_0)(C_1{}^*C_{k-1}) + \ldots = \\ &= \alpha_0 t_k C_0{}^*C_k + \alpha_1 t_k C_0{}^*C_k + \ldots = t_k(\textstyle\sum_i \alpha_i)C_0{}^*C_k.\end{aligned}$$

In order to simplify the notations we call ϕ the linear form on $Z(2,2m)$ given by $\phi(C_i{}^*C_j) = 1$ for all $0 \leqq i \leqq j \leqq m$. We have shown that $(C_0{}^*C_0)v_k = t_k\phi(v_k)C_0{}^*C_k$ for $0 \leqq k \leqq m$. As $C_0{}^*C_k$ plays a special role in the multiplication by $C_0{}^*C_0$, we call it the special element of V_k $(0 \leqq k \leqq m)$.

We know ([1] th. 1) that every derivation δ of $Z(2,2m)$ satisfies $\omega \circ \delta = 0$.

The following lemmas 4 to 7 will describe the action of a derivation δ on the subspaces $V_0, V_1, \dots, V_m, \dots, V_{2m}$ of $Z(2,2m)$.

Lemma 4. *For every derivation δ of $Z(2,2m)$, we have $\delta(C_0{*}C_0) = \alpha C_0{*}C_1$ for some $\alpha \in \mathbb{R}$.*

Proof. Call $\delta(C_0{*}C_0) = v_1 + v_2 + \dots + v_{2m}$ with $v_i \in V_i$ $(i = 1, \dots, 2m)$. Then

$$2(C_0{*}C_0)(v_1 + \dots + v_{2m}) = v_1 + \dots + v_{2m} \quad \text{or}$$
$$2\phi(v_1)t_1C_0{*}C_1 + \dots + 2\phi(v_m)t_mC_0{*}C_m = v_1 + \dots + v_m + \dots + v_{2m}.$$

Equating the components we have:

$$\begin{cases} 2\phi(v_k)t_kC_0{*}C_k = v_k & (1 \leqq k \leqq m) \\ v_{m+1} = \dots = v_{2m} = 0. \end{cases}$$

The first equality reads $2\phi(v_1)t_1C_0{*}C_1 = v_1$, thereby $t_1 = {}^1/_2$. Hence $v_1 = \alpha C_0{*}C_1$ for some $\alpha \in \mathbb{R}$. The equations corresponding to $2 \leqq k \leqq m$ have only the trivial solution $v_k = 0$. In fact, call $v_k = \mu_0 C_0{*}C_k + {}$ $+ \mu_1 C_1{*}C_{k-1} + \dots$. Then we have $2(\mu_0 + \mu_1 + \dots)t_kC_0{*}C_k = \mu_0C_0{*}C_k + {}$ $+ \mu_1C_1{*}C_{k-1} + \dots$ which implies

$$\begin{cases} 2t_k(\mu_0 + \mu_1 + \dots) = \mu_0 \\ \mu_1 = \dots = 0 \end{cases}$$

This system reduces to $2t_k\mu_0 = \mu_0$ and so $\mu_0 = 0$ because $t_k = \binom{2m}{k}^{-1}\binom{m}{k} < \frac{1}{2}$ when $2 \leqq k \leqq m$. Then $v_k = 0$ for all $2 \leqq k \leqq m$. It rests $\delta(C_0{*}C_0) = v_1 = {}$ $= \alpha C_0{*}C_1$, for some real number α.

Lemma 5. *For every $1 \leqq k \leqq m-1$, we have*

$$\delta(C_0{*}C_k) = \alpha_k C_0{*}C_k + \alpha t_1 \frac{t_{k+1}}{t_k - t_{k+1}} C_0{*}C_{k+1} + \alpha t_1 C_1{*}C_k,$$

where $\alpha_k \in R$ and α is as in lemma 4.

Proof. Again, call $\delta(C_0{*}C_k) = u_1 + \dots + u_m + \dots + u_{2m}, u_i \in V_i$. The equality $(C_0{*}C_0)(C_0{*}C_k) = t_k(C_0{*}C_k)$ implies

$$\alpha(C_0{*}C_1)(C_0{*}C_k) + (C_0{*}C_0)(u_1 + \dots + u_{2m}) = t_k(u_1 + \dots + u_{2m})$$

or

$$\alpha t_1 t_k(C_1{*}C_k) + \sum_{i=1}^{m} \phi(u_i)t_iC_0{*}C_i = t_k\Big(\sum_{i=1}^{2m} u_i\Big). \text{ But } C_1{*}C_k \in V_{k+1},$$

so we must have:

$$\begin{cases} \phi(u_i)t_i(C_0{}^*C_i) = t_k u_i, \; i = 1, \ldots, m, \; i \neq k+1, \\ \phi(u_{k+1})t_{k+1}(C_0{}^*C_{k+1}) + \alpha t_1 t_k(C_1{}^*C_k) = t_k u_{k+1}, \\ u_{m+1} = \ldots = u_{2m} = 0. \end{cases}$$

The equations in the first row, with $i \neq k$, have only the trivial solution $u_i = 0$, as in the preceding lemma. The equation $\phi(u_k)t_k C_0{}^*C_k = t_k u_k$ reduces to $\phi(u_k)C_0{}^*C_k = u_k$ which gives $u_k = \alpha_k C_0{}^*C_k$ for some real number α_k. The equation in the middle has the following solution: if

$$u_{k+1} = \lambda_0(C_0{}^*C_{k+1}) + \lambda_1(C_1{}^*C_k) + \lambda_2(C_2{}^*C_{k-1}) + \ldots,$$

then

$$\begin{aligned} &(\lambda_0 + \lambda_1 + \lambda_2 + \ldots)t_{k+1}(C_0{}^*C_{k+1}) + \lambda t_1 t_k(C_1{}^*C_k) = \\ &= t_k(\lambda_0(C_0{}^*C_{k+1}) + \lambda_1(C_1{}^*C_k) + \lambda_2(C_2{}^*C_{k-1}) + \ldots) \end{aligned}$$

and so

$$\begin{cases} (\lambda_0 + \lambda_1 + \lambda_2 + \ldots)t_{k+1} = t_k\lambda_0 \\ \alpha t_1 t_k = t_k \lambda_1 \\ t_k\lambda_2 = \ldots = 0 \end{cases}$$

From this system, we have $\lambda_2 = \ldots = 0$, $\lambda_1 = \alpha t_1 = \alpha/2$ and the first equality reduces to $\lambda_0 = \alpha t_1 \dfrac{t_{k+1}}{t_k - t_{k+1}}$. Hence the result.

The effect of δ on the vector $C_0{}^*C_m$ is given by

$$\delta(C_0{}^*C_m) = \alpha_m(C_0{}^*C_m) + \alpha t_1(C_1{}^*C_m)$$

where α_m is some real number. The proof is similar to that given in lemma 5.

Having obtained the effect of δ on the vectors $C_0{}^*C_k$ $(1 \leqq k \leqq m)$ we can now obtain $\delta(C_i{}^*C_j)$ for $1 \leqq i \leqq j \leqq m$.

Lemma 6. *For every* $1 \leqq i \leqq j \leqq m-1$, *we have*

$$\delta(C_i{}^*C_j) = (\alpha_i + \alpha_j)C_i{}^*C_j + \alpha t_1 \left[\frac{t_{i+1}}{t_i - t_{i+1}}(C_{i+1}{}^*C_j) + \frac{t_{j+1}}{t_j - t_{j+1}}(C_i{}^*C_{j+1})\right],$$

where α_i *and* α_j *are as in lemma 5.*

Proof. We have $(C_0{}^*C_i)(C_0{}^*C_j) = t_it_j(C_i{}^*C_j)$ and so

$$\delta(C_i{}^*C_j) = \frac{1}{t_it_j}[\delta(C_0{}^*C_i)(C_0{}^*C_j) + (C_0{}^*C_i)\delta(C_0{}^*C_j)] =$$

$$= \frac{1}{t_it_j}\left[\left[\alpha_i(C_0{}^*C_i) + \alpha t_1\left[\frac{t_{i+1}}{t_i - t_{i+1}}(C_0{}^*C_{i+1}) + (C_1{}^*C_i)\right]\right](C_0{}^*C_j) +\right.$$

$$\left. + (C_0{}^*C_i)\left[\alpha_j(C_0{}^*C_j) + \alpha t_1\left[\frac{t_{j+1}}{t_j - t_{j+1}}(C_0{}^*C_{j+1}) + (C_1{}^*C_j)\right]\right]\right] =$$

$$= \frac{1}{t_it_j}\left[\alpha_it_it_j(C_i{}^*C_j) + \alpha t_1\frac{t_{i+1}^2t_j}{t_i - t_{i+1}}(C_{i+1}{}^*C_j) + \alpha t_1t_{i+1}t_j(C_{i+1}{}^*C_j) +\right.$$

$$\left. + \alpha_jt_it_j(C_i{}^*C_j) + \alpha t_1\frac{t_{i+1}^2}{t_j - t_{j+1}}(C_i{}^*C_{j+1}) + \alpha t_1t_it_{j+1}(C_i{}^*C_{j+1})\right] =$$

$$= (\alpha_i + \alpha_j)(C_i{}^*C_j) + \alpha t_1\left(\frac{t_{i+1}^2}{t_i(t_i - t_{i+1})} + \frac{t_{i+1}}{t_i}\right)(C_{i+1}{}^*C_j) +$$

$$+ \alpha t_1\left(\frac{t_{j+1}^2}{t_j(t_j - t_{j+1})} + \frac{t_{j+1}}{t_j}\right)(C_i{}^*C_{j+1}) =$$

$$= (\alpha_i + \alpha_j)(C_i{}^*C_j) + \alpha t_1\left[\frac{t_{i+1}}{t_i - t_{i+1}}(C_{i+1}{}^*C_j) + \frac{t_{j+1}}{t_j - t_{j+1}}(C_i{}^*C_{j+1})\right].$$

In a similar way we prove the relations

$$\delta(C_i{}^*C_m) = (\alpha_i + \alpha_m)(C_i{}^*C_m) + \alpha t_1\frac{t_{i+1}}{t_i - t_{i+1}}(C_{i+1}{}^*C_m) \quad (1 \leqq i \leqq m-1)$$

and

$$\delta(C_m{}^*C_m) = 2\alpha_m(C_m{}^*C_m).$$

The effect of δ on the canonical basis of $Z(2,\dot{2}m)$ will be completely known after the following lemma.

Lemma 7. *The real numbers $\alpha_j (j = 1, \ldots, m)$ appearing in the formulae for $\delta(C_0{}^*C_j)$ satisfy $\alpha_j = j\alpha_1$ $(j = 1, \ldots, m)$.*

Proof. The equality is trivial for $j = 1$ and suppose we have already proved for $1 \leqq i < m$. From the equality

$$(C_1{}^*C_i)^2 = t_{i+1}^2(C_{i+1}{}^*C_{i+1}),$$

we obtain:

$$2(C_1{}^*C_i)\delta(C_1{}^*C_i) = t_{i+1}^2\delta(C_{i+1}{}^*C_{i+1})$$

or

$$2(C_1{}^*C_i)\,[(\alpha_1+\alpha_i)(C_1{}^*C_i)+\alpha\,\frac{t_1t_2}{t_1-t_2}(C_2{}^*C_i)+\alpha\,\frac{t_1t_{i+1}}{t_i-t_{i+1}}(C_1{}^*C_{i+1})]=$$

$$=t_{i+1}^2[2\alpha_{i+1}(C_{i+1}{}^*C_{i+1})+2\,\frac{t_1t_{i+2}}{t_{i+1}-t_{i+2}}(C_{i+1}{}^*C_{i+2})].$$

The comparison of components in the directions of $C_{i+1}{}^*C_{i+1}$ and $C_{i+1}{}^*C_{i+2}$ gives $\alpha_1+\alpha_i=\alpha_{i+1}$ (our desired result) and an identity in the t-roots, as in [1], th. 3.

The results of the preceding lemmas can be put together in the following set of equations:

$$(*)\begin{cases}\delta(C_0{}^*C_0)=\alpha(C_0{}^*C_1)\\ \delta(C_0{}^*C_k)=k\beta(C_0{}^*C_k)+\alpha t_1\left[\dfrac{t_{k+1}}{t_k-t_{k+1}}(C_0{}^*C_{k+1})+C_1{}^*C_k\right]\\ \delta(C_i{}^*C_j)=(i+j)\,\beta(C_i{}^*C_j)+\alpha t_1\left[\dfrac{t_{i+1}}{t_i-t_{i+1}}(C_{i+1}{}^*C_j)+\right.\\ \qquad\qquad\left.+\dfrac{t_{j+1}}{t_j-t_{j+1}}(C_i{}^*C_{j+1})\right]\end{cases}$$

where $1\leqq k\leqq m$, $1\leqq i\leqq j\leqq m$, $t_{m+1}=0$ and $\alpha,\beta\in R$.

Theorem 2. *The derivation algebra of $Z(2,2m)$ has dimension 2. In particular, every derivation of $Z(2,2m)$ is the duplicate of one and only one derivation of $G(2,2m)$.*

Proof. The preceding lemmas provide the relations (*). It is easy to see that if we choose arbitrarily $\alpha,\beta\in R$ and define $\delta: Z(2,2m)\to Z(2,2m)$ by the relations (*), we obtain a derivation. This means exactly that the derivation algebra of $Z(2,2m)$ has dimension 2. Since every duplicate of a derivation of $G(2,2m)$ yields a derivation of $Z(2,2m)$ and the derivation algebra of $G(2,2m)$ has dimension 2 (cf. [1]) we get the desired result.

4. Multiallelism and polyploidy

In the general case of multiallelism and polyploidy, we follow the same ideas of §§ 2,3.

The gametic algebra $G(n+1,2m)$ corresponding to a $n+1$-allelic and $2m$-ploid population has a canonical basis consisting of all monomials

$X_0^{i_0}X_1^{i_1}\dots X_n^{i_n}$ in commuting variables such that $i_0+i_1+\dots+i_n=m$ ([4], [5], [1]). This basis is ordered lexicographically by the exponents, that is, $X_0^{i_0}X_1^{i_1}\dots X_n^{i_n}$ precedes $X_0^{j_0}X_1^{j_1}\dots X_n^{j_n}$ when the first non vanishing difference i_k-j_k $(k=0,1,\dots,n)$ is positive. The multiplication in $G(n+1,2m)$ is given by

$$(X_0^{i_0}X_1^{i_1}\dots X_n^{i_n})(X_0^{j_0}X_1^{j_1}\dots X_n^{j_n})=$$
$$\begin{cases}\binom{2m}{m}^{-1}\binom{i_0+j_0}{m}X_0^{i_0+j_0-m}X_1^{i_1+j_1}\dots X_n^{i_n+j_n} & \text{if } m\leqq i_0+j_0\\ 0 & \text{if } i_0+j_0<m.\end{cases}$$

In particular,

$$X_0^m(X_0^{j_0}X_1^{j_1}\dots X_n^{j_n})=\binom{2m}{m}^{-1}\binom{m+j_0}{m}X_0^{j_0}X_1^{j_1}\dots X_n^{j_n},$$

which says the t-roots of $G(n+1,2m)$ are 1, $^1/_2$, $\dots$, $^1/_{\binom{2m}{m}}$ with multiplicities $1, n, \dots, \binom{m+n-1}{m}$ respectively.

Now we consider the duplicate $Z(n+1,2m)$ of $G(n+1,2m)$. One canonical basis of $Z(n+1,2m)$ is the set of all "double monomials" $(X_0^{i_0}X_1^{i_1}\dots X_n^{i_n})*(X_0^{j_0}X_1^{j_1}\dots X_n^{j_n})$ where the first one precedes the second, and, of course, $i_0+\dots+i_n=j_0+\dots+j_n=m$. We recall (see [1]) that $\dim G(n+1,2m)=\binom{m+n}{m}$ and so $\dim Z(n+1,2m)={}^1/_2\left[\binom{m+n}{m}^2+\binom{m+n}{m}\right]$ and that the weight function ω is defined by $\omega(X_0^m*X_0^m)=1$ and 0 otherwise.

Let V_{2m-r} be the subspace of $Z(n+1,2m)$ generated by the double monomials $(X_0^{i_0}\dots X_n^{i_n})*(X_0^{j_0}\dots X_n^{j_n})$ such that $i_0+j_0=r$. As $0\leqq i_0\leqq m$, $0\leqq j_0\leqq m$, we must have $0\leqq 2m-r\leqq 2m$. We list now some properties of the subspaces $V_0, V_1, \dots V_{2m}$.

(1) First of all, we have the direct sum decomposition $Z(n+1,2m)= = V_0\oplus V_1\oplus\dots\oplus V_{2m}$, by the own definition of the subspaces. In addition, $V_1\oplus\dots\oplus V_{2m}$ is the kernel of the weight function ω.

(2) V_0 is generated by the idempotent $X_0^m*X_0^m$.

(3) V_1 is generated by the double monomials $X_0^m*X_0^{m-1}X_i$ $(i=1,\dots,n)$ and so $\dim V_1=n$.

(4) Every element of $V_{m+1}\oplus\dots\oplus V_{2m}$ is an absolute divisor of zero in $Z(n+1,2m)$. In order to prove this, it is enough to prove that each double monomial belonging to one of these subspaces is an absolute divisor of zero. If

$$(X_0^{i_0}X_1^{i_1}\dots X_n^{i_n})*(X_0^{j_0}X_1^{j_1}\dots X_n^{j_n})\in V_k$$

and $m+1 \leqq k \leqq 2m$, then $i_0+j_0<m$ (definition of V_k) and so, given an arbitrary double monomial,

$$\mu = (X_0^{r_0}X_1^{r_1} \dots X_n^{r_n}) * (X_0^{s_0}X_1^{s_1} \dots X_n^{s_n}),$$

we have

$$\mu[X_0^{i_0}X_1^{i_1} \dots X_n^{i_n}) * (X_0^{j_0}X_1^{j_1} \dots X_n^{j_n})] =$$
$$= [X_0^{r_0}X_1^{r_1} \dots X_n^{r_n})(X_0^{s_0}X_1^{s_1} \dots X_n^{s_n})]*[X_0^{i_0}X_1^{i_1} \dots X_n^{i_n})(X_0^{j_0}X_1^{j_1} \dots X_n^{j_n})] =$$
$$= [(X_0^{r_0}X_1^{r_1} \dots X_n^{r_n})(X_0^{j_0}X_1^{j_1} \dots X_n^{j_n})]*0 = 0.$$

(5) If $0 \leqq k \leqq m$, V_k is an invariant subspace of the linear mapping $z \rightarrow (X_0^{m} * X_0^{m})z$, $z \in Z(n+1, 2m)$. In fact, if we take a double monomial $(X_0^{i_0}X_1^{i_1} \dots X_n^{i_n})*(X_0^{j_0}X_1^{j_1} \dots X_n^{j_n})$ in V_k, then $2m-i_0-j_0=k$, which implies $i_0+j_0 \geqq m$ and so

$$(X_0^{m} * X_0^{m})[(X_0^{i_0}X_1^{i_1} \dots X_n^{i_n})*(X_0^{j_0}X_1^{j_1} \dots X_n^{j_n})] =$$
$$= X_0^{m} * \binom{2m}{m}^{-1}\binom{i_0+j_0}{m} X_0^{i_0+j_0-m}X_1^{i_1+j_1} \dots X_n^{i_n+j_n} =$$
$$= t_k X_0^{m} * X_0^{i_0+j_0-m}X_1^{i_1+j_1} \dots X_n^{i_n+j_n} \in V_k,$$

because $2m-m-i_0-j_0+m=2m-i_0-j_0=k$. Observe that in general the double monomials $(X_0^{i_0}X_1^{i_1} \dots X_n^{i_n})*(X_0^{j_0}X_1^{j_1} \dots X_n^{j_n}) \in V_k$, $k=2m-i_0-j_0$, are not proper vectors of the above linear mapping. The elements $X_0^{m} * X_0^{m-k}X_1^{i_1} \dots X_n^{i_n} \in V_k$ are proper vectors, so there are $\binom{n+k-1}{k}$ linearly independent proper vectors in V_k. These double monomials will be called special.

(6) We introduce the following equivalence relation in the basis of $V_k (0 \leqq k \leqq m)$: Two double monomials μ and $\mu' \in V_k$ are equivalent if and only if $(X_0^{m} * X_0^{m})(\mu-\mu')=0$. As the special double monomials are proper vectors of the linear mapping $z \rightarrow (X_0^{m} * X_0^{m})z$, we see immediately that any two of them are not equivalent. On the other hand, every double monomial is equivalent to one of the special double monomials. In fact, $(X_0^{i_0}X_1^{i_1} \dots X_n^{i_n})*(X_0^{j_0}X_1^{j_1} \dots X_n^{j_n})$ is equivalent to $X_0^{m} * X_0^{i_0+j_0-m}X_1^{i_1+j_1} \dots X_n^{i_n+j_n}$, a special one (see (5) above). It is also clear that two double monomials $(X_0^{i_0} \dots X_n^{i_n})*(X_0^{j_0} \dots X_n^{j_n})$ and $(X_0^{r_0} \dots X_n^{r_n})*(X_0^{s_0} \dots X_n^{s_n})$ are equivalent if and only if $i_k+j_k=r_k+s_k$ for all $k=0, 1, \dots, n$. Hence every double monomial $\mu \in V_k$ is equivalent to one and only one special double monomial in V_k.

From this, it is possible to separate the basis of V_k in equivalence classes, one for each special double monomial and consequently we have a direct sum decomposition of V_k as follows: Call $\mu_1, \dots, \mu_s$ the special double monomials in V_k where $s=\binom{n+k-1}{k}$, and let V_{ki} be the subspace of V_k generated by the equivalence class of μ_i $(i=1, \dots, s)$. Then $V_k = V_{k1} \oplus \dots \oplus V_{ks}$.

(7) Observe now that each V_{ki} is again an invariant subspace of the linear mapping $z \to (X_0^{m*}X_0^m)z$, $z \in Z(n+1,2m)$. Moreover if we call ϕ the linear form on $Z(n+1,2m)$ taking the value 1 on each double monomial of the canonical basis, we have for any $z \in V_{ki}$, $(X_0^{m*}X_0^m)z = t_k\phi(z)\mu_i$.

$$(X_0^{m*}X_0^m)z = t_k\phi(z)\mu_i.$$

(8) We call V_k^{sp} the subspace of V_k generated by the special double monomials in the basis of V_k. In particular $V_1^{sp} = V_1$. The subspace $V_0^{sp} \oplus V_1^{sp} \oplus \ldots \oplus V_m^{sp}$ of $Z(n+1,2m)$ is isomorphic, as a vector space, to $G(n+1,2m)$. The isomorphism is given by

$X_0^{m*}X_0^{m-k}X_1^{i_1}\ldots X_n^{i_n} \to X_0^{m-k}X_1^{i_1}\ldots X_n^{i_n} \in G(n+1,2m)$, $i_1+\ldots+i_n = k$ and $0 \leqq k \leqq m$.

(9) In the following, double monomials will be denoted by

$$\mu = X_0^{m-r}X_{i_1}\ldots X_{i_r}{}^*X_0^{m-s}X_{j_1}\ldots X_{j_s}$$

where $r \leqq s$ and $1 \leqq i_1 \leqq \ldots \leqq i_r \leqq n$, $1 \leqq j_1 \leqq \ldots \leqq j_s \leqq n$. Given such μ, with $r+s \leqq 2m-1$ and $r < s$, we can define $\mu_{(1)}, \ldots, \mu_{(n)}$ by $\mu_{(i)} = X_0^{m-r-1}X_{i_1}\ldots X_i \ldots X_{i_r}{}^*X^{m-s}X_{j_1}\ldots X_{j_s}$ and, if $m < s$, we can define $\mu^{(1)}, \ldots, \mu^{(n)}$ by

$$\mu^{(i)} = X_0^{m-r}X_{i_1}\ldots X_{i_r}{}^*X_0^{m-s-1}X_{j_1}\ldots X_i\ldots X_{j_s}$$

In any case where $\mu^{(i)}$ and $\mu_{(i)}$ both exist, they are in V_{r+s+1} and are equivalent ((6) above). We can, of course, iterate this process. In particular we have

$$X_0^{m-r}X_{i_1}\ldots X_{i_r}{}^*X_0^{m-s}X_{j_1}\ldots X_{j_s} = (X_0^{m*}X_0^{m-s}X_{j_1}\ldots X_{j_s})_{(i_1)\ldots(i_r)}.$$

Recall that for every derivation δ of $Z(n+1,2m)$ we have $\omega\cdot\delta = 0$.

Lemma 8. *For every derivation δ of $Z(n+1,2m)$, $\delta(X_0^{m*}X_0^m) \in V_1$.*

Proof. Call $\delta(X_0^{m*}X_0^m) = A + v_2 + \ldots + v_m + v_{m+1} + \ldots + v_{2m}$ where $A \in V_1$ and $v_k \in V_k$ $(k = 2, \ldots, 2m)$. The idempotence of $X_0^{m*}X_0^m$ implies

$$2(X_0^{m*}X_0^m)(A + v_2 + \ldots + v_m + \ldots + v_{2m}) = A + v_2 + \ldots + v_m + \ldots + v_{2m}$$

But $v_{m+1}, \ldots, v_{2m}$ are absolute divisors of zero so we are reduced to

$$2(X_0^{m*}X_0^m)A + 2(X_0^{m*}X_0^m)v_2 + \ldots + 2(X_0^{m*}X_0^m)v_m = A + v_2 + \ldots + v_{2m}.$$

As each V_k $(0 \leqq k \leqq m)$ is invariant ((5) above), we must have

$$\begin{cases} 2(X_0^{m*}X_0^m)A = A \\ 2(X_0^{m*}X_0^m)v_k = v_k \quad (k = 2, \ldots, m) \\ v_{m+1} = \ldots = v_{2m} = 0. \end{cases}$$

The elements of V_1 are all proper vectors of the linear mapping $z \to (X_0^{m*} X_0^m)z$, corresponding to the proper value $t_1 = {}^1/_2$, so the first equality is an identity in A. The equations corresponding to $k = 2, \ldots, m$ have only the trivial solution $v_k = 0$ because otherwise v_k would be a proper vector corresponding to the proper value $t_1 = {}^1/_2$ and so $v_k \in V_1 \cap V_k = 0$, a contradiction.

From now on, we will call

$\delta(X_0^{m*} X_0^m) = A = \sum_{i=1}^{n} \alpha_i X_0^{m*} X_0^{m-1} X_i$ where $\alpha_1, \ldots, \alpha_n$ are real numbers.

Lemma 9. *Let δ be a derivation of $Z(n+1,2m)$ and*

$$\mu = X_0^{m*} X_0^{m-k} X_{j_1} \ldots X_{j_k} \in V_k \quad (1 \leqq k \leqq m-1)$$

a special double monomial. Then

$$\delta(\mu) = P_{j_1 \cdots j_k} + t_1 \left[\sum_{i=1}^{n} \alpha_i \mu_{(i)} + \frac{t_{k+1}}{t_k - t_{k+1}} \sum_{i=1}^{n} \alpha_i \mu^{(i)} \right]$$

where $P_{j_1 \cdots j_k}$ is some element of V_k^{sp} depending on μ.

Proof. Call $\delta(\mu) = v_1 + v_2 + \ldots + v_{2m}$ with $v_i \in V_i$ $(i = 1, \ldots, 2m)$. To $(X_0^{m*} X_0^m)\mu = t_k \mu$ we apply δ and obtain

$$A\mu + (X_0^{m*} X_0^m)(v_1 + \ldots + v_{2m}) = t_k(v_1 + \ldots + v_{2m}).$$

By the invariance of the subspaces V_i, the fact that $A\mu \in V_{k+1}$ and $v_{m+1}, \ldots, v_{2m}$ are absolute divisors of zero, we must have

$$\begin{cases} (X_0^{m*} X_0^m) v_i = t_k v_i \quad (i = 1, \ldots, m \text{ but } i \neq k+1) \\ A\mu + (X_0^{m*} X_0^m) v_{k+1} = t_k v_{k+1} \\ v_{m+1} = \ldots = v_{2m} = 0.. \end{cases}$$

In the first set of equations we distinguish $i = k$ and $i \neq k$. When $i \neq k$, we must have $v_i = 0$ as the only solution, because otherwise v_i would be a proper vector of the linear mapping $z \to (X_0^{m*} X_0^m)z$ corresponding to the proper value t_k and so $v_i \in V_i \cap V_k = 0$, a contradiction.

Now the case $i = k$. We will prove that v_k is a linear combination of the special double monomials in V_k. For this let $\mu_1, \ldots, \mu_s$ be the special double monomials in V_k, so $s = \binom{n+k-1}{k}$. Let now V_{k_i} be the subspace of V_k generated by the equivalence class of μ_i ((6) above), so $V_k = V_{k1} \oplus \ldots \oplus V_{ks}$. We have the decomposition $v_k = v_{k1} + \ldots + v_{ks}$, $v_{ki} \in V_{ki}$. The equation

$(X_0^{m*} X_0^m) v_k = t_k v_k$ becomes $(X_0^{m*} X_0^m)(v_{k1} + \ldots + v_{ks}) = t_k(v_{k1} + \ldots + v_{ks})$.

But each V_{ki} is invariant ((7) above) so the equation splits in the following system of s equations:

$$\begin{cases} (X_0^{m*}X_0^m)v_{k1} = t_k v_{k1} \\ \vdots \\ (X_0^{m*}X_0^m)v_{ks} = t_k v_{ks} \end{cases}$$

Take one of these equations, say $(X_0^{m*}X_0^m)v_{ki} = t_k v_{ki}$ $(1 \leqq i \leqq s)$. Call $\mu_i^1 = \mu_i, \mu_i^2, \ldots, \mu_i^p$ the double monomials equivalent to μ_i, that is, the basis of V_{ki}. We must have $v_{ki} = \beta_1\mu_i^1 + \beta_2\mu_i^2 + \ldots + \beta_p\mu_i^p$ for some real numbers $\beta_1, \ldots, \beta_p$. Then:

$$t_k(\beta_1\mu_i^1 + \ldots + \beta_p\mu_i^p) = (\beta_1 + \ldots + \beta_p)t_k\mu_i^1$$

which implies, by comparison of coordinates, that $\beta_2 = \ldots = \beta_p = 0$ hence $v_{ki} = \beta_1\mu_i$. It follows that v_k is a linear combination of the special double monomials $\mu_1, \ldots, \mu_s$, that is, $v_k \in V_k^{sp}$. We denote, from now on, v_k by $P_{j_1 \cdots j_k}$.

We turn now to the more difficult equation

$$A\mu + (X_0^{m*}X_0^m)v_{k+1} = t_k v_{k+1}.$$

As we have noticed before the lemma, we have in V_{k+1} the special double monomials $\mu^{(1)}, \ldots, \mu^{(n)}$ and the non-special ones $\mu_{(1)}, \ldots, \mu_{(n)}$. In V_{k+1} there are $\binom{k+1+n-1}{k+1} = \binom{k+n}{k+1}$ special double monomials and so we may suppose that $\mu^{(1)}, \ldots, \mu^{(n)}$ are the first n of them. Having made this convention, we decompose V_{k+1} according to (6) above:

$$V_{k+1} = V_{k+1,1} \oplus \ldots \oplus V_{k+1,n} \oplus \ldots \oplus V_{k+1,r} \text{ where } r = \tbinom{n+k}{k+1}.$$

For $1 \leqq i \leqq n$, $V_{k+1,i}$ has a basis formed by $\mu^{(i)}$, $\mu_{(i)}$ and some other double monomials. Decompose v_{k+1} of the above equation as

$$v_{k+1} = v_{k+1,1} + \ldots + v_{k+1,n} + \ldots + v_{k+1,r}.$$

We have

$$A\mu = \sum_{i=1}^{n} \alpha_i(X_0^{m*}X_0^{m-1}X_i)(X_0^{m*}X_0^{m-k}X_{j_1} \ldots X_{j_k}) =$$

$$= t_1 t_k \sum_{i=1}^{n} \alpha_i(X_0^{m-1}X_i)^*(X_0^{m-k}X_{j_1} \ldots X_{j_k}) = t_1 t_k \sum_{i=1}^{n} \alpha_i\mu_{(i)},$$

which belongs to $V_{k+1,1} \oplus \ldots \oplus V_{k+1,n}$. Our equation becomes:

$$t_1 t_k \sum_{i=1}^{n} \alpha_i\mu_{(i)} + (X_0^{m*}X_0^m)(v_{k+1,i} + \ldots + v_{k+1,r}) = t_k(v_{k+1,1} + \ldots + v_{k+1,r})$$

and so we must have:

$$\begin{cases} t_1 t_k \alpha_i \mu_{(i)} + (X_0^{m*} X_0^m) v_{k+1,i} = t_k v_{k+1,i} & (1 \leqq i \leqq n) \\ (X_0^{m*} X_0^m) v_{k+1,i} = t_k v_{k+1,i} & (n+1 \leqq i \leqq r) \end{cases}$$

The last r-n equations of this system have only the trivial solution $v_{k+1,i} = 0$ by the proper vector argument used above. We analyse now an equation corresponding to $1 \leqq i \leqq n$. Decompose $v_{k+1,i}$ as $\lambda \mu_{(i)} + \lambda' \mu^{(i)} + u$ where $\lambda, \lambda' \in R$ and u is a linear combination of monomials in the basis of $V_{k+1,i}$, different from $\mu_{(i)}$ and $\mu^{(i)}$. So:

$$\alpha_i t_1 t_k \mu_{(i)} + t_{k+1}(\lambda + \lambda' + \phi(u))\mu^{(i)} = t_k(\lambda \mu_{(i)} + \lambda' \mu^{(i)} + u).$$

By comparison of coordinates we have:

$$\begin{cases} \alpha_i t_1 t_k = \lambda t_k \\ t_{k+1}(\lambda + \lambda' + \phi(u)) = t_k \lambda' \\ 0 = t_k u \end{cases}$$

Hence $u = 0$, $\lambda = \alpha_i t_1$ and $\lambda' = \alpha_i t_1 \dfrac{t_{k+1}}{t_k - t_{k+1}}$.

This means that

$$v_{k+1,i} = t_1 \alpha_i \mu_{(i)} + t_1 \frac{t_{k+1}}{t_k - t_{k+1}} \alpha_i \mu^{(i)}$$

which gives

$$v_{k+1} = \sum_{i=1}^{r} v_{k+1,i} = \sum_{i=1}^{n} v_{k+1,i} = t_1 \left[\sum_{i=1}^{n} \alpha_i \mu_{(i)} + \frac{t_{k+1}}{t_k - t_{k+1}} \sum_{i=1}^{n} \alpha_i \mu^{(i)} \right].$$

The following lemma 10 has a similar but easier proof.

Lemma 10. *Let δ be a derivation of $Z(n+1,2m)$, $\mu = X_0^{m*} X_{j_1} \dots X_{j_m} \in V_m$ a special double monomial. Then $\delta(\mu) = P_{j_1 \dots j_m} + t_1 \sum_{i=1}^{n} \alpha_i \mu_{(i)}$ where $P_{j_1 \dots j_m}$ is some element of V_m^{sp} depending on μ.*

If we make the convention $t_{m+1} = 0$, then lemma 10 can be absorbed by lemma 9.

We remark that $A\mu = t_1 t_k \sum_{i=1}^{n} \alpha_i \mu_{(i)}$ and so it is usefull to denote by $\overline{A\mu}$ the "conjugate" $t_1 t_k \sum_{i=1}^{n} \alpha_i \mu^{(i)} \in V_{k+1}^{sp}$. With these notations we have for $\mu = X_0^{m*} X_0^{m-k} X_{j_1} \dots X_{j_k} \in V_k^{sp}$ that

$$\delta(\mu) = P_{j_1 \cdots j_k} + \frac{1}{t_k}\left[A\mu + \frac{t_{k+1}}{t_k - t_{k+1}}\overline{A\mu}\right].$$

In the following we will abbreviate

$$\frac{1}{t_k}\left(A\mu + \frac{t_{k+1}}{t_k - t_{k+1}}\overline{A\mu}\right) \in V_{k+1}$$

by (A, μ). Observe that (A, μ) is a bilinear function of A and μ.

Having obtained the effect of δ on special double monomials, we can now obtain its effect on non-special ones. If

$$\mu = X_0^{m-r}X_{i_1} \dots X_{i_r} * X_0^{m-s}X_{j_1} \dots X_{j_s}, \text{ with } r \leqq s,$$

then taking

$$\mu_1 = X_0^m * X_0^{m-r}X_{i_1} \dots X_{i_r} \in V_r^{sp} \text{ and } \mu_2 = X_0^m * X_0^{m-s}X_{j_1} \dots X_{j_s} \in V_s^{sp}$$

we have $\mu_1\mu_2 = t_r t_s \mu$ and so

$$\delta(\mu) = \frac{1}{t_r t_s}[\mu_1(P_{j_1 \cdots j_s} + (A, \mu_2)) + \mu_2(P_{i_1 \cdots i_r} + (A, \mu_1))] =$$

$$= \frac{1}{t_r t_s}[(\mu_1 P_{j_1 \cdots j_s} + \mu_2 P_{i_1 \cdots i_r}) + (\mu_1(A, \mu_2) + \mu_2(A, \mu_1))] \in V_{r+s} \oplus V_{r+s+1}.$$

From Lemma 9, we have $\delta(X_0^m * X_0^{m-1}X_i) = P_i + (A, X_0^m * X_0^{m-1}X_i)$ for every $1 \leqq i \leqq n$, where $P_i \in V_1$. Suppose now

$$\mu = X_0^m * X_0^{m-k}X_{i_1} \dots X_{i_k} \in V_k^{sp}.$$

Then μ is equivalent to

$$\tilde{\mu}_{(i_k)} = X_0^{m-1}X_{i_k} * X_0^{m-k+1}X_{i_1} \dots X_{i_{k-1}} \text{ and so } (X_0^m * X_0^m)\mu = (X_0^m * X_0^m)\tilde{\mu}_{(i_k)}.$$

This equality gives the following equality between components of their derivatives in the subspace V_k:

$$t_1 t_{k-1}(X_0^m * X_0^m)P_{i_1 \cdots i_k} =$$

$$= (X_0^m * X_0^m)\left[(X_0^m * X_0^{m-1}X_{i_k})\, P_{i_1 \cdots i_{k-1}} + (X_0^m * X_0^{m-k+1}X_{i_1} \dots X_{i_{k-1}})\, P_{i_k}\right].$$

But $(X_0^m * X_0^m)\, P_{i_1 \cdots i_k} = t_k P_{i_1 \cdots i_k}$ because $P_{i_1 \cdots i_k} \in V_k^{sp}$ and this implies

$$P_{i_1 \cdots i_k} = \frac{1}{t_1 t_{k-1} t_k}(X_0^m * X_0^m)\left[X_0^m * X_0^{m-1}X_{i_k})\, P_{i_1 \cdots i_{k-1}} + \right.$$

$$\left. + (X_0^m * X_0^{m-k+1}X_{i_1} \dots X_{i_{k-1}})\, P_{i_k}\right],$$

a recurrence relation which shows that each $P_{i_1 \cdots i_k}$ $(2 \leqq k \leqq m)$ can be expressed linearly as a function of the elements $P_1, \dots, P_n \in V_1$.

Theorem 3. The duplication mapping is an isomorphism of Lie algebras for every $G(n+1,2m)$.

Proof. From the preceding lemmas, we see that given a derivation δ of $Z(n+1,2m)$, we can associate to it a sequence $(A, P_1, \ldots, P_n)$ of elements of V_1, given by

$$A = \delta(X_0^{m*}X_0^m) \text{ and } \delta(X_0^m * X_0^{m-1}X_i) = P_i + (A, X_0^m * X_0^{m-1}X_i) \text{ for } 1 \leqq i \leqq n.$$

Conversely if we give a sequence $(A, P_1, \ldots, P_n)$ of elements of V_1, we define δ by $\delta(X_0^m * X_0^m) = A$, $\delta(X_0^m * X_0^{m-1}X_i) = P_i + (A, X_0^m * X_0^{m-1}X_i)$, extending to the whole basis by the recurrence formulae appearing in the lemmas. It is not difficult to prove that δ is indeed a derivation of $Z(n+1,2m)$. The correspondence $\delta \to (A, P_1, \ldots, P_n)$ is clearly linear and bijective. This means that the dimension of the derivation algebra of $Z(n+1,2m)$ is n^2+n, which shows that the duplication mapping is an isomorphism.

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