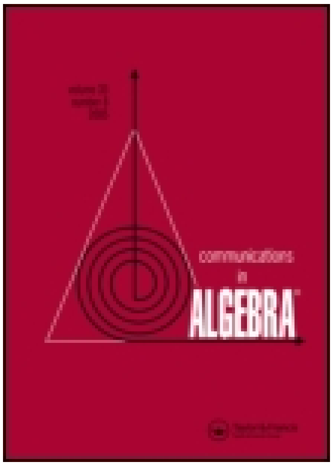

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R. Costa [a], L.S. Ikemoto [a] & A. Suazo [b]

[a] Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281 - Agêencia Cidade de São Paulo, São Paulo, 05315-970, Brazil E-mail:

[b] Facultad de Ciencias, Universidad de La Serena Benavente, Benavente, La Serena, 980, Chile E-mail:

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ON THE MULTIPLICATION ALGEBRA OF A BERNSTEIN ALGEBRA*

R. Costa† and L. S. Ikemoto‡

Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 66281 – Agência Cidade de São Paulo
05315-970 – São Paulo – Brazil
e-mail: rcosta@ime.usp.br and ikemoto@ime.usp.br

A. Suazo§

Facultad de Ciencias – Universidad de La Serena
Benavente, 980 – La Serena, Chile
e-mail: asuazo@elqui.cic.userena.cl

Abstract

This paper deals with the way in which some restrictions on the structure of Bernstein algebras and on the multiplication algebra of a Bernstein algebra interrelate, the first class of restrictions being expressed as the vanishing of some products between Peirce subspaces, and the second by bounds on the dimension of the multiplication algebra (or some of its key subspaces). The course of this enquiry leads to the introduction in Section 4 of some new numerical invariants of Bernstein algebras, namely, the dimensions of the subspaces U^{2k} and $L_U^{2k}(V)$.

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1 Preliminaries

This paper is a natural continuation of [3], where the authors established basic properties of the multiplication algebra $M(A)$ of a Bernstein algebra A. All the notation and terminology used here can be found in that article, as well as the proofs of the results about $M(A)$ that we assume here. In this article we introduce some new invariants for Bernstein algebras and establish some connections with other known invariants.

We recall that a baric algebra over a field F is a pair (A, ω), where A is a not necessarily associative algebra over F and $\omega : A \longrightarrow F$ is a nonzero homomorphism: in this case ω is called the weight function of (A, ω). Moreover, $N = \ker \omega$ is a two-sided ideal of A of codimension 1. A baric algebra (A, ω) is Bernstein if A is commutative and $(x^2)^2 = \omega(x)x^2$, for all $x \in A$. From now on we consider only finite dimensional Bernstein algebras over fields of characteristic not 2. Given a Bernstein algebra (A, ω), the elements x^2 with $\omega(x) = 1$ are all the nonzero idempotents in A, and given a nonzero idempotent e, we have the Peirce decomposition relative to this idempotent:

$$A = Fe \oplus U_e \oplus V_e \tag{1}$$

where $U_e = \{x \in N : 2xe = x\}$ and $V_e = \{x \in N : xe = 0\}$. Unless necessary, we omit the subscript e in U_e and V_e. The relations

$$U^2 \subseteq V,\ UV \subseteq U,\ V^2 \subseteq U,\ UV^2 = 0 \tag{2}$$

as well as the identities for $u \in U$ and $v \in V$

$$u^3 = 0;\ u(uv) = 0;\ uv^2 = 0;\ u^2(uv) = 0;\ u^2v^2 = 0;\ (uv)^2 = 0 \tag{3}$$

hold in A. The dimensions of U and V in (1) are invariant under change of idempotents, and the pair $(1 + \dim U, \dim V)$, which therefore is well defined, is called the *type* of A. The subspace $L = \{u \in U : uU = 0\}$ is an ideal of A that is independent of the chosen idempotent. Moreover, A/L is a Bernstein Jordan algebra, that is, its elements satisfy the identity $x^2(yx) = (x^2y)x$. Nevertheless, the smallest ideal I of A such that A/I is Jordan is that generated by all the elements $x^3 - \omega(x)x^2$, $x \in A$. See, for instance, [4, th. 3.4.19] or [1, Section 3]. When $U^2 = 0$ in (2), A is called *exceptional* and when $UV = V^2 = 0$, A is *normal*. These conditions are independent of the chosen idempotent in A. Other characterizations of these algebras can be found in [4] or [6], as well as the basic theory of Bernstein algebras.

Given an arbitrary algebra A over the field F, its multiplication algebra $M(A)$ (over F) is the subalgebra of End(A) generated by the operators L_x and R_x, defined by $L_x(a) = xa$, $R_x(a) = ax$, $a, x \in A$. In [3], the authors proved the following results about $M(A)$ when A is Bernstein:

(a) $M(A)$ is baric with weight function ϖ, where $\varpi(L_x) = \omega(x)$ on the generators L_x of $M(A)$ and the operator $2L_e^2 - L_e$ is an idempotent of weight 1. This operator is the projection of A onto Fe in (1). We have the decomposition $M(A) = F(2L_e^2 - L_e) \oplus (N : A)$, where $(N : A) = \{\sigma \in M(A) : \sigma(A) \subseteq$

$N\} = \{\sigma \in M(A) : \omega \circ \sigma = 0\}$ and $N = \ker\omega$. Moreover, $4L_e - 4L_e^2$ is another idempotent of $M(A)$, orthogonal to $2L_e^2 - L_e$. The operator $4L_e - 4L_e^2$ is the projection of A onto U in (1), so that $4L_e - 4L_e^2 \neq 0$ if and only if $U \neq 0$. Any element $\sigma \in M(A)$ can be decomposed in the form $\sigma = \alpha(2L_e^2 - L_e) + \beta(4L_e - 4L_e^2) + \theta$, where $\theta \in \tilde{N}$, $\alpha, \beta \in F$ and $\tilde{N}$ is the ideal of $M(A)$ generated by the set $\{L_x : x \in N\}$. In general, $M(A)$ has not a unity, so the projection of A onto V in (1) does not belong to $M(A)$.

(b) For a fixed idempotent $e \in A$, the algebra $M(A)$ can be decomposed as $M(A) = F(2L_e^2 - L_e) \oplus \tilde{U} \oplus \tilde{V}$ where $\tilde{U} = \{\sigma \in (N : A) : \sigma(2L_e^2 - L_e) = \sigma\} = \{\sigma \in (N : A) : \sigma(N) = 0\} = \{\psi_x : x \in U \oplus U^2\}$, with $\psi_x(e) = x$ and $\psi_x(N) = 0$ and $\tilde{V} = \{\sigma \in (N : A) : \sigma(2L_e^2 - L_e) = 0\} = \{\sigma \in (N : A) : \sigma(e) = 0\}$. The application $\psi : U \oplus U^2 \to \tilde{U}$ given by $x \mapsto \psi_x$ is an isomorphism of vector spaces. In particular, $\dim \tilde{U} = \dim U + \dim U^2$ and it is an invariant of A.

(c) The subspaces $\tilde{U}$ and $\tilde{V}$ of $M(A)$ verify the following relations

$$\tilde{U}^2 = 0,\ \tilde{U}\tilde{V} = 0,\ \tilde{V}\tilde{U} \subseteq \tilde{U},\ \tilde{V}^2 \subseteq \tilde{V} \tag{4}$$

so $\tilde{U}$ is an ideal of zero square and $\tilde{V}$ is a subalgebra of $M(A)$. Consider $\sigma \in M(A)$ decomposed in the form $\sigma = \alpha(2L_e^2 - L_e) + \psi + \theta$, where $\psi \in \tilde{U}$ and $\theta \in \tilde{V}$. If we denote θ by $[\sigma]$, we have, from (4), that the mapping $\sigma \mapsto [\sigma]$ is a homomorphism (of algebras) from $M(A)$ onto $\tilde{V}$.

(d) The idempotent $4L_e - 4L_e^2 \in \tilde{V}$ gives rise to the usual Peirce decomposition of $\tilde{V}$: $\tilde{V} = \tilde{V}_{11} \oplus \tilde{V}_{10} \oplus \tilde{V}_{01} \oplus \tilde{V}_{00}$. We can characterize the elements of $\tilde{V}_{ij}$ using the Peirce decomposition of A in the following way:

$$\sigma \in \tilde{V}_{11} \Leftrightarrow (4L_e - 4L_e^2)\sigma = \sigma(4L_e - 4L_e^2) = \sigma \Leftrightarrow \sigma(U) \subseteq U \text{ and } \sigma(V) = 0 \tag{5}$$
$$\sigma \in \tilde{V}_{10} \Leftrightarrow (4L_e - 4L_e^2)\sigma = \sigma \text{ and } \sigma(4L_e - 4L_e^2) = 0 \Leftrightarrow \sigma(U) = 0 \text{ and } \sigma(V) \subseteq U \tag{6}$$
$$\sigma \in \tilde{V}_{01} \Leftrightarrow (4L_e - 4L_e^2)\sigma = 0 \text{ and } \sigma(4L_e - 4L_e^2) = \sigma \Leftrightarrow \sigma(U) \subseteq V \text{ and } \sigma(V) = 0 \tag{7}$$
$$\sigma \in \tilde{V}_{00} \Leftrightarrow (4L_e - 4L_e^2)\sigma = \sigma(4L_e - 4L_e^2) = 0 \Leftrightarrow \sigma(U) = 0 \text{ and } \sigma(V) \subseteq V \tag{8}$$

Observe that $4L_e - 4L_e^2 \in \tilde{V}_{11}$ and $\tilde{V}_{ij}\tilde{V}_{kl} \subseteq \delta_{jk}\tilde{V}_{il}$. The decomposition of $M(A)$ given by

$$M(A) = F(2L_e^2 - L_e) \oplus \tilde{U} \oplus \tilde{V}_{11} \oplus \tilde{V}_{10} \oplus \tilde{V}_{01} \oplus \tilde{V}_{00} \tag{9}$$

is called the *complete Peirce decomposition* of $M(A)$ relative to the idempotent $e \in A$. The $\tilde{V}_{ij}$ component of $\sigma \in M(A)$ will be denoted by $[\sigma]_{ij}$. It was proved in [3] that the dimensions of the subspaces $\tilde{V}_{ij}$ in (9) are invariant under change of idempotents in A hence we have some new invariants for A.

2 On the subspace $\tilde{V}$

While the subspace $\tilde{U}$ in (9) has a very simple structure, since it is isomorphic to $U \oplus U^2$ under the mapping $x \in U \oplus U^2 \mapsto \psi_x \in \tilde{U}$, the subspace $\tilde{V}$ reflects the complexity of the Bernstein algebras. The subspaces $\widetilde{V}_{ij}$ in the decomposition of $M(A)$ given in (9) provide some information about the algebra A. The coming propositions show us how. Let $x = \alpha e + u + v \in A$, according to (1). The decomposition of L_x relative to the direct sum given in (9) is

$$L_x = \alpha(2L_e^2 - L_e) + \tfrac{1}{2}\psi_u + [L_x] \tag{10}$$

where $[L_x] = [L_x]_{11} + [L_x]_{10} + [L_x]_{01} + [L_x]_{00}$ with

$$[L_x]_{11} = \tfrac{1}{2}\alpha(4L_e - 4L_e^2) + 2L_vL_e \tag{11}$$
$$[L_x]_{10} = (2L_eL_u - \tfrac{1}{2}\psi_u) + (L_v - 2L_vL_e) \tag{12}$$
$$[L_x]_{01} = L_u - 2L_eL_u \tag{13}$$
$$[L_x]_{00} = 0 \tag{14}$$

The proof of these igualities can be easily obtained using the characterizations of the elements of $\widetilde{V}_{ij}$ given above and relations (3). To exemplify, $L_u - 2L_eL_u \in \widetilde{V}_{01}$ because $(L_u - 2L_eL_u)(e) = 0$, $(L_u - 2L_eL_u)(U) \subseteq U^2 \subseteq V$ and $(L_u - 2L_eL_u)(V) = 0$.

The set $\{L_u - 2L_eL_u : u \in U\}$ is a vector subspace of $\widetilde{V}_{01}$ and the mapping $u \in U \mapsto L_u - 2L_eL_u \in \widetilde{V}_{01}$ is linear. By (7), we have $L_u - 2L_eL_u = 0$ if and only if $(L_u - 2L_eL_u)(U) = 0$, that is, if and only if $u \in L$, where $L = \mathrm{ann}_U U = \{u \in U : uU = 0\}$. So, we have the following result:

Proposition 1 *The vector subspace of $\widetilde{V}_{01}$ consisting of all operators of the form $L_u - 2L_eL_u$, $u \in U$, has the same dimension as U/L and so, the dimension of this subspace of $\widetilde{V}_{01}$ is an invariant of A. In particular,* $\dim \widetilde{V}_{01} \geq \dim U - \dim L$. ■

For all Bernstein algebras of type $(1 + r, s)$ with $r \geq 1$ we have $\widetilde{V}_{11} \neq 0$ because $4L_e - 4L_e^2 \in \widetilde{V}_{11}$. But the other 3 direct summands of $\widetilde{V}$ in (9) may be zero, according to the following propositions.

Proposition 2 *With the previous notations, we have the equivalence between*

(i) A is normal

(ii) $\widetilde{V}_{10} = 0$.

Proof: (i) ⇒ (ii): We have $[L_x] = [L_x]_{11} + [L_x]_{10} + [L_x]_{01}$, for all $x \in A$. As $UV = V^2 = 0$, we have $[L_x]_{10} = 0$, and so, $[L_x] \in \widetilde{V}_{11} \oplus \widetilde{V}_{01}$, which is a subalgebra of $M(A)$. Then, for $x_1, \ldots, x_k \in A$, $[L_{x_1} \ldots L_{x_k}] = [L_{x_1}] \ldots [L_{x_k}] \in \widetilde{V}_{11} \oplus \widetilde{V}_{01}$, that is, $\widetilde{V}_{10} = \widetilde{V}_{00} = 0$.

(ii) ⇒ (i): If $\widetilde{V}_{10} = 0$ then, by (12), for all $u \in U$ and $v \in V$ we have $2L_eL_u = \frac{1}{2}\psi_u$ and $L_v = 2L_vL_e$. If we evaluate on $v' \in V$, we obtain $uv' = 2L_eL_u(v') = \frac{1}{2}\psi_u(v') = 0$ and $vv' = L_v(v') = 2L_vL_e(v') = 0$, and so $UV = V^2 = 0$. ■

Proposition 3 *The following conditions about the Bernstein algebra A are equivalent:*

(i) A is exceptional

(ii) $\widetilde{V}_{01} = 0$

Proof: (i) $\Rightarrow$ (ii): As in the proof of the Proposition 2, it's sufficient to prove that $[L_x] \in \widetilde{V}_{11} \oplus \widetilde{V}_{10}$. As $[L_x]_{01}(U) \subseteq U^2 = 0$, by (7), we have $[L_x]_{01} = 0$.

(ii) $\Rightarrow$ (i): For all $u \in U$, we have $L_u = 2L_eL_u$, that is, $uU = L_u(U) = 2L_eL_u(U) = 0$ and so $U^2 = 0$. ■

Proposition 4 *With the same hypothesis, the following conditions are equivalent:*
(i) $U(UV) = 0$
(ii) $\widetilde{V}_{00} = 0$

Proof: (i) $\Rightarrow$ (ii): We prove by induction on k that $[L_{x_1} \dots L_{x_k}]_{00} = 0$ and that there exists $u_0 \in U$ such that $[L_{x_1} \dots L_{x_k}]_{01} = L_{u_0} - 2L_eL_{u_0}$. The case $k = 1$ is verified using the decomposition of L_x in (10). Suppose that $[\sigma]_{01} = L_{u_0} - 2L_eL_{u_0}$ and $[\sigma]_{00} = 0$, for $\sigma = L_{x_1} \dots L_{x_k}$. If $x = \alpha e + u + v$ then

$$[\sigma L_x]_{01} = [\sigma]_{01}[L_x]_{11} + [\sigma]_{00}[L_x]_{01} = (L_{u_0} - 2L_eL_{u_0})(\tfrac{1}{2}\alpha(4L_e - 4L_e^2) + 2L_vL_e) = \\ = \tfrac{1}{2}\alpha(L_{u_0} - 2L_eL_{u_0}) + (L_{u_0} - 2L_eL_{u_0})(2L_vL_e)$$

As $U(UV) = 0$, it follows that $(L_{u_0} - 2L_eL_{u_0})(2L_vL_e) = 0$. So $[\sigma L_x]_{01} = L_{\frac{1}{2}\alpha u_0} - 2L_eL_{\frac{1}{2}\alpha u_0}$. Moreover, $[\sigma L_x]_{00} = [\sigma]_{01}[L_x]_{10} = (L_{u_0} - 2L_eL_{u_0})((2L_eL_u - \frac{1}{2}\psi_u) + (L_v - 2L_vL_e)) = 0$, because $(L_{u_0} - 2L_eL_{u_0})(L_v - 2L_vL_e)(V) \subseteq UV^2 = 0$ and $(L_{u_0} - 2L_eL_{u_0})(2L_eL_u - \frac{1}{2}\psi_u) \subseteq U(UV) = 0$.

(ii) $\Rightarrow$ (i): If $\widetilde{V}_{00} = 0$ then $[L_{x_1}L_{x_2}]_{00} = 0$, for all $x_1, x_2 \in A$. For $x_1 = u_1$, $x_2 = u_2 \in U$, we get $0 = [L_{u_1}L_{u_2}]_{00} = [L_{u_1}]_{01}[L_{u_2}]_{10} = (L_{u_1} - 2L_eL_{u_1})(2L_eL_{u_2} - \frac{1}{2}\psi_{u_2})$, that is, $u_1(u_2v) = 0$, for all $v \in V$ and so $U(UV) = 0$. ■

Corollary 1 *If* $A = Fe \oplus U \oplus V$ *is a Bernstein algebra with* $U(UV) = 0$ *then* $\dim \widetilde{V}_{01} = \dim U - \dim L$.

Proof: In the proof of the previous proposition, we established that all the elements of $\widetilde{V}_{01}$ have the form $L_u - 2L_eL_u$, with $u \in U$. Use now Proposition 1. ■

Corollary 2 *If* $\widetilde{V}_{01} = 0$ *or* $\widetilde{V}_{10} = 0$ *then* $\widetilde{V}_{00} = 0$. ■

Remark: We may guess, from Proposition 4, that the dimension of $U(UV)$ does not depend on the choice of the idempotent. Moreover, Bernstein algebras for which $U(UV) \neq 0$ appear only in dimension greater than 6. These facts will be proved in Section 4.

3 On the dimension of $M(A)$

If A is a Bernstein algebra of type $(1 + r, s)$ then evidently $\dim M(A) \leq (1 + r + s)^2$. We prove in this section some results about the problem of estimating the dimension of $M(A)$. Firstly, we recall that in [3] it was proved that if A has type $(1, s)$ then $\dim M(A) = 1$. Also in [3] the authors have calculated $M(A)$ when A

has type $(1 + r, 0)$. In this section we are interested in the cases $r \neq 0$ and $s \neq 0$. Bernstein algebras of a fixed type and having multiplication algebra with minimum dimension are described in this way:

Proposition 5 *If $A = Fe \oplus U \oplus V$ is a Bernstein algebra of type $(1 + r, s)$, where $r \neq 0$, then*

(a) $\dim M(A) \geq r + 2$

(b) $\dim M(A) = r + 2$ *if and only if* $N^2 = 0$

Proof:

(a) As $M(A) = F(2L_e^2 - L_e) \oplus \widetilde{U} \oplus \widetilde{V}$, $\dim \widetilde{U} = \dim(U \oplus U^2) \geq r$ and $4L_e - 4L_e^2$ is a nonzero element in $\widetilde{V}$ (because $U \neq 0$), we have $\dim M(A) \geq 2 + r$.

(b) If $N^2 = 0$ then A is normal and exceptional and by Propositions 2, 3 and 4, $\widetilde{V}_{10} = \widetilde{V}_{01} = \widetilde{V}_{00} = 0$ and $\widetilde{V}_{11} = F(4L_e - 4L_e^2)$, so $\dim M(A) = r + 2$. If $N^2 \neq 0$ then A is not normal or not exceptional, so $\widetilde{V}_{10} \neq 0$ or $\widetilde{V}_{01} \neq 0$ and then $\dim \widetilde{V} \geq 2$. ∎

In the case of nuclear algebras, we can improve the bound of Proposition 5. We recall that A is nuclear if $A^2 = A$ or equivalently $U^2 = V$ in (2). The Peirce decomposition of A is $A = Fe \oplus U \oplus U^2$.

Proposition 6 *If $A = Fe \oplus U \oplus U^2$ is a nuclear Bernstein algebra of type $(1+r, s)$, with $s \neq 0$, then* $\dim M(A) \geq 3 + r + s = 2 + \dim A$.

Proof: It's enough to prove that $\dim \widetilde{V} \geq 2$. The condition $s \neq 0$ implies that A is not exceptional and so $\widetilde{V}_{01} \neq 0$ by Proposition 3. As $\dim \widetilde{V}_{11} \geq 1$, we have $\dim \widetilde{V} \geq 2$. ∎

Here is an a example where the equality $\dim M(A) = 2 + \dim A$ of Proposition 6 holds.

Example: Let A be a (nuclear) Bernstein algebra of type $(1 + r, 1)$ with basis $\{e, u_1, \ldots, u_r, v\}$, weight function ω defined by $\omega(e) = 1$, $\omega(u_i) = 0$ $(i = 1, \ldots, r)$ and $\omega(v) = 0$ and multiplication table

$$e^2 = e;\ eu_i = \tfrac{1}{2}u_i\ (i = 1, \ldots, r);\ u_1^2 = v;\ \text{other products are zero} \tag{15}$$

This algebra is normal and the ideal $L = \operatorname{ann}_U U$ has dimension $r-1$. So, $\dim M(A) = 2 + 2r + 1 - (r - 1) = 3 + r = 2 + \dim A$.

Nuclear algebras with $\dim M(A) = 2 + \dim A$ are characterized as follows.

Proposition 7 *Let $A = Fe \oplus U \oplus U^2$ be a nuclear Bernstein algebra of type $(1+r, s)$ with $s \geq 1$ such that $\dim M(A) = 2 + \dim A$. Then $s = 1$ and A is isomorphic to the algebra (15).*

Proof: Firstly, we prove that $s = 1$. If $u \in U$ is such that $u^2 \neq 0$ then $L_u - 2L_eL_u \neq 0$. In fact, $(L_u - 2L_eL_u)(u) = u^2 \neq 0$. As $\dim M(A) = 2 + \dim A$ (recalling that $\dim \widetilde{U} = \dim U + \dim U^2$ and $\dim \widetilde{V}_{11} \geq 1$), we have $\dim \widetilde{V}_{01} = 1$, that is, $\widetilde{V}_{01}$ is generated by $L_u - 2L_eL_u$. So, given $x, y \in U$, there are $\lambda_1, \lambda_2 \in F$ such that $L_x - 2L_eL_x = \lambda_1(L_u - 2L_eL_u)$ and $L_y - 2L_eL_y = \lambda_2(L_u - 2L_eL_u)$. Then

$$\begin{aligned} xy &= (L_x - 2L_eL_x)(y) = \lambda(L_u - 2L_eL_u)(y) = \lambda_1 uy = \\ &= \lambda_1(L_y - 2L_eL_y)(u) = \lambda_1\lambda_2(L_u - 2L_eL_u)(u) = \lambda_1\lambda_2 u^2. \end{aligned}$$

This equality shows that u^2 generates U^2 and so A has type $(1 + r, 1)$. So, there is a basis $\{e, u_1, \ldots, u_r, v\}$ of A with $u_1 = u$ and $v = u^2$, and its multiplication table is given by

$$e^2 = e;\ eu_i = \tfrac{1}{2}u_i\ u_i^2 = \lambda_i v\ (i = 1, \ldots, r);\ \text{other products are zero}$$

where $\lambda_1 \neq 0$. If there is some $i > 1$ with $\lambda_i \neq 0$ then $L_{u_1} - 2L_eL_{u_1}$ and $L_{u_i} - 2L_eL_{u_i}$ would be linearly independent and so, the dimension of $\widetilde{V}_{01}$ would be greater than 1. Then $\lambda_2 = \ldots = \lambda_r = 0$ and the algebra is isomorphic to the algebra given in (15). ∎

Corollary 3 *For every $n \geq 3$ there is a unique (up to isomorphisms) nuclear Bernstein algebra A of dimension n such that $\dim M(A) = 2 + \dim A$. This algebra is of type $(n-1, 1)$.* ∎

The concepts of direct product and indecomposable baric algebras were introduced in [1] and [2]. The direct product of the baric algebras (A_1, ω_1) and (A_2, ω_2) is the subalgebra $A_1 \vee A_2$ of $A_1 \times A_2$, formed by the pairs (a_1, a_2) such that $\omega_1(a_1) = \omega_2(a_2)$. The weight function of $A_1 \vee A_2$ is defined by $\omega(a_1, a_2) = \omega_1(a_1) = \omega_2(a_2)$. A baric algebra that can be written as $A_1 \vee A_2$ where A_1 and A_2 both have dimensions greater than or equal to 2, is called *decomposable*. Otherwise, it is *indecomposable*.

Corollary 4 *The unique indecomposable nuclear Bernstein algebra A such that $\dim M(A) = 2 + \dim A$ has type $(2, 1)$ and is given by the table:*

$$e^2 = e;\ eu = \tfrac{1}{2}u;\ u^2 = v;\quad \textit{other products are zero} \tag{16}$$

where $\{e, u, v\}$ is a convenient basis, $\omega(e) = 1$, $\omega(u) = \omega(v) = 0$.

Proof: We have seen in Corollary 3 that A must be given by (15). If r is strictly greater than 1 then A will be the direct product (in baric sense) of the 3-dimensional algebra given in (16) by an algebra of type $(r, 0)$. ∎

The class of normal Bernstein algebras can be characterized in several ways, see for instance [4, Th. 3.4.15, Th. 3.4.17]. Now, we can introduce one more characterization.

Theorem 1 *The following conditions on the Bernstein algebra $A = Fe \oplus U \oplus V$ are equivalent:*

(i) A is normal;

(ii) $\dim M(A) = 2 + 2\dim U + \dim U^2 - \dim L$. ■

Proof: (i) ⇒ (ii): We have, in general, $M(A) = F(2L_e^2 - L_e) \oplus \tilde{U} \oplus \tilde{V}_{11} \oplus \tilde{V}_{10} \oplus \tilde{V}_{01} \oplus \tilde{V}_{00}$. By Proposition 2, $\tilde{V}_{10} = 0$ and so also $\tilde{V}_{00} = 0$. Let $x = \alpha e + u + v \in A$. As $L_v L_e(U) \subseteq UV = 0$, we have, for normal Bernstein algebras, a particular case of (10):

$$[L_x] = \tfrac{1}{2}\alpha(4L_e - 4L_e^2) + (L_u - 2L_e L_u)$$

If $x_i = \alpha_i e + u_i + v_i$ $(i = 1, \ldots, k)$ then

$$[L_{x_1} \ldots L_{x_k}] = (\tfrac{1}{2})^k \alpha_1 \ldots \alpha_k (4L_e - 4L_e^2) + (\tfrac{1}{2})^{k-1}\alpha_2 \ldots \alpha_k (L_{u_1} - 2L_e L_{u_1})$$

So, $\tilde{V} = F(4L_e - 4L_e^2) \oplus \{L_u - 2L_e L_u : u \in U\}$, and $\tilde{V}_{11} = F(4L_e - 4L_e^2)$ has dimension 1. Therefore $\dim \tilde{V} = \dim(\tilde{V}_{11} \oplus \tilde{V}_{01}) = 1 + \dim \tilde{V}_{01} = 1 + \dim U - \dim L$ by Corollary 1 to Proposition 4, and $\dim M(A) = 2 + 2\dim U + \dim U^2 - \dim L$ by (9).

(ii) ⇒ (i): Let A be a Bernstein algebra such that $\dim M(A) = 2 + 2\dim U + \dim U^2 - \dim L$. We have $\dim \tilde{V} = 1 + \dim U - \dim L$. As $\dim \tilde{V}_{11} \geq 1$ (because $4L_e - 4L_e^2 \in \tilde{V}_{11}$) and $\dim \tilde{V}_{01} \geq \dim U - \dim L$ we get $\dim \tilde{V}_{10} = 0$, that is, A is normal, by Proposition 2. ■

4 On the subspaces U^{2k} and $L_U^{2k}(V)$

Let $A = Fe \oplus U \oplus V$ be a Bernstein algebra. It's known that the dimensions of the subspaces $U, V, U^2, UV + V^2$ are invariant under change of idempotents in A. On the other hand, the dimension of some subspaces like UV, V^2 depends on the idempotent. In this section we prove two results concerning the invariance of the dimension of certain subspaces of a Bernstein algebra, obtained from the Peirce decomposition. Using the relations given in (4), it can be proved, by induction, that

$$U \supseteq U^3 \supseteq \ldots \supseteq U^{2t-1} \subseteq U^{2t+1} \supseteq \ldots \tag{17}$$

$$V \supseteq U^2 \supseteq \ldots \supseteq U^{2t} \subseteq U^{2t+2} \supseteq \ldots \tag{18}$$

where $U^{i+1} = U^i U$ and $U^1 = U$.

Suppose now that A is Bernstein Jordan. It's known that A satisfies the equation $x^3 - \omega(x)x^2 = 0$, see for instance [1, Prop.3.1]. In this case, the elements of $N = \ker \omega$ satisfy the identity $x^3 = 0$, and its linearization, the Jacobi identity $a_1(a_2a_3) + a_2(a_3a_1) + a_3(a_1a_2) = 0$. Using it, we have:

Proposition 8 *If X is a vector subspace of N then $X^i X^j \subseteq X^{i+j}$, for all $i, j \geq 1$.*

Proof: By induction on i. If $i = 1$, it is the definition ($XX^j = X^{j+1}$). Suppose that $i > 1$ and $X^r X^j \subseteq X^{r+j}$ holds, for $r < i$ and $j \geq 1$. As $X^i X^j = (XX^{i-1})X^j$, it's sufficient to show that $(xy)z \in X^{i+j}$ if $x \in X$, $y \in X^{i-1}$ and $z \in X^j$. Using the Jacobi identity and the induction hypothesis, it results that $(xy)z = -x(yz) - y(xz) \in X(X^{i-1}X^j) + X^{i-1}(XX^j) \subseteq X^{i+j}$ and so, $X^iX^j \subseteq X^{i+j}$, for all $i, j \geq 1$. ■

Lemma 1 *If $A = Fe \oplus U \oplus V$ is a Bernstein Jordan algebra and $N_1 = U \oplus U^2$ then $N_1^k = U^k \oplus U^{k+1}$, for all $k \geq 1$.*

Proof: By induction on k. For $k = 1$, it's clear. If $N_1^k = U^k \oplus U^{k+1}$ then $N_1^{k+1} = N_1^kN_1 = (U^k \oplus U^{k+1})(U \oplus U^2) = U^{k+1} + U^{k+2} + U^kU^2 + U^{k+1}U^2$. Using the relations given in (17) and (18) and Proposition 8, we have that $U^kU^2 \subseteq U^{k+2}$ and $U^{k+1}U^2 \subseteq U^{k+3} \subseteq U^{k+1}$. So, $N_1^{k+1} = U^{k+1} \oplus U^{k+2}$. ■

The following result can be found also in [5, Prop.2].

Corollary 5 *In a finite dimensional Bernstein Jordan algebra $A = Fe \oplus U \oplus V$, the dimension of the subspace U^k $(k = 1, 2, \ldots)$ is independent of the choice of the idempotent in A.*

Proof: It is sufficient to observe that if e_0 is another idempotent and $A = Fe_0 \oplus U_0 \oplus V_0$ is the Peirce decomposition of A relative to e_0 then $A^2 = Fe \oplus U \oplus U^2 = Fe_0 \oplus U_0 \oplus U_0^2$ and the restriction of ω to A^2 has kernel $N_1 = U \oplus U^2$ $(= U_0 \oplus U_0^2)$. By the previous lemma, $N_1^k = U^k \oplus U^{k+1}$ $(= U_0^k \oplus U_0^{k+1})$, so $\dim U^k = \dim U_0^k$, by induction. ■

For arbitrary subespaces $X, Y \subseteq N$, we denote by $L_X(Y)$ the subespace XY, and recursively, $L_X^{k+1}(Y) = XL_X^k(Y)$, for $k \geq 1$. Then we have:

Corollary 6 *If $A = Fe \oplus U \oplus V$ is a finite dimensional Bernstein Jordan algebra then the dimension of the subspace $L_U^k(V)$ is independent of the choice of the idempotent in A, for all $k \geq 1$.*

Proof: Let $N = U \oplus V$ and $N_1 = U \oplus U^2$. The subspaces $L_{N_1}^k(N^2)$ are independent of the choice of the idempotent, so their dimension is invariant. We prove, by induction on k, that $L_{N_1}^k(N^2) = U^{k+2} \oplus L_U^{k+1}(V)$. For $k = 1$, $N_1N^2 = (U \oplus U^2)(UV \oplus U^2) = U(UV) + U^3 + U^2(UV) = L_U^2(V) \oplus U^3$ because $U^2(UV) \subseteq U^3$. If $L_{N_1}^k(N^2) = U^{k+2} \oplus L_U^{k+1}(V)$ then $L_{N_1}^{k+1}(N^2) = (U \oplus U^2)(U^{k+2} \oplus L_U^{k+1}(V)) = U^{k+3} + L_U^{k+2}(U) + U^2U^{k+2} + U^2L_U^{k+1}(V) = U^{k+3} + L_U^{k+2}(U)$, because $U^2U^{k+2} \subseteq U^{k+4} \subseteq L_U^{k+2}(V)$ and $U^2L_U^{k+1}(V) \subseteq U^2U^{k+1} \subseteq U^{k+3}$, by (2) and Proposition 8. So, by the previous corollary, we have that the dimension of $L_U^k(V)$ is invariant under change of idempotents. ■

Corollary 7 *If $A = Fe \oplus U \oplus V$ is an arbitrary Bernstein algebra, the dimensions of the subspaces $L_U^{2k}(V)$ and U^{2k}, for $k \geq 1$, are invariant under change of idempotents.*

Proof: We consider the Bernstein algebra $\overline{A} = A/L$, where $L = \mathrm{ann}_U(U)$. This algebra is Jordan and has the Peirce decomposition $\overline{A} = F\overline{e} \dotplus \overline{U} \dotplus \overline{V}$ relative to $\overline{e} = e + L$. By the previous corollaries, $\overline{U}^{2k} = U^{2k} + L$ and $L^{2k}_{\overline{U}}(\overline{V}) = L^{2k}_U(V) + L$ have invariant dimensions. As $U^{2k} \cap L = 0$ by (18), $L^{2k}_U(V) \cap L = 0$ by (2) and L has invariant dimension, we have the desired result. ■

Proposition 9 *If* $\dim A \leq 6$ *then* $U(UV) = 0$ *hence* $\widetilde{V}_{00} = 0$ *in (9).*

Proof: Suppose $U(UV) \neq 0$. So, there are elements $u_1, u_2 \in U$ and $v \in V$ such that $u_1(u_2v) \neq 0$. Then we have $\{u_1, u_2, u_1v, u_2v\}$ linearly independent in U. In fact, if $\alpha_1u_1 + \alpha_2u_2 + \beta_1u_1v + \beta_2u_2v = 0$, multiplying by u_2v, we obtain $\alpha_1u_1(u_2v) = 0$, hence $\alpha_1 = 0$, using the linearisations of identities (3). In the same way, multiplying by u_1v, we get $\alpha_2 = 0$. Multiplying $\beta_1u_1v + \beta_2u_2v = 0$ by u_1 we get $\beta_2 = 0$ and so, also $\beta_1 = 0$. Moreover, we have, again by linearisations of identities (3), that $u_1(u_2(u_1(u_2v))) = -u_1(u_2(u_2(u_1v))) = \frac{1}{2}u_1(u_2^2(u_1v)) = -\frac{1}{2}(u_1v)(u_1u_2^2) = 0$. Then, $\{u_1(u_2v), v\}$ is also linearly independent: if $u_1(u_2v) = \lambda v$ with $\lambda \neq 0$ then $v = \lambda^{-1}u_1(u_2v) = \lambda^{-2}(u_1(u_2(u_1(u_2v)))) = 0$. Consequently, $\dim U \geq 4$ and $\dim V \geq 2$ so that $\dim A \geq 7$ if $U(UV) \neq 0$. ■

Example: For the following Bernstein algebra of dimension 7, we have $U(UV) \neq 0$: A is the vector space with basis $\{e, u_1, u_2, u_3, u_4, v_1, v_2\}$ and multiplication table

$$eu_i = \tfrac{1}{2}u_i;\ u_2u_3 = -u_1u_4 = v_2;\ u_1v_1 = u_3;\ u_2v_1 = u_4;\ \text{other products are zero}$$

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