

2/26/17

Example = Exercise 3 of chpt 6
Example 1 p. 46

F field, $\mathcal{F} = F_n$ column vectors $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\begin{aligned} [xyz] &:= yx^t z - xy^t z \\ &= \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{n \times 1} \underbrace{[x_1 \dots x_n]}_{1 \times n} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} - \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \underbrace{[y_1 \dots y_n]}_{n \times 1} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \\ &\quad (x, z) = \sum_{i=1}^n x_i z_i \\ &= (x, z)y - (y, z)x \end{aligned}$$

also = $\underbrace{(yx^t - xy^t)}_{n \times n} z$

~~skew~~ skewsymmetric matrix: $A^t = -A$ $yx^t - xy^t$ is skewsymmetric
 $n \times n$ notation: $M_{n,\text{skew}}(F)$

FACT Every skew-symmetric matrix is a linear combination
of $yx^t - xy^t$ $A = \sum_{j=1}^m (y_j x_j^t - x_j y_j^t)$

PROOF?

We "identify" $L(x,y)$ with ~~$yxt - xy^t$~~
 ~~$yxt - xy^t$~~ = all skew symmetric matrices

and $\text{span}\{L(x,y) : x,y \in \mathbb{F}\}$ with $\text{span}\{yxt - xy^t : x,y \in \mathbb{F}\}$

a Lie algebra

(by (6.3)(uc) on p. 44)

$$[L(x,y), L(u,v)] = L([xyu], v) + L(u, [xvy])$$

a Lie algebra

(skew symmetric matrices
form a Lie algebra)
under $AB - BA$

$$\text{let } \phi: \left(\sum_{j=1}^m L(x_j, y_j) \right) = \sum_{j=1}^m (y_j x_j^t - x_j y_j^t)$$

ϕ is well-defined

ϕ is linear

ϕ is one-to-one

ϕ is onto

$$\phi \left(\left[\sum_{j=1}^m L(x_j, y_j), \sum_{R=1}^p L(u_R, v_R) \right] \right) = \left[\sum_{j=1}^m (y_j x_j^t - x_j y_j^t), \sum_{R=1}^p (v_R u_R^t - u_R v_R^t) \right]$$

i.e. ϕ is multiplicative

$$\text{i.e. } \phi([X, Y]) = [\phi(X), \phi(Y)]$$

i.e.

ϕ is an isomorphism of Lie algebras

\mathcal{H} and $M_{n,\text{Skew}}(\mathbb{F})$

well-defined

~~$\phi(\sum L(x_j, y_j)) = \phi($~~ Suppose

$$X = \sum_j L(x_j, y_j) = \sum_k L(u_k v_k^t) = Y$$

to prove

$$\sum_j (\cancel{L(x_j, y_j)} - (y_j x_j^t - x_j y_j^t)) = \sum_k w_k u_k^t - u_k v_k^t$$

$$\phi(X) = \phi(Y)$$

Recall $L(x, y)z = (y x^t - x y^t)z$

$$Xz = \bar{\Phi}(X)z$$

(X acting
on z) $\bar{\Phi}(X)$ acting on z
(by matrix multiplication)

$$Yz = \bar{\Phi}(Y)z$$

but $Xz = Yz$ so $\bar{\Phi}(X)z = \bar{\Phi}(Y)z$

so $(\bar{\Phi}(X) - \bar{\Phi}(Y))$ is a matrix
which kills all z

so $\bar{\Phi}(X) - \bar{\Phi}(Y)$ is the zero matrix.

linear

$$\begin{aligned}
 \phi(X+Y) &= \phi\left(\sum_{j=1}^m L(x_j, y_j) + \sum_{n=1}^p L(u_n, v_n)\right) \\
 &= \phi(L(x_1, y_1) + \dots + L(x_m, y_m) + L(u_1, v_1) + \dots + L(u_p, v_p)) \\
 &= \underbrace{(y_1^t x_1^t - x_1^t y_1^t) + \dots + (y_m^t x_m^t - x_m^t y_m^t)}_{\phi(X)} + \underbrace{(v_1^t u_1^t - u_1^t v_1^t) + \dots + (v_p^t u_p^t - u_p^t v_p^t)}_{\phi(Y)} \\
 &= \cancel{\sum_{i=1}^m}
 \end{aligned}$$

one-to-one

$$\phi(X) = \phi(Y)$$

recall $Xz = \bar{\Phi}(X)z$ $Yz = \bar{\Phi}(Y)z$

so $\bar{\Phi}(X) = \bar{\Phi}(Y) \Rightarrow X = Y$

onto If $A = \sum (y_i x_i^t - x_i^t y_i^t)$ then $\phi\left(\sum L(x_i, y_i)\right) = A$.

multiplicative

$$X = \sum L(x_j, y_j) \quad Y = \sum L(u_n, v_n)$$

$$\begin{aligned}
 \phi[X, Y] &= \sum \sum_k \phi \left[\underbrace{L(x_j, y_j), L(u_k, v_k)}_{L([xy]_k, [uv]_k)} \right] \\
 &\quad L([xy]_k, v) + L(u_k, [xy]_k)
 \end{aligned}$$

suffices to prove $\phi([L(x, y), L(u, v)]) = [yxt - xy^t, vut - uv^t]$

i.e. $\phi(L([xy]_k, v) + L(u, [xy]_k)) = (yxt - xy^t)(vut - uv^t)$

$$[xy]_k = L(x, y)u = (yxt - xy^t)u$$

$$\begin{aligned}
 &-(vut - uv^t)(yxt - xy^t) \\
 &= (yxt - xy^t)(vut - uv^t) \\
 &- (vut - uv^t)(yxt - xy^t)
 \end{aligned}$$

(3)

(5)

$$\begin{aligned}\phi(L([xyu], v)) &= \phi(L((yx^t - xy^t)u, v)) \\ &= v((yx^t - xy^t)u)^t - ((yx^t - xy^t)u)v^t \\ &= \boxed{v u^t (x y t - y x^t) - (y x^t - x y^t) u v^t} \quad ①\end{aligned}$$

$$\begin{aligned}\phi(L(u, [xyv])) &= \phi(L(u, (yx^t - xy^t)v)) \\ &= \cancel{(yx^t - xy^t)v u^t} - u v^t (x y t - y x^t) \quad ②\end{aligned}$$

Need to prove $\textcircled{1} + \textcircled{2} = \textcircled{3}$

(6)

$\mathcal{F} = \mathbb{F}_n$ column vectors

let $\mathcal{L} = \mathcal{H} \oplus \mathcal{F}$ be the standard embedding of \mathcal{F}

$$\begin{aligned}\tilde{\phi} &: \mathcal{L} \rightarrow M_{n+1, \text{skew}}(\mathbb{F}) \\ \tilde{\phi} \left(\sum_{i=1}^m L(x_i, y_i), \vec{x} \right) &= \begin{bmatrix} \overset{n}{\underset{n}{\phi(x)}} & \vec{x}^{\overset{n \times 1}{\rightarrow}} \\ -x^t & \underset{1 \times 1}{O} \end{bmatrix} \\ \vec{x} &= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\end{aligned}$$

$\tilde{\phi}$ is a Lie algebra isomorphism of \mathcal{L} onto $M_{n+1, \text{skew}}(\mathbb{F})$