Assign # 3 p. 14 Discussion # 1–10 5/2/17 (1)

(1) If I is an ideal in Lie algebra L then

(i) each $L^{(i)}$ is an ideal

(ii) each L^{i} is an ideal

Hint: Jacobi identity and induction.

(2) L is solvable $\iff$ $\exists$ subalgebras $L = L_0 \supset L_1 \supset \cdots \supset L_k = 0$

such that L_i / L_{i+1} is abelian

$\boxed{\Rightarrow}$ if L is solvable, show $L^{(i)} / [L^{(i)}, L^{(i)}]$

is abelian. (This is a general fact: $L / [L, L]$ is abelian — show this)

$\boxed{\Leftarrow}$ $L_k = 0$ so L_k is solvable. L_{k-1}/L_k is

abelian, therefore solvable, so L_{k-1} is solvable

by Proposition 3.2. Continue to get L_0 solvable.

(3) If char $F = 2$, then $sl(2, F)$ is nilpotent

Hint: since $2 = 0$, show $L^1 = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} : \alpha \in F \right\}$

and $L^2 = [L, L^1] = 0$.

4 (i) L is solvable $\iff$ ad L is solvable

(ii) L is nilpotent $\iff$ ad L is nilpotent.

Since homomorphic images ~~are~~ ~~so~~ of solvable are solvable and homomorphic images of nilpotent are nilpotent the $\boxed{\Rightarrow}$ in (i) & (ii) follow.

To prove the converse direction ($\boxed{\Leftarrow}$) use Prop 3.1 for solvable and Proposition 3.2 for nilpotent. (Note that ker ad $= Z(L)$.)

5 (i) L is non-abelian Lie algebra with basis x, y with $[x,y] = x$. Then L is solvable but not nilpotent

[Show $L^{(1)} = sp\{x\}$ so $L^{(2)} = 0$

and $L^1 = sp\{x\}$ and $L^2 = [L^1, L] = sp\{x\} = L^1$]

(ii) Let L be a Lie alg with basis x, y, z

$[x,y] = z$ $[x,z] = y$ $[y,z] = 0$. Then L is solvable but not nilpotent.

[Show $L^{(1)} = sp\{z, y\}$ and $L^{(2)} = 0$

Show $L' = L^{(1)} = sp(z, y)$ and $L^2 = [L, L'] = L'$]

6. If L is a Lie algebra

(i) I, J nilpotent ideals $\Rightarrow$ $I + J$ is a nilpotent ideal

(ii) L possesses a unique maximal nilpotent ideal

(iii) If L is non-abel with basis x, y such that $[x, y] = x$ find the unique maximal nilpotent ideal

(iv) If L has basis x, y, z with $[x, y] = z$, $[x, z] = y$ and $[y, z] = 0$, find the unique maximal nilpotent ideal.

[For (i) let $K = I + J$ show that

$$K^n \subseteq I^n + J^n + I^{n-1} + J^{n-1}$$

so $K^m = 0$ for some m.

For (ii) use Zorn's lemma to show that L contains a maximal nilpotent ideal.

To show the maximal ideal is unique, (~~For Zorn's lemma~~ see pages 3 - 4 of Third meeting October 14, 2016 Nilradical (informal notes) Meyberg pages 6-7 www.math.uci.edu/~brusso/nilradical.pdf)

Here are some details in the use of Zorn's lemma.

Let $\mathcal{S}$ = the set of all nilpotent ideals of L

Let $\mathcal{C}$ be a "chain" in $\mathcal{S}$, i.e. $\mathcal{C} = \{B_\alpha\}_{\alpha \in \Delta} \subset \mathcal{S}$

and $\forall \alpha, \beta \in \Delta$ either $B_\alpha \subset B_\beta$ or $B_\beta \subset B_\alpha$

Show that $B := \bigcup_\alpha B_\alpha$ is a nilpotent ideal. *

B is obviously an upper bound for $\mathcal{C}$

so Zorn's lemma applies to get a maximal nilpotent ideal.

* To show that B is nilpotent, you use Engel's theorem, which states that B is nilpotent $\Leftrightarrow$ $\forall x \in B$ $\operatorname{ad} x : B \to B$ is nilpotent

(details of this and parts (iii) & (iv) are left to you.

7. If L is nilpotent, K a proper subalgebra of L then $K \subsetneq N_L(K)$ $(= \{x \in L : [x, K] \subset K\})$

(I have no hints for this problem!)

8. If L is nilpotent, then L has an ideal of codimension 1

(This looks like it uses Theorem 3.3)

9. Every nilpotent Lie algebra has an outer derivation.

(The proof is outlined in the text using the result of Exercise 8 and the space $C_L(K)$ where K is the ideal in L with codimension 1)

10. L is a Lie algebra, K an ideal, L/K is nilpotent and $\operatorname{ad} x|_K$ is nilpotent for all $x \in L$. Prove that L is nilpotent.

(This looks like it uses Engel's theorem)