Discussion of Assignment 2 part 1

Exercises 1–12 p. 9–10

① Let D be a derivation of L and let $x \in L$ & $E = [\operatorname{ad} x, D] = (\operatorname{ad} x)D - D(\operatorname{ad} x)$

Show that $E = \operatorname{ad}(-Dx)$; thus the set of inner derivations of L is an ideal in the Lie algebra of all derivations on L.

② Need to prove $sl(n,F) = [gl(n,F), gl(n,F)]$

This was proved in Problem 9 of Assignment 1 for characteristic 0 (actually for characteristic $\neq 2$)

To prove this for arbitrary characteristic you can read the paper

"On matrices of trace zero" by A. Albert & B. Muckenhoupt Mich. Math J vol $\underline{4}$ 1957 pp 1–3 (posted online)

This can be someone's project!

③ Prove (i) $Z(gl(n,F)) = s(n,F)$ (= scalar matrices)

(ii) $Z(sl(n,F)) = 0$ unless char F divides n

Proof of (ii) for $n=2$ is done by direct calculation.

Then the proof of (i) for arbitrary n is done by induction.

$\underline{n=2}$ if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} x & y \\ u & v \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \forall\ x, y, u, v \in F$

then $\begin{pmatrix} ax+bu & ay+bv \\ cx+du & cy+dv \end{pmatrix} = \begin{pmatrix} xa+yc & xb+yd \\ ua+vc & ub+vd \end{pmatrix}$

By judicious choices of x, y, u, v one shows

$b=0$, $c=0$, & $d=a$.

For the induction hypothesis let A be $n \times n$ matrix

$\vec{b}$ $n \times 1$ matrix $\vec{c}$ $1 \times n$ matrix d 1×1 matrix

& suppose $\begin{pmatrix} A & \vec{b} \\ \vec{c} & d \end{pmatrix}\begin{pmatrix} X & \vec{y} \\ \vec{u} & v \end{pmatrix} = \begin{pmatrix} X & \vec{y} \\ \vec{u} & v \end{pmatrix}\begin{pmatrix} A & \vec{b} \\ \vec{c} & d \end{pmatrix}$

Then $AX + b \cdot \vec{u} = XA + \vec{y} \cdot \vec{c}$

take $\vec{u} = \vec{y} = 0 \Rightarrow A =$ scalar $n \times n$ (by induction)

Then take $X=0$ $u=0$ to get $\vec{y} \cdot \vec{c} = 0 \quad \forall \vec{y}$

$\Rightarrow \vec{c} = 0$

Continue to get $\vec{b} = 0$ and A is diagonal

proving (i) by induction.

The proof of (ii) is left to you !!!

4 postponed to later !!

5 $\dim L = 3 \quad L = [L,L] \overset{?}{\Longrightarrow} L$ simple

Let I be an ideal in L Need to prove

$I = (0)$ or $I = L$, i.e. I has dimension 0 or 3

Suppose L has dimension 1 or 2. We shall derive a contradiction.

Let $\phi : L \to L/I$ be the canonical quotient homomorphism. It is easy to show that

$$\phi(L) = \phi([L,L]) = [\phi(L), \phi(L)]$$

Now $\tilde{L} := L/I$ has dimension 1 or 2

and $[\tilde{L}, \tilde{L}] = \tilde{L}$

Look at p. 5 of Humphreys book

If $\dim \tilde{L} = 1$, then $\tilde{L} = sp\{x\}$, $[\lambda x, \mu x] = \lambda\mu[x,x] = 0$

so $[\tilde{L}, \tilde{L}] = 0$ a contradiction

If $\dim \tilde{L} = 2$, $\tilde{L}$ is not abelian since $[\tilde{L}, \tilde{L}] = \tilde{L}$

so $\tilde{L}$ has a basis x, y with $[x,y] = x$

Then $[\alpha x+\beta y, \gamma x+\delta y] = (\alpha\delta-\beta\gamma)x$

So $[\tilde{L},\tilde{L}]$ is one dimensional. But

$[\tilde{L},\tilde{L}] = \tilde{L}$ is 2 dimensional, contradiction

This completes the proof that $I=(0)$ or $I=L$.

In particular, since $L=sl(2,F)$ has dimension 3

and $[L,L]=L$ then $sl(2,F)$ is simple. $\boxed{//}$

(6) postponed to later

(7) (i) $\vec{t}(n,F) \overset{?}{=} N_{gl(n,F)}(\vec{t}(n,F)) = \{x\in gl(n,F) : [x,\vec{t}]\subset\vec{t}\}$

Proof. Since $\vec{t}$ is a subalgebra $\vec{t}\subset N_{gl(n,F)}(\vec{t})$

Conversely let $x\in gl(n,F)$ be such that $[x,\vec{t}]\subset\vec{t}$

Need to prove $x\in\vec{t}$

Consider the case $n=2$. Then use induction

to go from n to $n+1$, namely

if $\begin{pmatrix} X & \xi \\ \eta & a\end{pmatrix} \in gl(n+1,F)$ where X is $n\times n$ ξ is $n\times 1$, η is $1\times n$ a is 1×1

and $\begin{pmatrix} X & \xi \\ \eta & a\end{pmatrix}\begin{pmatrix} A & \alpha \\ 0 & b\end{pmatrix} - \begin{pmatrix} A & \alpha \\ 0 & b\end{pmatrix}\begin{pmatrix} X & \xi \\ \eta & a\end{pmatrix} \in \vec{t}(n+1,F)$

$\forall \begin{pmatrix} A & \alpha \\ 0 & b \end{pmatrix} \in \vec{t}(n+1, F)$ A is $n \times n$ α is $n \times 1$
b is 1×1

then show that $XA = AX$ ~~so~~ $\in \vec{t}(n, F)$ (by induction)

and $\eta = 0$ so $\begin{pmatrix} X & \xi \\ \eta & a \end{pmatrix} \in \vec{t}(n+1, F)$, as required.

The proofs of (ii) & (iii) are similarly done

(ii) $\vec{d} = N_{gl(n,F)}(\vec{d})$

(iii) $\vec{t} = N_{gl(n,F)}(\vec{n})$

and are left to you to do.

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8 The technique is similar to that of Problem 7

9 Proposition 2.2 on p. 7 of Humphreys was proved in Math 199A Fall 2016 for all non-associative algebras. See

Meyberg Notes p. 3 Theorem 1

www.math.uci.edu/~brusso/ch1marked.pdf

and Second Meeting October 7, 2016 Meyberg chapter 1 Exercises 1-3 Solutions

www.math.uci.edu/~brusso/meybergch1ex1-3.pdf