(#7) <u>Assignment 3</u> Further Discussion # 7–10 exercises p14 _ 7–10.pdf ①

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L is nilpotent, K subalgebra, $K \neq L$

$$\Rightarrow \quad K \subsetneq \underset{\parallel}{N_L(K)}$$

$$\{y \in L : [y, K] \subset K\}$$

(Look at the proof of Theorem 3.3)

L nilpotent $\Rightarrow$ all $\operatorname{ad} x$ are nilpotent

just a subspace

Consider $T_x : L/K \to L/K$ for $x \in K$

$$y + K \to [xy] + K$$

T_x is a nilpotent endomorphism

& $\tilde{L} = \{T_x : x \in K\}$ is a Lie algebra of nilpotent endomorphisms on L/K

so $\vec{\exists}$ by Theorem 3.3 $\exists\ z + K \neq K \quad z \in L$

such that $T_x(z+K) = 0 \quad \forall x \in K$

i.e. $[x, z] + K = 0$ so $[x, z] \in K \quad \forall x \in K$

i.e. $z \in N_L(K)$ But $z \notin K$. $\boxtimes$

8 L nilpotent $\Rightarrow$ L has an ideal of codimension 1

Proof: Look at the proof of Theorem 3.3 again.

Take a maximal proper subalgebra $K \subset L$.

Since $K \subset N_L(K)$ and $N_L(K)$ is a subalgebra,

either $K = N_L(K)$ or $N_L(K) = L$
(K is "self-normalizing")

If $N_L(K) = L$, as in the proof of Theorem 3.3

K is an ideal of codimension one.

What about the case $K = N_L(K)$

This is ruled out by Exercise 7.

#9 Every NON ABELIAN nilpotent Lie algebra L has an outer derivation.

proof By Exercise 8 $L = K + Fx$ for some ideal K. Recall $C_L(K) = \{y \in L : [y, K] = 0\}$

claim $C_L(K) \neq 0$ if $C_L(K) = 0$, $\cancel{[[L,K]] \neq 0}$

$\cancel{[[K+Fx, K]]} = \forall y \in L, [y, K] = 0$

$y = \underset{\in K}{z} + \lambda x \quad 0 = \overset{w \in K}{[y, w]} = [z + \lambda x, w] = [z, w] + \lambda [x, w]$

$\forall z, w \in K, \lambda \in F \quad [z, w] + \lambda [x, w] = 0$

$z = 0$, $\lambda = 1$ $\quad [x, K] = 0$

$[K, K] = 0$
$[x, K] = 0$
$[L, K] = 0$

$L' = [L, L] = [K + Fx, K + Fx]$
$= 0$

So, if L is not abelian, then $C_L(K) \neq 0$

$$L = L^0 \supset L^1 \supset L^2 \supset \cdots \supset L^k \supset L^{k+1} \cdots = 0$$

$\exists\ n$ s.t. $C_L(K) \subset L^n$

$C_L(K) \not\subset L^{n+1}$

$$\boxed{\begin{array}{ll} C_L(K) \subset L = L^0 & \\ \text{if } C_L(K) \not\subset L^1 & \text{then } C_L(K) \subset L^0,\ C_L(K) \not\subset L^1 \\ \text{if } C_L(K) \subset L^1,\ C_L(K) \not\subset L^2 & \text{then } C_L(K) \subset L^1,\ C_L(K) \not\subset L^2 \\ & \text{etc.} \end{array}}$$

let $z \in C_L(K)$, $z \notin L^{n+1}$

define $\delta : L \to L$ by $\delta(K) = 0$ $\delta(x) = z$

Show (i) δ is a derivation

(ii) δ is not an inner derivation

(c)

$$\delta[y_1+\lambda_1 x,\ y_2+\lambda_2 x] \overset{?}{=} [y_1+\lambda_1 x,\ \lambda_2 z] + [\lambda_1 z,\ y_2+\lambda_2 x]$$

$$\delta[y_1+\lambda_1 x,\ y_2+\lambda_2 x] = \delta\left([y_1,y_2] + \lambda_1[x,y_2] + \lambda_2[y_1,x]\right) = \lambda_1\,\delta[x,y_2] + \lambda_2\,\delta[y_1,x] = 0$$

$$[y_1+\lambda_1 x,\ \lambda_2 z] + [\lambda_1 z,\ y_2+\lambda_2 x] = \lambda_2[y_1,z] + \lambda_1\lambda_2[x,z] + \lambda_1[z,y_2] + \lambda_1\lambda_2[z,x]$$

$$= [z,\ -\lambda_2 y_1 - \cancel{\lambda_1\lambda_2 x} + \lambda_1 y_2 + \cancel{\lambda_1\lambda_2 x}]$$

$$= [z,\ \lambda_1 y_2 - \lambda_2 y_1] = 0$$

since $z \in C_L(K)$

This proves (i)

(ii) was not finished — see the next page now suppose δ is an inner derivation $\delta = ad\,(w + \mu x)$

$$\delta(y+\lambda x) = [\cancel{\ }\overset{w}{} + \mu x, y + \lambda x] \quad \forall\ \overset{\in K}{y}, \lambda \in F$$

$$\overset{\parallel}{\lambda z} = [w,y] + \mu[x,y] + \lambda[w,x]$$

$\lambda = 0$ $\quad 0 = \cancel{[w,y]}\ [w + \mu x, y]$

$\left.\begin{array}{l} y = 0 \\ \lambda = 1 \end{array}\right\}$ $\quad \lambda z = \lambda[w,x]$

$$z = [w,x] \in [K,x] \in [L,L] = L' \supset L^2$$

$z \notin L^{n+1}$

Take $y + \lambda x \in L^n$ $\quad$ (since $L^n \neq 0$)

$$\delta(y+\lambda x) = [w + \mu x, y + \lambda x] \in L^{n+1}$$

$\overset{\parallel}{\lambda z}$ $\quad$ Need to know λ can be $\neq 0$

i.e. $\cancel{K \neq L^n}$ $K \neq L^n$

$L = K + Fx$

$$L' = [K + Fx, K + Fx] = [K,K] + [x,K] = K' + [x,K]$$

$$\cancel{[x, u + \sigma x] = [x,u]}$$ NO GO

To be continued

10 L Lie alg, K ideal, L/K nilpotent

$\forall x \in L$ $\operatorname{ad} x|_K$ is nilpotent $\overset{?}{\Longrightarrow}$ L nilpotent.

To apply Engel, need $\operatorname{ad} x : L \to L$ nilpotent

$$L = \underset{\substack{\| \\ \ker \pi}}{K} \oplus \underset{S}{\cancel{L/K}} \qquad L \xrightarrow{\pi} L/K$$

$S \simeq L/K$ as vector spaces

$\operatorname{ad}(x+K) : L/K \to L/K$ is nilpotent

NO GO

$z = k + u$

$[z_1, z_2] = [k_1 + u_1, k_2 + u_2]$

$= \underbrace{[k_1, k_2] + [u_1, k_2] + [k_1, u_2]}_{\in K} + \underbrace{[u_1, u_2]}_{?}$

NOGO

$\operatorname{ad}(x+K)\,(y+K) = \operatorname{ad} x\,(y) + K$

$\operatorname{ad}(x+K)^n\,(y+K) = (\operatorname{ad} x)^n\,(y) + K$

Since L/K is nilpotent

For each $x \in L$,

$$(\operatorname{ad} x)^n (y) \in K \quad \text{for some } n \quad \left(\begin{array}{c}\text{the same}\\ \text{for all } x\end{array}\right)$$

$\forall x \in L$

$$(\operatorname{ad} x)^n (L) \subset K$$

so $$(\operatorname{ad} x)^k \underbrace{(\operatorname{ad} x)^n (L)}_{\subset K} = 0 \quad \text{for some } k$$

So $(\operatorname{ad} x)^{k+n} = 0$ and each element of L is ad-nilpotent. $\blacksquare$