4/16/17 ①

Discussion of Assignment 1
Exercises 1–12 p. 5–6

$\boxed{\text{part 1 \# 1–5}}$

(1) $a \times b = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$

$a = (a_1, a_2, a_3) \qquad b = (b_1, b_2, b_3)$

$$\left.\begin{array}{l} a \times a = 0 \qquad a \times (b + c) = a \times b + a \times c \\ [a, [bc]] + [b, [c,a]] + [c, [a,b]] = 0 \end{array}\right] \text{calculations}$$

let $x_1 = (1,0,0),\ x_2 = (0,1,0),\ x_3 = (0,0,1)$

$$x_i \times x_j = [x_i x_j] = \sum_{k=1}^{3} a_{ij}^k x_k$$

Of course $a_{ii}^k = 0 \qquad i, k \in \{1,2,3\}$

	a_{12}^k	a_{21}^k	a_{13}^k	a_{31}^k	a_{23}^k	a_{32}^k
$k=1$	0	0	0	0	1	−1
2	0	0	−1	1	0	0
3	1	−1	0	0	0	0

▨

(2) Write x_1, x_2, x_3 for x, y, z

so $[x_1 x_2] = x_3$, $[x_1 x_3] = x_2$, $[x_2 x_3] = 0$

$$x = \sum_{i=1}^{3} \alpha_i x_i \Rightarrow [x, x] = \sum_{i,j=1}^{3} \alpha_i \alpha_j [x_i x_j] = \left\{\begin{array}{l} \alpha_1 \alpha_2 x_3 + \\ \alpha_1 \alpha_3 x_2 + \alpha_2 \alpha_1 (-x_3) \\ \quad + \alpha_3 \alpha_1 (-x_2) = 0 \end{array}\right.$$

For Jacobi it suffices to prove $[x_i, [x_j x_k]] + [x_j, [x_k x_i]] + [x_k, [x_i, x_j]] = 0$

$i, j, k \in \{1, 2, 3\}$

check the 27 cases!

▨

3

$$\mathrm{ad}\,x \sim \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{ad}\,h \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\mathrm{ad}\,y \sim \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

IIII

4 L is the Lie algebra with basis x, y

such that $[x,y] = x$

x, y span L so $\mathrm{ad}x, \mathrm{ad}y$ span $\mathrm{ad}(L) \subset \mathrm{Der}(L)$

and $x \longrightarrow \mathrm{ad}\,x$ is a homomorphism.

$$\mathrm{ad}\,x \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathrm{ad}\,y \sim \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$ w/r ordered basis x, y

$$[\mathrm{ad}x, \mathrm{ad}y] = \mathrm{ad}[x,y] = \mathrm{ad}\,x$$

so $\mathrm{span}\{\mathrm{ad}x, \mathrm{ad}y\} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in F \right\}$

This is a Lie algebra isomorphic to L

since L' has dimension 2.

IIII

5 $$\underbrace{(a_{ij})(b_{ij})}_{(c_{ij})} - \underbrace{(b_{ij})(a_{ij})}_{(d_{ij})} = (\underbrace{c_{ij} - d_{ij}}_{e_{ij}})$$

$$e_{ij} = \sum_k (a_{ik} b_{kj} - b_{ik} a_{kj})$$

suppose $a_{ij} = 0$ if $i > j$ and $b_{ij} = 0$ if $i > j$

Then $e_{ij} = \sum_{i \le k \le j} a_{ik} b_{kj} - \sum_{i \le k \le j} b_{ik} a_{kj}$

So if $i \geq j$, then e_{ij} is the empty sum, so 0

This proves $\boxed{\vec{t} \text{ is a Lie algebra}}$ (subalgebra of $gl(n,F)$) $\boxed{\text{and } [\vec{t}, \vec{t}\,] \subset \vec{n}}$

and also if $i = j$, $e_{ii} = 0$ also.

Similarly suppose $a_{ij} = 0$ if $i \geq j$ and $b_{ij} = 0$ if $i \geq j$

then $e_{ij} = \sum_{i < k < j} (a_{ik} b_{kj} - b_{ik} a_{kj}) = 0$ if $i \geq j$

$\boxed{\text{This proves } \vec{n} \text{ is a Lie algebra}}$ (subalgebra of $gl(n,F)$)

It is obvious that $\vec{\delta}(n,F)$ is a Lie algebra

and $\vec{t} = \vec{n} + \vec{\delta}$

we still need to prove $[\vec{\delta}, \vec{n}] = \vec{n}$

and $[\vec{t}, \vec{t}] = \vec{n}$

If $(a_{ij}) \in \vec{\delta}$ and $(b_{ij}) \in \vec{n}$

$$(a_{ij})(b_{ij}) - (b_{ij})(a_{ij}) = (e_{ij})$$

$$e_{ij} = \sum_k (a_{ik} b_{kj} - b_{ik} a_{kj})$$

we have $b_{ij} = 0$ if $i \geq j$ and $a_{ij} = 0$ if $i \neq j$

so $e_{ij} = a_{ii} b_{ij} - b_{ij} a_{jj} = b_{ij}(a_{ii} - a_{jj}) = 0$ if $i \geq j$

so $[\vec{\delta}, \vec{n}] \subset \vec{n}$

To prove $[\vec{\delta}, \vec{n}] \supset \vec{n}$ note that

$$\left[(a_{ij}), \begin{bmatrix} c_1 & & 0 \\ & c_2 & \\ 0 & & \ddots \; c_n \end{bmatrix} \right] = \begin{bmatrix} 0 & a_{12}(c_2-c_1) & a_{13}(c_3-c_1) & \cdots & a_{1n}(c_n-c_1) \\ 0 & 0 & a_{23}(c_3-c_2) & \cdots & a_{2n}(c_n-c_2) \\ 0 & 0 & 0 & \cdots & a_{3n}(c_n-c_3) \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-1,n}(c_n-c_{n-1}) \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

so given $(a_{ij}) \in \vec{n}$,

$$\left(a_{ij}(c_j - c_i)\right) \in \vec{n} \text{ and } \left[(a_{ij}(c_j - c_i)), \begin{bmatrix} c_1 & & 0 \\ 0 & \ddots & c_n \end{bmatrix} \right] = (a_{ij})$$

proving $[\vec{\delta}, \vec{n}] \supset \vec{n}$ and $[\vec{\delta}, \vec{n}] = \vec{n}$

Finally $\vec{n} = [\vec{\delta}, \vec{n}] \subseteq [\vec{t}, \vec{t}] \subseteq \vec{n} \Rightarrow \vec{n} = [\vec{t}, \vec{t}]$