Discussion of Assignment 1 4/17/17

Exercises 1–12 p. 5 part 2 # 6–9

(6) Let $x \in gl(n, F)$. Let L_x = the operator : $F^n \to F^n$
(have distinct eigenvalues $a_1, \ldots, a_n$) $\underset{n\times 1}{\xi} \to \underset{n\times n}{x}\underset{n\times 1}{\xi}$

By the Corollary on p. 261 of the text for Math 121AB ("Linear Algebra" by Friedberg, Insel, Spence) L_x is "diagonalizable". Then by the Definition on page 245, there is an ordered basis such that the matrix of L_x with respect to that basis is diagonal. Then by Theorem 2.23 on page 112 there is an invertible matrix Q such that $x = Q^{-1} d Q$ where $d = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$ and $a_1, \ldots, a_n$ are the distinct eigenvalues of x.

Now consider $\mathrm{ad}\,x : gl(n, F) \to gl(n, F)$. Prove that

(*) $\mathrm{ad}\,x(Q^{-1} y Q) = Q^{-1}(\mathrm{ad}(d)(y))Q$

Let $Sy = Q^{-1} y Q$. Thus $\mathrm{ad}\,x = S^{-1}\,\mathrm{ad}(d)\,S$

i.e. $\mathrm{ad}\,x$ is similar to $\mathrm{ad}(d)$

so the eigenvalues of $\mathrm{ad}\,x$ are the same as the eigenvalues of $\mathrm{ad}(d)$

To find the eigenvalues of ad(d) consider first the case $n=2$. Then

$$\operatorname{ad}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 0 & (a_1-a_2)x_{12} \\ (a_2-a_1)x_{21} & 0 \end{pmatrix}$$

and the matrix of $\operatorname{ad}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ w/r the ordered basis $\begin{smallmatrix}1 & 0 \\ 0 & 0\end{smallmatrix}$ $\begin{smallmatrix}0 & 1 \\ 0 & 0\end{smallmatrix}$ $\begin{smallmatrix}0 & 0 \\ 1 & 0\end{smallmatrix}$ $\begin{smallmatrix}0 & 0 \\ 0 & 1\end{smallmatrix}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_1-a_2 & 0 & 0 \\ 0 & 0 & a_2-a_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This proves the exercise for $n=2$ and indicates how to proceed for $n>2$.

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(7) $$x = \left(x - \frac{\operatorname{tr}(x)}{n} I\right) + \frac{\operatorname{tr}(x)}{n} I$$

$$gl(n,F) = sl(n,F) \oplus \vec{s}(n,F)$$

8. For D_ℓ, $\dim V = 2\ell$ and $x \in \mathfrak{o}(2\ell, F)$

$$\iff \quad x = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \quad \text{with} \quad sx = -x^t s \qquad s = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}$$

$$\iff \quad \begin{aligned} p &= -p^t \\ m &= -q^t \\ n &= -n^t \end{aligned}$$

The contribution to the dimension of $\mathfrak{o}(2\ell, F)$ from

p is $\dfrac{\ell^2 - \ell}{2}$ (p consists of all $\ell \times \ell$ anti-symmetric matrices)

n is $\dfrac{\ell^2 - \ell}{2}$ (n . . . .)

$m - m^t$ is ℓ^2 (m consists of all $\ell \times \ell$ matrices)

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add to get $2\ell^2 - \ell$

9. If L is a Lie algebra of class A_ℓ

see the file "Lie Algebra and Representation Theory"

Example 2.3 page 20 (filename AndreasCap.pdf)

which is ~~posted~~ attached and will be posted on the webpage. This also indicates how to show this for classes B_ℓ, C_ℓ, D_ℓ.