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The Bernstein Problem in Dimension 6

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The solution of the Bernstein problem in the regular and exceptional cases, in all dimensions n, was made by Yu. Lyubich. A. Grishkov proved that there are no nonregular nonexceptional nuclear Bernstein algebras of type $(4, 2)$ with stochastic realization and therefore the Bernstein problem of type $(4, 2)$ was completely solved by the present author (*J. Algebra*, to appear). The aim of this paper is to describe explicitly all simplicial stochastic nonexceptional nonregular Bernstein algebras of type $(3, 3)$. Since every nonregular nonexceptional Bernstein algebra of dimension 6 is either of type $(4, 2)$ or of type $(3, 3)$, the Bernstein problem in dimension 6 is completely solved in this paper.

1. INTRODUCTION

In works [1–4], S. N. Bernstein raised and partially solved an important problem concerning mathematical expression of fundamental laws of biological heredity. Let us, following [21], describe the statement of the Bernstein problem.

The state of a population in any generation can be described by a *stochastic* (or *probabilistic*) vector $x = (x_i)_{i=1}^n$, so all the $x_i \geq 0$ and $s(x) \equiv \sum_i x_i = 1$. The set of all states is the basic simplex $\Delta^{n-1} \subset \mathbb{R}^n$. The vertices $(e_i)_{i=1}^n$ of this simplex are the *types* of individuals in this population. Let us denote by $p_{ij,k}$ the probability that an individual of type e_k appears in the next generation from parents whose types are e_i and e_j, so

$$p_{ij,k} \geq 0 \qquad \text{and} \qquad \sum_k p_{ij,k} = 1. \tag{1}$$

Moreover, if the material or paternal origin does not play a role in the production of offsprings' types, then

$$p_{ij,k} = p_{ji,k}, \qquad i, j, k = 1, 2, \ldots, n. \tag{2}$$

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Let the population be *panmixic*, i.e., the mating in it is at random. Then in the absence of selection the state $x' = (x'_i)_{i=1}^n$ in the next generation will be

$$x'_k = \sum_{i,j} p_{ij,k} x_i x_j \qquad (1 \le k \le n). \tag{3}$$

These formulas define a mapping $V\colon \Delta^{n-1} \to \Delta^{n-1}$ called the *evolutionary operator* $(e.o.)$ of the given population. A state x is called an *equilibrium* if $Vx = x$. An evolutionary operator V is called *stationary* if for every $x \in \Delta^{n-1}$ the corresponding state in the next generation is an equilibrium, or equivalently $V^2 = V$.

The Bernstein problem is to describe explicitly all stationary evolutionary operators.

The following interpretation of evolutionary operator V, given by the formula (1), is very useful in the solution of this problem. One can define in the space $\mathbb{R}^n$ a commutative product

$$e_i e_j = \sum_k p_{ij,k} e_k \qquad (1 \le i, j \le n). \tag{4}$$

In such a way we obtain a commutative (nonassociative) baric algebra A over $\mathbb{R}$. The pair (A, s) is a *simplicial stochastic algebra* (see [21, p. 150] for a general definition). This means that the product of any elements $x, y \in \Delta^{n-1}$ belongs to Δ^{n-1} (we say "stochastic" for simplicity). This commutative algebra A has a nonzero homomorphism $s\colon A \to \mathbb{R}$. There is a remarkable connection between the stationary properties of an evolutionary operator and some property of the corresponding algebra A.

The following theorem was established by Lyubich [12].

THEOREM 1.1. *The operator V is stationary if and only if the identity $(x^2)^2 = s(x)^2 x^2$ holds.*

One of most important conclusions that can be drawn from the preceding theorem is that the Bernstein problem is equivalent to finding, for every Bernstein algebra, all stochastic bases with their corresponding multiplication constants.

Let us recall some definitions and facts concerning Bernstein algebras (see [21, 22] for more information). A *Bernstein algebra* (A, ω) over $\mathbb{R}$ is a commutative *baric algebra*, that is, $\omega\colon A \to \mathbb{R}$ is a nonzero algebra homomorphism) satisfying the identity

$$(x^2)^2 = \omega(x)^2 x^2 \tag{5}$$

for every $x \in A$. We remark that Bernstein algebras are not necessarily algebras with a *stochastic realization*; that is, there exist Bernstein algebras such that for every basis the simplex Δ spanned by this basis is not invariant with respect to the multiplication, i.e., $\Delta \cdot \Delta \not\subset \Delta$ (see [8]). A linearly independent set Φ in A is said to be *stochastic* if $\Delta(\Phi) \cdot \Delta(\Phi) \subset \Delta(\Phi)$. Therefore, a Bernstein algebra is an algebra with a stochastic realization if and only if it has a stochastic basis.

A Bernstein algebra has a nonzero idempotent element e. Every such elements yields the Peirce decomposition $A = \mathbb{R}e \oplus U_e \oplus V_e$, where

$$\ker(\omega) = U_e \oplus V_e, \qquad U_e = \left\{x \in A \,/\, ex = \tfrac{1}{2}x\right\},$$
$$V_e = \{x \in A \,/\, ex = 0\}.$$

We will let $N = \ker(\omega)$ and denote by $I(A)$ the set of idempotent elements.

It was proved in [12] that $\dim U_e$ (and then $\dim V_e$) does not depend on the choice of the idempotent element, so we can define type $A = (m, \delta)$, where $m - 1 = \dim U_e$ and $\delta = \dim V_e$ (so $n = \dim A = m + \delta$). Products between elements of U_e and V_e satisfy the relations

$$U_e^2 \subset V_e, \qquad U_e V_e \subset U_e, \qquad V_e^2 \subset U_e, \tag{6}$$

$$(u^2)^2 = u^2(uv) = u^3 = u(uv) = u(v^2) = (uv)^2 = u^2 v^2 = 0 \tag{7}$$

for all $u \in U_e$ and $v \in V_e$. On the other hand, $I(A) = \{e + u + u^2 \,/\, u \in U_e\}$ and if $\bar{e} = e + \bar{u} + \bar{u}^2$ is another idempotent, then

$$U_{\bar{e}} = \{u + 2u\bar{u} \,/\, u \in U_e\}, \qquad V_{\bar{e}} = \left\{v - 2(\bar{u} + \bar{u}^2)v \,/\, v \in V_e\right\}. \tag{8}$$

Moreover $\dim U_e^2$ and $\dim(U_e V_e + V_e^2)$ do not depend on the choice of the idempotent element. So we will use the following definitions introduced by Lyubich (see [12, 16, 21]): a Bernstein algebra in which $U_e V_e + V_e^2 = (0)$ is called *regular* and a Bernstein algebra in which $U_e^2 = (0)$ is said to be *exceptional*. Every Bernstein algebra of types (m, δ) with $m - 1$ or δ less than or equal to 1 is regular or exceptional. Further, an algebra A where $A^2 = A$ is called *nuclear*.

We now define the basic concept of *invariant linear form*. This is a linear form, f, on a Bernstein algebra which satisfies the identity

$$f(x^2) = \omega(x) f(x) \tag{9}$$

for all $x \in A$ (see [21] for more information). If we denote by J_A the set of invariant linear forms of A, then its orthogonal is

$$J_A^{\perp} = \left(U_e V_e + V_e^2\right) \oplus V_e. \tag{10}$$

Finally, we will denote by U_0 the subset $U_e \cap \operatorname{ann} U_e$. This subset is independent of the idempotent considered and is an ideal of A. On the other hand, $\bar{A} = A/U_0$ is a Jordan–Bernstein algebra (see [5] for general information about U_0 and Jordan–Bernstein algebras).

We call a *s.e.o. regular*, *exceptional*, or of *type* (m, δ) if the corresponding stochastic Bernstein algebra is regular, exceptional, or of type (m, δ), respectively. So a s.e.o. is called *degenerate* if there exists i such that $x_i' = 0$.

Bernstein solved the Bernstein problem for $n = 3$. (The case $n = 2$ is trivial.) For all $n > 3$ the Bernstein problem in the so-called regular case and the exceptional case was completely solved by Yu. Lyubich in a series of papers (see also his book [21]). In [7] we solved the problem for type $(3, 2)$ in the nonregular and nonexceptional case and therefore we solved completely the problem for $n = 5$. Finally, in [10] we solved the Bernstein problem for type $(n - 2, 2)$, for all n, in the nonregular nonexceptional nonnuclear case.

Now, we will recall some definitions of Lyubich that are very useful in the solution of this problem (see [21, pp. 228, 233] for more information). Let V be a Bernstein evolutionary operator. Consider an arbitrary face Γ of the basis simplex Δ and let $C_\Gamma = \operatorname{Int} \Gamma \in \operatorname{Im} V$ (as usual, $\operatorname{Int} \Gamma$ is the interior of the face Γ with respect to its affine hull). We will call the face Γ *essential* if $C_\Gamma \neq \emptyset$. An essential face Γ is called a *k-dimensional* essential face if C_Γ has dimension k as topological space (we say "k-essential" for simplicity). On the other hand, if $x = \sum_{i=1}^{n} \alpha_i e_i$, $\alpha_i \geq 0$ the *support* of x, denoted $\operatorname{supp}(x)$, is the set of basis vector (or their corresponding indices) for which the coefficient is positive, i.e., $i \in \operatorname{supp}(x)$ if and only if $\alpha_i > 0$.

We can prove the following generalization of Lemma 5.7.2 of [21]:

LEMMA 1.1. *Let V be a Bernstein evolutionary operator of type (m, δ). Then it contains at least $m - k$ k-dimensional essential faces, for $k = 0, 1, \ldots, m - 1$.*

Now, we obtain the following

LEMMA 1.2. *Let V be a Bernstein evolutionary operator, Γ and Γ' two essential faces of V, and Γ 0-dimensional. Then $\Gamma \cap \Gamma' \neq \emptyset$, implies that $\Gamma \subset \Gamma'$.*

Besides, if Γ' is also 0-essential, then either $\Gamma = \Gamma'$ or $\Gamma \cap \Gamma' = \emptyset$.

Proof. Let x be an element of $\Gamma \cap \Gamma'$. Since every essential face is invariant with respect to the evolutionary operator, we have that $x^2 \in \Gamma \cap \Gamma'$. Consequently, $\operatorname{supp}(x^2) \subset \Gamma \cap \Gamma'$. On the other hand, $x^2 \in C_\Gamma$ and therefore $\operatorname{supp}(x^2)$ is equal to $\{e_i / e_i \in \Gamma\}$. ∎

We remark that if $\Phi = \{e_1, \ldots, e_n\}$ is a basis of a Bernstein algebra with $\lambda_{ij,k}$ the corresponding multiplication constants, i.e., $e_i e_j = \sum_l \lambda_{ij,l} e_l$, then the basis is stochastic if and only if $\lambda_{ij,k} \geq 0$ and $\sum_l \lambda_{ij,l} = 1$ for $i, j, k = 1, \ldots, n$. Thus, let Φ be stochastic; we can consider $\lambda_{ij,k}$ as the hereditary coefficients of a stationary evolutionary operator. In the following we will take the elements of a stochastic basis as vectors of the algebra or as types of the s.e.o. indiscriminately. Also, we will call a stochastic basis *degenerate* if its corresponding s.e.o. is degenerate. A vector e_k of the stochastic basis is a *disappearing type* if $x'_k = 0$ in the corresponding s.e.o., so e_k is disappearing if and only if $\lambda_{ij,k} = 0$ for $i, j = 1, \ldots, n$.

LEMMA 1.3 [8]. *If $\Phi = \{e_1, \ldots, e_n\}$ is a stochastic basis of a Bernstein algebra, then $\omega(e_i) = 1$ for $i = 1, \ldots, n$.*

Therefore, we obtain the following

LEMMA 1.4. *If $\Phi = \{e_i\}$ is a basis of a Bernstein algebra and $\omega(e_i) = 1$ for all i, then the basis Φ is stochastic if and only if the coordinates of $e_{j_1} e_{j_2}$ with respect to Φ are either greater than or equal to 0 for all j_1 and j_2.*

2. STOCHASTIC BASIS FOR TYPE $(3, 3)$

It can be proved that if A is a nonregular nonexceptional Bernstein algebra of dimension 6, then the type of A is either $(4, 2)$ or $(3, 3)$. (Remember that the Bernstein problem in the regular and exceptional case was solved by Lyubich.) The type $(4, 2)$ was solved, in [10], in the nonregular nonexceptional nonnuclear case. Also, A. Grishkov proved that if a Bernstein algebra is of type $(4, 2)$ nuclear and stochastic, then it is regular. Thus, the Bernstein problem for type $(4, 2)$ has been completely solved. In this section we will describe all stochastic bases of a nonregular nonexceptional Bernstein algebra of type $(3, 3)$ and therefore, the Bernstein problem will be completely solved in this paper for $n = 6$.

For a finite sequence $\Phi = \{x_1, x_2, \ldots, x_r\}$ of elements of a vector space we write $\langle x_1, x_2, \ldots, x_r \rangle$ as the vector subspace spanned by Φ and denote by $[x_1, x_2, \ldots, x_r]$ the convex hull of Φ.

LEMMA 2.1 [20]. *Let A be a nonregular nonexceptional Bernstein algebra of type $(3, n-3)$. If e is an idempotent element of A, then $U_e^3 = (0)$ and $U_e V_e \subset U_0$.*

Proof. It is enough to prove that $U_0 \neq (0)$. If $V_e^2 \neq (0)$, then $(0) \neq V_e^2 \subset U_0$. In the other case, $V_e^2 = (0)$ and as the algebra is nonexceptional $U_e V_e \neq (0)$. Therefore, there exist u and v such that $uv \neq 0$. If u and uv are linearly independent, they are a basis of U_e and by (7) it follows that

$uv \in U_0$. If u and uv are linearly dependent, we can assume $u = uv$, then $u = (uv)v \in U_0$. ∎

An immediate consequence of the previous lemma is that if A is a nonregular nonexceptional Bernstein $\mathbb{R}$-algebra of type $(3, n-3)$, then $\dim U_0 = 1$ and type $A^2 = (3, 1)$.

In the following A will be a Bernstein $\mathbb{R}$-algebra of type $(3, n-3)$ and with $\dim U_0 = 1$. A basis $\Psi = \{e, u_1, u_2, v_1, \ldots, v_{n-3}\}$ of A is called *standard* if e is an idempotent element, u_1, u_2 a basis of U_e, $v_1, \ldots, v_{n-3}$ a basis of V_e, $u_2 \in U_0$, and $u_1^2 = v_1$. (See Proposition 4 of [7] for more information about standard bases of A.) We will denote by $\vartheta_{j,k}$ and $\theta_{i,k}$ the scalars where $v_j v_k = \vartheta_{j,k} u_2$ and $u_i v_k = \theta_{i,k} u_2$ for $j, k = 1, \ldots, n-3$ and $i = 1, 2$. For every standard basis Ψ, we denote by Θ the scalar

$$\sum_{2 \le k \le n-3} \left(\sum_{i=1,2} |\theta_{i,k}| + \sum_{1 \le j \le k} |\vartheta_{j,k}| \right).$$

PROPOSITION 2.1 [6]. *Let* $\{e_1, u_1, u_2, v_1, \ldots, v_{n-3}\}$ *be a standard basis of* A. *Then*

(i) *a basis of* A *of the form* $\{e_1, u_1', u_2', v_1', \ldots, v_{n-3}'\}$ *is standard if and only if there exist* $\alpha, \beta \in \mathbb{R}^*$ *and* $\xi \in \mathbb{R}$ *such that* $u_1' = \alpha u_1 + \xi u_2$, $u_2' = \beta u_2$ *and* $v_1' = \alpha^2 v_1$;

(ii) *if* $\bar{e} = e + \bar{u} + \bar{u}^2$ *is another idempotent of* A, $\bar{u} = \gamma u_1 + \mu u_2$, *then the vectors*

$$\bar{u}_1 = u_1 + 2\gamma v_1, \qquad \bar{u}_2 = u_2, \qquad \bar{v}_1 = v_1$$

$$\bar{v}_i = v_i - 2\left(\theta_{1,i}\gamma + \theta_{2,i}\mu + \vartheta_{1,i}\gamma^2\right)u_2, \qquad i = 2, \ldots, n-3$$

form a standard basis of A.

LEMMA 2.2. *Let* e_1, e_2 *be two linearly independent elements of* A *such that with respect to a standard basis* $e_1 = e + \mu_1 u_2 + \beta v_2$, $e_2 = e + \mu_2 u_2 + v_2$, *and* $e + u_2 \in [e_1, e_2]$. *Then the following conditions are equivalent*:

(i) *The set* $[e_1, e_2]$ *is* 0-*essential.*

(ii) $\beta < 0$ *and* $\mu_1 = 1 - 2\beta\theta_{2,2}$, $\mu_2 = 1 - 2\theta_{2,2}$ $\vartheta_{2,2} = 4(\theta_{2,2})^2$.

LEMMA 2.3. *Let* e_1, e_2 *be two linearly independent elements of* A *such that with respect to a standard basis* $e_1 = e + \lambda u_1 + \mu_1 u_2 + \lambda^2 v_1 + \beta v_2$, $e_2 = e + \lambda u_1 + \mu_2 u_2 + \lambda^2 v_1 + v_2$, *and* $e + \lambda u_1 + \lambda^2 v_1 \in [e_1, e_2]$. *Then the following conditions are equivalent*:

(i) *The set* $[e_1, e_2]$ *is* 0-*essential.*

(ii) $\beta < 0$ *and* $\mu_1 = \beta\mu_2$, $\mu_2 = -2\lambda(\theta_{1,2} + \lambda\vartheta_{1,2})$, $\vartheta_{2,2} = -2\mu_2\theta_{2,2}$.

LEMMA 2.4. *Let* e_1, e_2, e_3 *be three linearly independent elements of* A *of weight* 1 *such that* $[e_1]$ *and* $[e_2, e_3]$ *are* 0*-essential*, $[e_1, e_2, e_3]$ *is* 1*-essential, and* $e_2, e_3 \notin A^2$. *Then* $f(e_1) = f(e_2) = f(e_3)$ *for all* $f \in J_A \cap U_0^{\perp}$.

We now suppose that dim $A = 6$ and $\Phi = \{e_i\}$ is a stochastic basis of A. Since dim $A^2 = 4$ the stochastic basis has at most two disappearing types. If Φ has two disappearing types, we can assume that they are e_5 and e_6; then $\{e_1, e_2, e_3, e_4\}$ is a stochastic basis of A^2. If Φ has exactly one disappearing type, we can assume that it is e_6; then $\{e_1, \dots, e_5\}$ is a stochastic basis of a subalgebra of A that contains A^2, and therefore is a stochastic basis of a Bernstein subalgebra of type $(3, 2)$ with dim $U_0 = 1$.

By Lemma 2.1, A^2 is a regular Bernstein algebra of type $(3, 1)$. As a reformulation of a result obtained by Lyubich in [13] (see also [8]), we have the following

THEOREM 2.1. *The stochastic bases of* A^2 *have the forms*

$$\begin{aligned} e_1 &= e \\ e_2 &= e + u_2 \\ e_3 &= e + u_1 + \alpha_1 v_1 \\ e_4 &= e + u_1 + \alpha_2 v_1, \end{aligned}$$

where $\alpha_1 \le 0$ *and* $1 \le \alpha_2$.

By Lemma 1.1 we have the following.

LEMMA 2.5. *Every stochastic basis in* A, dim $A = 5$, *contains an idempotent element*.

LEMMA 2.6. *Every stochastic basis in* A, dim $A = 5$ *contains two elements in* A^2.

Proof. Suppose $\Phi = \{e_i\}$ is a stochastic basis in A, with a unique element in A^2. By Lemma 1.1 for an arrangement of the stochastic basis the faces $[e_1]$, $[e_2, e_3]$, and $[e_4, e_5]$ are 0-essential, and $[e_1, e_2, e_3]$ is 1-essential. Suppose now that e is the idempotent in $[e_2, e_3]$. Since the vectors of Φ have weight 1, they are of the form $e_i = e + z_i$ with $z_i \in \ker(\omega)$. By Proposition 2.1 and Lemma 2.3 there exists $v_2 \notin A^2$ such that $e_2 = e + v_2$ and $e_3 = e + \beta v_2$ and by Lemma 2.4 we obtain that $e_1 = e + u_2$ with $u_2 \in U_0$. Also by Proposition 2.1 the idempotent of $[e_4, e_5]$ is of the form $e + u_1 + v_1$ and therefore there exist scalars such that

$$\begin{aligned} e_4 &= e + u_1 + \mu_1 u_2 + \alpha_1 v_1 + \beta_1 v_2, \\ e_5 &= e + u_1 + \mu_2 u_2 + \alpha_2 v_1 + \beta_2 v_2. \end{aligned}$$

Since the coordinates of the vectors ee_4 and ee_5 with respect to the stochastic basis are greater than or equal to 0, we obtain that $\mu_1 = \mu_2 = 0$. Finally, considering the coordinates of $e_1 e_4^2$, we obtain a contradiction. ▮

THEOREM 2.2. *Let A be of dimension* 5. *If $\{e_1, \dots, e_5\}$ is a stochastic basis of A, with at most three elements in A^2, then there exist a standard basis and an arrangement of the stochastic basis, such that the basis has one of the two following forms*:

1. *The form is*

$$\begin{aligned} e_1 &= e + u_2 \\ e_2 &= e + \beta v_2 \\ e_3 &= e + v_2 \\ e_4 &= e + u_1 + \alpha_1 v_1 \\ e_5 &= e + u_1 + \alpha_2 v_1, \end{aligned}$$

where $-1 \le \beta < 0$, $\alpha_1 \le 0$, $1 \le \alpha_2$, $\theta_{1,2} = \vartheta_{2,2} = \vartheta_{1,2} = 0$, *and* $-1 \le 2\theta_{2,2} \le 1$.

2. *The form is*

$$\begin{aligned} e_1 &= e \\ e_2 &= e + u_2 \\ e_3 &= e + u_1 + \alpha_1 v_1 + \beta_1 v_2 \\ e_4 &= e + u_1 + \alpha_2 v_1 + \beta_2 v_2 \\ e_5 &= e + u_1 + v_2, \end{aligned}$$

where $\beta_2((\beta_1 - 1)\alpha_2 - (\beta_2 - 1)\alpha_1) \ne 0$, *the vectors of* $\mathbb{R}^2$, $(0,0)$ *and* $(1,0)$, *belong to the set* $[(\alpha_1, \beta_1), (\alpha_2, \beta_2), (0,1)]$, *and A is regular*.

Proof. If A is nonregular, then the theorem was proved in [7]. In the rest of the proof we will suppose that A is regular. Now we have two cases:

1. If $\{e_i\}$ is a stochastic basis with exactly three elements in A^2, then we can assume that $e_1, e_2, e_3 \in A^2$. There exists a unique element that belongs to A^2 and $[e_4, e_5]$, which we denote by e'. The set $\{e_1, e_2, e_3, e'\}$ is a stochastic basis of A^2. By Theorem 2.1 and Lemmas 1.3 and 1.4, we obtain the desired result.

2. If $\{e_i\}$ is a stochastic basis with exactly two elements in A^2, then by Lemma 1.1 after a suitable arrangement we have one of the two following possibilities:

   (a) Faces $[e_1], [e_2], [e_3, e_4]$ are 0-essential.

   (b) Faces $[e_1], [e_2], [e_3, e_4, e_5]$ are 0-essential.

(a) If e^* is the idempotent of $[e_3, e_4]$, then there exist $v_2^* \notin A^2$ and $\beta_1 < 0$ such that $e_3 = e^* + \beta_1 v_2^*$ and $e_4 = e^* + v_2^*$. Either e_1 or e_2 is not in $\mathbb{R}e^* + U_0 + V_{e^*}$, for otherwise there exists f, an invariant linear form such that $f(e_i) < f(e_5)$ for $i = 1, 2, 3, 4$, and therefore the vector e_5 would be an idempotent, but this is impossible since $e_5 \notin A^2$. Thus, we can assume that $e_1 = e^* + u_1^* + v_1^*$ and $e_2 = e^* + \lambda_1 u_1^* + u_2^* + \lambda_1^2 v_1^*$ with $u_1^* \notin U_0$ and $u_2^* \in U_0$. The vector u_2^* is different from 0, for otherwise e_5 would be an idempotent. Therefore the stochastic basis with respect to the standard basis $\{e^*, u_1^*, u_2^*, v_1^*, v_2^*\}$ is of the form

$$e_1 = e^* + u_1^* + v_1^*, \qquad e_2 = e^* + \lambda_1 u_1^* + u_2^* + \lambda_1^2 v_1^*, \qquad e_3 = e^* + \beta_1 v_2^*,$$
$$e_4 = e^* + v_2^*, \qquad e_5 = e^* + \lambda_2 u_1^* + \mu_1 u_2^* + \alpha_1 v_1^* + \beta_2 v_2^*,$$

where $\beta_1 < 0$ and $\beta_2 \neq 0$. Now, as the multiplication constants of the stochastic basis are greater than or equal to 0, we obtain

$$\begin{aligned} e_1 e_3 &\to \mu_1, \alpha_1 \leq 0, \\ e_5^2 &\to \mu_1 = 0 \text{ and } 0 \leq \lambda_2 \leq 1, \\ e_1 e_2 &\to \lambda = 1, \\ e_2 e_3 &\to \lambda_2 = 0. \end{aligned}$$

Therefore, considering $e = e_1$, we obtain by Proposition 2.1 that the basis has the desired form.

(b) For a suitable standard basis $e_1 = e^* + u_1^* + v_1^*$, $e_2 = e^* + \lambda u_1^* + u_2^* + \lambda^2 v_1^*$, and $e_i = e^* + \alpha_{i-2} v_1^* + \beta_{i-2} v_2^*$ for $i = 3, 4, 5$. Multiplying e_1 by e_2 we obtain that $\lambda = 1$. Therefore, for $e = e_1$ the stochastic basis has the form of the theorem. ∎

2.1. *The Degenerate Case with Two Disappearing Types*

If the algebra A is nonregular and $\dim A = 6$, then every stochastic basis of A with two disappearing types has the form

$$\begin{aligned} e_1 &= e \\ e_2 &= e + u_2 \\ e_3 &= e + u_1 + \alpha_1 v_1 \\ e_4 &= e + u_1 + \alpha_2 v_1 \\ e_5 &= e + \lambda_1 u_1 + \mu_1 u_2 + v_2 \\ e_6 &= e + \lambda_2 u_1 + \mu_2 u_2 + v_3, \end{aligned}$$

where $\Theta \neq 0$ and the following inequalities hold,

$$\alpha_1 \leq 0, \qquad 1 \leq \alpha_2, \tag{11}$$

$$0 \leq \lambda_i; \qquad 0 \leq \delta_i \leq 2, \tag{12}$$

$$-1 \leq \delta_i + 2\theta_{2,i+1} \leq 1, \tag{13}$$

$$0 \leq \delta_i + 2(\theta_{1,i+1} + \alpha_j \vartheta_{1,i+1}) \leq 1, \tag{14}$$

$$0 \leq \frac{\delta_i + \delta_j}{2} + (\lambda_i \theta_{1,j+1} + \mu_i \theta_{2,j+1}) + (\lambda_j \theta_{1,i+1} + \mu_j \theta_{2,i+1}) + \vartheta_{i+1,j+1} \leq 1, \tag{15}$$

for $\delta_i \in \{\mu_i, \lambda_i + \mu_i\}$ and $i, j \in \{1, 2\}$.

If $\Phi = \{e_i\}$ is a stochastic basis of A with e_5 and e_6 the disappearing types, then e_1, e_2, e_3, e_4 is a stochastic basis of A^2.

By Lemma 1.4 the inequalities (11)–(15) hold.

2.2. *The Degenerate Case with Exactly One Disappearing Type*

If the algebra A is nonregular and dim $A = 6$, then every stochastic basis of A with exactly one disappearing type has one of the following forms:

1. If there are two integers r, s such that x'_r and x'_s are proportional and nondegenerate, then

$$\begin{aligned}
e_1 &= e + u_2\\
e_2 &= e + u_1 + \alpha_1 v_1\\
e_3 &= e + u_1 + \alpha_2 v_1\\
e_4 &= e + \beta v_2\\
e_5 &= e + v_2\\
e_6 &= e + \lambda u_1 + \mu u_2 + v_3,
\end{aligned}$$

where $\Theta \neq 0$ and the following inequalities hold,

$$\theta_{1,2} = \vartheta_{1,2} = \vartheta_{2,2} = 0, \tag{16}$$

$$-1 \leq \beta < 0; \qquad \alpha_1 \leq 0; \qquad 1 \leq \alpha_2, \tag{17}$$

$$0 \leq \lambda; \qquad 0 \leq \delta + 2\beta\vartheta_{2,3},\ \delta + 2\vartheta_{2,3} \leq 2, \tag{18}$$

$$-1 \leq 2\theta_{2,2},\ \delta + 2\theta_{2,3} \leq 1; \qquad 0 \leq \delta + 2(\theta_{1,3} + \alpha_i \vartheta_{1,3}) \leq 1, \tag{19}$$

$$0 \leq \delta + 2(\lambda\theta_{1,3} + \mu\theta_{2,3}) + \vartheta_{3,3} \leq 1, \tag{20}$$

for $\delta \in \{\mu, \lambda + \mu\}$ and $i \in \{1, 2\}$.

2. If there is no pair of integers r, s such that x'_r and x'_s are proportional and nondegenerate, then

$$\begin{aligned} e_1 &= e \\ e_2 &= e + u_2 \\ e_3 &= e + u_1 + \alpha_1 v_1 + \beta_1 v_2 \\ e_4 &= e + u_1 + \alpha_2 v_1 + \beta_2 v_2 \\ e_5 &= e + u_1 + v_2 \\ e_6 &= e + \lambda u_1 + \mu u_2 + v_3, \end{aligned}$$

where $\Theta \neq 0$ and the following inequalities hold,

$$\theta_{1,2} = \theta_{2,2} = \vartheta_{1,2} = \vartheta_{2,2} = 0, \tag{21}$$

$$\beta_2((\beta_1 - 1)\alpha_2 - (\beta_2 - 1)\alpha_1) \neq 0, \tag{22}$$

$$(0,0), (1,0) \in [(\alpha_1, \beta_1), (\alpha_2, \beta_2), (0,1)], \tag{23}$$

$$0 \leq \lambda; \qquad 0 \leq \delta \leq 2; \qquad -1 \leq \delta + 2\theta_{2,3} \leq 1, \tag{24}$$

$$0 \leq \delta + 2(\theta_{1,3} + \alpha_i \vartheta_{1,3} + \beta_i \vartheta_{2,3}) \leq 1, \tag{25}$$

$$0 \leq \delta + 2(\lambda\theta_{1,3} + \mu\theta_{2,3}) + \vartheta_{3,3} \leq 1, \tag{26}$$

for $\alpha_3 = 0$, $\beta_3 = 1$, $\delta \in \{\mu, \lambda + \mu\}$, and $i \in \{1, 2, 3\}$.

We remark that if $\Phi = \{e_1, \ldots, e_6\}$ is a stochastic basis of A with e_6 the unique disappearing type, then this basis has at most three elements in A^2 and $\Phi' = \{e_1, \ldots, e_5\}$ is a stochastic basis of a subalgebra of A that contains A^2. Therefore, Φ' has one of the two forms of Theorem 2.2

2.3. *The Nondegenerate Case* (*Stochastic Bases without Disappearing Types*)

If A is nonregular and $\{e_1, e_2, \ldots, e_6\}$ is a stochastic basis of A with the corresponding s.e.o. nondegenerate, then there exist a standard basis and an arrangement of the stochastic basis, such that the basis is of one of the following three forms:

1. The form is

$$\begin{aligned} e_1 &= e + u_1 + \alpha_1 v_1 \\ e_2 &= e + u_1 + \alpha_2 v_1 \\ e_3 &= e + \beta v_2 \\ e_4 &= e + v_2 \\ e_5 &= e + \mu_1 u_2 + \gamma v_3 \\ e_6 &= e + \mu_2 u_2 + v_3, \end{aligned}$$

where $\Theta \neq 0$ and the following inequalities hold,

$$\theta_{1,2} = \vartheta_{1,2} = \vartheta_{1,3} = \vartheta_{2,2} = 0, \tag{27}$$

$$\theta_{1,3} = \theta_{2,3}; \qquad \vartheta_{3,3} = 4(\theta_{2,3})^2;$$

$$\mu_1 = 1 - 2\gamma\theta_{2,3}; \qquad \mu_2 = 1 - 2\theta_{2,3}, \tag{28}$$

$$-1 \leq \beta, \gamma < 0; \qquad \alpha_1 \leq 0; \qquad 1 \leq \alpha_2, \tag{29}$$

$$0 \leq \mu_i + 2\beta_j(\mu_i\theta_{2,2} + \gamma_i\vartheta_{2,3}) \leq 2, \tag{30}$$

for $\beta_1 = \beta$, $\gamma_1 = \gamma$, $\beta_2 = \gamma_2 = 1$, and $i, j \in \{1, 2\}$.

2. The form is

$$\begin{aligned} e_1 &= e + u_2 \\ e_2 &= e + \beta v_2 \\ e_3 &= e + v_2 \\ e_4 &= e + u_1 + \alpha_i v_1 + \gamma_1 v_3 \\ e_5 &= e + u_1 + \alpha_2 v_1 + \gamma_2 v_3 \\ e_6 &= e + u_1 + v_3, \end{aligned}$$

where $\Theta \neq 0$ and the following inequalities hold,

$$\theta_{1,2} = \theta_{1,3} = \theta_{2,3} = \vartheta_{1,3} = \vartheta_{1,2} = \vartheta_{2,2} = \vartheta_{2,3} = \vartheta_{3,3} = 0, \tag{31}$$

$$\gamma_2((\gamma_1 - 1)\alpha_2 - (\gamma_2 - 1)\alpha_1) \neq 0, \tag{32}$$

$$(0,0), (1,0) \in [(\alpha_1, \gamma_1), (\alpha_2, \gamma_2), (0,1)], \tag{33}$$

$$-1 \leq \beta < 0; \qquad -1 \leq 2\theta_{2,2}, \gamma_1, \gamma_2 \leq 1. \tag{34}$$

3. The form is

$$\begin{aligned} e_1 &= e + u_2 \\ e_2 &= e + u_1 + \alpha_1 v_1 \\ e_3 &= e + u_1 + \alpha_2 v_1 \\ e_4 &= e + \beta v_2 + \gamma v_3 \\ e_5 &= e + v_2 \\ e_6 &= e + v_3, \end{aligned}$$

where $\Theta \neq 0$ and the following inequalities hold,

$$\theta_{1,2} = \theta_{1,3} = \vartheta_{1,2} = \vartheta_{1,3} = \vartheta_{2,2} = \vartheta_{2,3} = \vartheta_{3,3} = 0, \tag{35}$$

$$-1 \leq \beta, \gamma < 0; \qquad \alpha_1 \leq 0; \qquad 1 \leq \alpha_2, \tag{36}$$

$$-1 \leq 2\theta_{2,2}, 2\theta_{2,3}, 2(\beta\theta_{2,2} + \gamma\theta_{2,3}) \leq 1. \tag{37}$$

Besides, every family of the above vectors is a stochastic basis of A if it verifies the corresponding inequalities.

Now, we will prove that these bases give us a full description of all nondegenerate stochastic bases of A.

2.3.1. *Stochastic Bases without Idempotents*

Let $\Phi = \{e_i\}$ be a stochastic basis of A without idempotent elements. By Lemma 1.1 (after a reordering of the stochastic basis) we have the following properties:

(i) $[e_1, e_2], [e_3, e_4], [e_5, e_6]$ are 0-essential faces and

(ii) $[e_1, e_2, e_3, e_4], [e_1, e_2, e_5, e_6]$ are 1-essential faces.

Now, we will distinguish two cases:

1. If $[e_1, e_2] \in A^2$, then $\{e_1, e_2, e_3^2, e_5^2\}$ is a stochastic basis of A^2. By Lemmas 2.2 and 2.3 and Theorem 2.1 we obtain that this basis belongs to the form (1). The result is analogous if $[e_3, e_4]$ or $[e_5, e_6]$ belongs to A^2.

2. If $[e_i, e_{i+1}] \notin A^2$ for $i = 1, 3, 5$, then there exists a standard basis where

$$
\begin{aligned}
e_1 &= e + \beta_1 v_2, & e_4 &= e + u_1 + \mu_4 u_2 + \alpha_4 v_1 + \beta_4 v_2,\\
e_2 &= e + v_2, & e_5 &= e + \mu_5 u_2 + \gamma_5 v_3,\\
e_3 &= e + u_1 + \mu_3 u_2 + \alpha_3 v_1 + \beta_3 v_2, & e_6 &= e + \mu_6 u_2 + v_3,
\end{aligned}
$$

and $e + u_1 + v_1 \in [e_3, e_4]$, $e + u_2 \in [e_5, e_6]$. Since ee_3 and ee_4 are contained in the face $[e_1, e_2, e_3, e_4]$, scalars μ_1 and μ_2 are equal to zero. Finally, considering the product $e_3^2 e_5^2$ we obtain a contradiction.

2.3.2. *Stochastic Bases with Exactly One Idempotent and One Element in A^2*

Reordering the basis we have the following cases:

1. *Faces* $[e_1], [e_2, e_3], [e_4, e_5, e_6]$ *are* 0-*essential*. If $[e_1, e_2, e_3]$ is a 1-essential face, then there exists a standard basis such that

$$
\begin{aligned}
e_1 &= e + u_2, & e_4 &= e + u_1 + \mu_4 u_2 + \alpha_4 v_1 + \beta_4 v_2 + \gamma_4 v_3,\\
e_2 &= e + \beta_2 v_2, & e_5 &= e + u_1 + \mu_5 u_2 + \alpha_5 v_1 + \beta_5 v_2 + \gamma_5 v_3,\\
e_3 &= e + v_2, & e_6 &= e + u_1 + \mu_6 u_2 + v_3,
\end{aligned}
$$

where $v_2^2 = 0$ and $[e_4, e_5, e_6]^2 = \{e + u_1 + v_1\}$. By Lemma 1.1 one of the faces $[e_2, e_3, e_4, e_5, e_6]$, $[e_1, e_4, e_5, e_6]$ is a 1-essential face. Now taking the products ee_k for $k = 4, 5, 6, e_4^4$, and $e_1 e_4^2$ we see that

$$\mu_4 = \mu_5 = \mu_6 = \beta_4 = \beta_5 = 0.$$

Therefore, the basis belongs to the form (2). If $[e_1, e_2, e_3]$ is not a 1-essential face, then $[e_2, e_3, e_4, e_5, e_6]$ and $[e_1, e_4, e_5, e_6]$ are 1-essential faces. We consider a suitable standard basis

$$\begin{aligned} e_1 &= e + u_1 + v_1, & e_4 &= e + \lambda_4 u_1 + \mu_4 u_2 + \alpha_4 v_1 + \beta_4 v_2 + \gamma_4 v_3, \\ e_2 &= e + \beta_2 v_2, & e_5 &= e + \lambda_4 u_1 + \mu_5 u_2 + \alpha_5 v_1 + \beta_5 v_2 + \gamma_5 v_3, \\ e_3 &= e + v_2, & e_6 &= e + \lambda_4 u_1 + \mu_6 u_2 + v_3, \end{aligned}$$

where $v_2^2 = 0$ and $e_4^2 = e + \lambda_4 u_1 + u_2 + \lambda_4^2 v_1$. Now we obtain a contradiction as the following schema shows:

$$ee_1 \to \exists k \in \{4, 5, 6\} / \mu_k \le 0,$$

$$ee_i (i = 4, 5, 6) \to \lambda_4 = 0,$$

$$e_1 e_4^2 \to v_1 \in [e_4, e_5, e_6] \text{ and } \beta_4 = \beta_5 = 0,$$

$$ee_1 \to \exists \beta \in \mathbb{R} / e + \beta v_1 \in [e_4, e_5, e_6].$$

2. *Faces* $[e_1], [e_2, e_3], [e_4, e_5]$ *are* 0-*essential*. If $[e_1, e_2, e_3]$ is a 1-essential face, then for a suitable standard basis

$$\begin{aligned} e_1 &= e + u_2, & e_4 &= e + u_1 + \mu_4 u_2 + v_1 + \gamma_4 v_3, \\ e_2 &= e + \beta_2 v_2, & e_5 &= e + u_1 + \mu_5 u_2 + v_1 + v_3, \\ e_3 &= e + v_2, & e_6 &= e + \lambda_6 u_1 + \mu_6 u_2 + \alpha_6 v_1 + \beta_6 v_2 + \gamma_6 v_3, \end{aligned}$$

where $v_2^2 = 0$, $e_5^2 = e + u_1 + v_1$, and $0 \le \lambda_6 \le 1$. The set

$$\langle e, u_1, u_2, v_2 \rangle \cap [e_4, e_5, e_6]$$

contains exactly one vector, which we denote by $e' = e + \lambda u_1 + \mu u_2 + \beta v_2$. As ee_4, ee_5, ee_6, and $e_1 e_4^2$ belong to $[e_1, e_2, e_3, e']$, we conclude that

$$\lambda_6 = 1, \qquad \mu_4 = \mu_5 = \mu_6 = \beta_6 = 0$$

and the basis belongs to the form (2). On the other hand, if $[e_1, e_i, e_{i+1}]$ is not a 1-essential face for i equal to 2 and 4, then $[e_j, e_{j+1}, e_6]$ is not a 1-essential face for j equal to 2 and 4 and also either $[e_1, e_2, e_3, e_6]$ or $[e_1, e_4, e_5, e_6]$ is a 1-essential face (we will suppose that $[e_1, e_2, e_3, e_6]$ is 1-essential, the other case being analogous). For a suitable standard basis

$$\begin{aligned} e_1 &= e + u_1 + v_1, & e_4 &= e + \lambda_4 u_1 + \mu_4 u_2 + \lambda_4^2 v_1 + \gamma_4 v_3, \\ e_2 &= e + \beta_2 v_2, & e_5 &= e + \lambda_4 u_1 + \mu_5 u_2 + \lambda_4^2 v_1 + v_3, \\ e_3 &= e + v_2, & e_6 &= e + \lambda_6 u_1 + \alpha_6 v_1 + \beta_6 v_2, \end{aligned}$$

where $0 \leq \lambda_6 \leq 1$, $\lambda_4 \neq 1$, $\beta_6 \neq 0$, and $e_4^2 = e + \lambda_4 u_1 + u_2 + \lambda_4^2 v_1$. The set $\{e_1, e_2, e_3, e_4^2, e_6\}$ is stochastic, but by Theorem 2.2 this is impossible.

2.3.3. Nondegenerate Stochastic Bases with One Idempotent and Exactly Two Elements in A^2

Reordering the basis we have the following possibilities:

1. faces $[e_1], [e_2], [e_3, e_4, e_5, e_6]$ are 0-essential;
2. faces $[e_1], [e_2], [e_3, e_4, e_5]$ are 0-essential;
3. faces $[e_1], [e_2], [e_3, e_4]$ are 0-essential;
4. faces $[e_1], [e_3, e_4], [e_5, e_6]$ are 0-essential and $e_2 \in A^2$;
5. faces $[e_1], [e_2, e_3, e_4], [e_5, e_6]$ are 0-essential and $e_2 \in A^2$.

1. If our basis verifies case 1, then A is regular, which is impossible.

2. When the basis verifies case 2, then with respect to a suitable standard basis

$$e_1 = e + \lambda_1 u_1 + u_2 + \lambda_1^2 v_1, \qquad e_4 = e + v_2,$$
$$e_2 = e + u_1 + v_1, \qquad e_5 = e + v_3,$$
$$e_3 = e + \beta_3 v_2 + \gamma_3 v_3, \qquad e_6 = e + \lambda_6 u_1 + \mu_6 u_2 + \alpha_6 v_1 + \gamma_6 v_3,$$

where $[e_3, e_4, e_5]^2 = \{e\}$, $-1 \leq \lambda_1 \leq 1$, $\lambda_6 \in [\lambda_1, 0, 1]$, and $\alpha_6 < 0$. Multiplying e_2 by e we see that $\mu_6 \leq 0$. So the products $e_1 e_2$ and $e_1 e$ imply $\lambda_1 = 1$ and $\lambda_6 = \mu_6 = 0$. Now considering the products of the elements of the stochastic basis we see that A is regular, but this is impossible.

3. If the basis verifies one of the cases 3, 4, or 5, then the set $[e_5, e_6] \cap (\langle e_3, e_4 \rangle + A^2)$ contains exactly one vector, which we denote by e'. It is clear that $\Psi = \{e_1, e_2, e_3, e_4, e'\}$ is stochastic. Using Theorem 2.2 we can see that the basis is one of the following:

(a) The basis is

$$e_1 = e + u_2, \qquad e_4 = e + v_2,$$
$$e_2 = e + u_1 + \alpha_2 v_1, \qquad e_5 = e + u_1 + \mu_5 u_2 + \alpha_5 v_1 + \gamma_5 v_3,$$
$$e_3 = e + \beta_3 v_2, \qquad e_6 = e + u_1 + \mu_6 u_2 + \alpha_6 v_1 + v_3,$$

where $v_2^2 = 0$ and $e' = e + u_1 + \alpha v_1$. Since ee_5 and ee_6 belong to $[\Psi]$, it follows that $\mu_5 = \mu_6 = 0$ and so the basis is of the form (2).

(b) The basis is

$$e_1 = e + u_1 + v_1, \qquad e_4 = e + v_2,$$
$$e_2 = e + u_1 + \alpha_2 v_1, \qquad e_5 = e + \mu_5 u_2 + \gamma_5 v_3,$$
$$e_3 = e + \beta_3 v_2, \qquad e_6 = e + \mu_6 u_2 + v_3,$$

where $v_2^2 = 0$ and $e' = e + u_2$. It is clear that the basis is of the form (1).

(c) The basis is

$$e_1 = e, \qquad e_4 = e + u_1 + v_2,$$
$$e_2 = e + u_2, \qquad e_5 = e + u_1 + \mu_5 v_2 + \alpha_5 v_1 + \beta_5 v_2 + \gamma_5 v_3,$$
$$e_3 = e + u_1 + \alpha_3 v_1 + \beta_3 v_2, \qquad e_6 = e + u_1 + \mu_6 u_2 + v_3,$$

where $e' = e + u_1 + \alpha v_1 + \beta v_2$. Since $e_1 e_5$ and $e_1 e_6$ belong to $[\Psi]$ it follows that $\mu_5 = \mu_6 = 0$. Also, we can see that the algebra is regular, but this is impossible.

(d) The basis is

$$e_1 = e + u_2, \qquad e_4 = \ + u_1 + \alpha_4 v_1 + v_2,$$
$$e_2 = e + u_1 + \alpha_2 v_1, \qquad e_5 = e + \gamma_5 v_3,$$
$$e_3 = e + u_1 + \alpha_3 v_1 + \beta_3 v_2, \qquad e_6 = e + v_3,$$

where $e' = e$. The basis is of the form (2).

2.3.4. Stochastic Nondegenerate Bases with One Idempotent and Exactly Three Elements in A^2

We can consider the basis with e_1 idempotent element and e_2 and e_3 in A^2. We denote by e' the unique element in $[e_4, e_5, e_6] \cap A^2$. It is clear that $\{e_1, e_2, e_3, e'\}$ is a stochastic basis of A^2 and $[e_1, e_2, \dots, e_6]^2 \subset [e_1, e_2, e_3, e']$. According to Theorem 2.1 we have two possibilities:

1. The basis is

$$e_1 = e, \qquad e_4 = e + u_1 + \mu_4 u_2 + v_2,$$
$$e_2 = e + u_2, \qquad e_5 = e + u_1 + \mu_5 u_2 + v_3,$$
$$e_3 = e + u_1 + \alpha_3 v_1, \qquad e_6 = e + u_1 + \mu_6 u_2 + \alpha_6 v_1 + \beta_6 v_2 + \gamma_6 v_3,$$

where $\beta_6 \gamma_6 \neq 0$ and $e' = e + u_1 + \alpha v_1$. The products $e_1 e_i$ for $i = 4, 5, 6$ imply that $\mu_4 = \mu_5 = \mu_6 = 0$. Considering the products of the vectors of the stochastic basis we can see that the algebra is regular, which is impossible.

2. The basis is

$$e_1 = e + u_2, \qquad e_4 = e + \mu_4 u_2 + v_2,$$
$$e_2 = e + u_1 + \alpha_2 v_1, \qquad e_5 = e + \mu_5 u_2 + v_3,$$
$$e_3 = e + u_1 + \alpha_3 v_1, \qquad e_6 = e + \mu_6 u_2 + \alpha_6 v_1 + \beta_6 v_2 + \gamma_6 v_3,$$

where $\beta_6\gamma_6 \neq 0$ and $e' = e$. It is easy to prove that $\mu_4 = \mu_5 = \mu_6 = 0$, considering the products ee_i for $i = 4, 5, 6$. Consequently, the basis is of the form (3).

3. THE BERNSTEIN PROBLEM FOR TYPE (3, 3)

Now we will formulate our main result about the Bernstein problem. It is a reformulation of the previous results.

THEOREM 3.1. *Let V be a nonregular nonexceptional nondegenerate stationary evolutionary operator of type* (3, 3). *Then it has one of the following three forms*:

1. *The form is*

$$
\begin{aligned}
x_1' &= \frac{1}{\alpha_2 - \alpha_1}(\alpha_2 s - p)p \\
x_2' &= \frac{1}{\alpha_2 - \alpha_1}(p - \alpha_1 s)p \\
x_3' &= \beta_2\big(s^2 - ps - \Lambda\big) \\
x_4' &= \beta_1\big(s^2 - ps - \Lambda\big) \\
x_5' &= \gamma_2\Lambda \\
x_6' &= \gamma_1\Lambda,
\end{aligned}
$$

where $s = \sum_{i=1}^{6} x_i$ *and* $p = x_1 + x_2$ *are invariant linear forms and*

$$p' = (1 + \gamma_1\theta)x_5 + (1 - \gamma_2\theta)x_6, \qquad q_1 = -\beta_1 x_3 + \beta_2 x_4,$$

$$q_2 = -\gamma_1 x_5 + \gamma_2 x_6, \tag{38}$$

$$\Lambda = sp' + (p + p' + q_2\theta)q_2\theta + p'q_1\eta + q_1q_2\vartheta, \tag{39}$$

$$0 < \beta_1 \leq \beta_2; \qquad 0 < \gamma_1 \leq \gamma_2; \qquad \beta_1 + \beta_2 = \gamma_1 + \gamma_2 = 1, \tag{40}$$

$$\alpha_1 \leq 0; \qquad 1 \leq \alpha_2; \qquad |\theta| + |\vartheta| + |\eta| \neq 0, \tag{41}$$

$$-1 \leq (-1)^{i+1}\gamma_i\theta + (-1)^j\beta_j\big(\eta + (-1)^i\gamma_i(\vartheta - \theta\eta)\big) \leq 1 \qquad (i, j = 1, 2). \tag{42}$$

2. *The form is*

$$
\begin{aligned}
x_1' &= x_1\big(s + (-\beta_1 x_2 + \beta_2 x_3)\theta\big)\\
x_2' &= \beta_2\big((x_2 + x_3)s - x_1(-\beta_1 x_2 + \beta_2 x_3)\theta\big)\\
x_3' &= \beta_1\big((x_2 + x_3)s - x_1(-\beta_1 x_2 + \beta_2 x_3)\theta\big)\\
x_4' &= \Gamma\big((\gamma_2 - 1)p + \alpha_2 s\big)p\\
x_5' &= \Gamma\big((1 - \gamma_1)p - \alpha_2 s\big)p\\
x_6' &= \Gamma\big((\alpha_1\gamma_2 - \alpha_2\gamma_1)s + (\gamma_1 - \gamma_2)p\big)p,
\end{aligned}
$$

where $s = \sum_{i=1}^{6} x_i$ *and* $p = x_4 + x_5 + x_6$ *are invariant linear forms and*

$$\Gamma^{-1} = (\gamma_2 - 1)\alpha_1 - (\gamma_1 - 1)\alpha_2, \tag{43}$$

$$0 < \beta_1 \le \beta_2; \qquad \beta_1 + \beta_2 = 1; \qquad \alpha_1 \le 0; \qquad 1 \le \alpha_2, \tag{44}$$

$$(0,0),(1,0) \in \big[(\alpha_1,\gamma_1),(\alpha_2,\gamma_2),(0,1)\big], \tag{45}$$

$$-1 \le \beta_2\theta, \gamma_1, \gamma_2 \le 1; \qquad \theta \ne 0; \qquad |\gamma_1| + |\gamma_2| \ne 0. \tag{46}$$

3. *The form is*

$$
\begin{aligned}
x_1' &= \frac{1}{\alpha_2 - \alpha_1}(\alpha_2 s - p)p\\
x_2' &= \frac{1}{\alpha_2 - \alpha_1}(p - \alpha_1 s)p\\
x_3' &= \big(s + (\beta\theta + \gamma\eta)x_4 + \theta x_5 + \eta x_6\big)x_3\\
x_4' &= \frac{1}{1 - \beta - \gamma}\big((x_4 + x_5 + x_6)s - ((\beta\theta + \gamma\eta)x_4 + \theta x_5 + \eta x_6)x_3\big)\\
x_5' &= \frac{-\beta}{1 - \beta - \gamma}\big((x_4 + x_5 + x_6)s - ((\beta\theta + \gamma\eta)x_4 + \theta x_5 + \eta x_6)x_3\big)\\
x_6' &= \frac{-\gamma}{1 - \beta - \gamma}\big((x_4 + x_5 + x_6)s - ((\beta\theta + \gamma\eta)x_4 + \theta x_6 + \eta x_6)x_3\big),
\end{aligned}
$$

where $s = \sum_{i=1}^{6} x_i$ *and* $p = x_1 + x_2$ *are invariant linear forms and*

$$-1 \le \beta, \gamma < 0; \qquad \alpha_1 \le 0; \qquad 1 \le \alpha_2, \tag{47}$$

$$-1 \le \theta, \eta, \beta\theta + \gamma\eta \le 1; \qquad |\theta| + |\eta| \ne 0. \tag{48}$$

Remark. By (10) and Lemma 2.1, if V is a nonregular nonexceptional s.e.o. of type $(3, 3)$, then its space of invariant linear forms, J_V, has dimension 2. Consequently if V belongs to one of the above three forms, then $J_V = \langle s, p\rangle$.

It remains to note that from 2.1 and 2.2 also we can describe explicitly all nonregular nonexceptional and degenerate s.e.o.'s of type $(3, 3)$.

We conclude that these evolutionary operators are not normal in the sense of [21, p. 167]. This is a new support for the Lyubich conjecture [21, p. 232]. According to the Lyubich conjecture, if V is a *normal* s.e.o., then the corresponding Bernstein algebra is regular. A nondegenerate s.e.o. V is normal if there is no pair j_1, j_2 such that x'_{j_1} and x'_{j_2} are proportional and there is no pair j_1, j_2 such that all x'_j's depend only on $x'_{j_1} + x'_{j_2}, x'_i$ $(i \neq j_1, j_2)$.

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