NORTH-HOLLAND

Structure of Bernstein Populations of Type $(3, n-3)$

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Submitted by Daniel Hershkowitz

ABSTRACT

This paper describes explicitly all simplicial stochastic nonregular nonexceptional Bernstein algebras of type $(3, n-3)$. Consequently, the Bernstein problem of type $(3, n-3)$ is settled, since the regular and exceptional cases have already been done. © 1998 Elsevier Science Inc.

1. INTRODUCTION

In [1–4], Bernstein raised and partially solved an important problem concerning the mathematical expression of some fundamental laws of biological heredity. Let us, following [14] and [20], describe the statement of the Bernstein problem.

A *state* of a population in any generation can be described by a *stochastic* (or *probabilistic*) vector $x = (x_i)_{i=1}^n$, so all the $x_i \geqslant 0$ and $s(x) \equiv \sum_i x_i = 1$. The set of all states is the basic simplex $\Delta^{n-1} \subset \mathbb{R}^n$. The vertices $(e_i)_{i=1}^n$ of this simplex are *types* of individuals in this population. Let us denote by $p_{ij,k}$ the probability that an individual of the type e_k appears in the next generation from parents whose types are e_i and e_j, so $p_{ij,k} \geqslant 0$ and $\sum_k p_{ij,k} = 1$; moreover, $p_{ij,k} = p_{ji,k}$, assuming that the maternal or paternal origin does

*Partially supported by PB94-1311-C03-01, Spain.

LINEAR ALGEBRA AND ITS APPLICATIONS 269:17–32 (1998)

655 Avenue of the Americas, New York, NY 10010

0024-3795/98/$19.00
PII S0024-3795(97)00001-3

not play a role in the production of offspring's types. Let the mating in the population be at random and without selection. Then the state $x' = (x'_i)_{i=1}^n$ in the next generation will be

$$x'_k = \sum_{i,j} p_{ij,k} x_i x_j \qquad (1 \leqslant k \leqslant n). \tag{1}$$

These formulas define a mapping $V : \Delta^{n-1} \to \Delta^{n-1}$ called the *evolutionary operator* (e.o. for short) of the given population. A state x is called an *equilibrium* if $Vx = x$. An evolutionary operator V is called *Bernstein* or *stationary* (s.e.o. for short) if for every $x \in \Delta^{n-1}$ the corresponding state in the next generation is an equilibrium, or equivalently $V^2 = V$.

The Bernstein problem is to describe explicitly all stationary evolutionary operators.

The following interpretation of evolutionary operator V, given by the formula (1), is very useful in the solution of this problem. One can define in the space $\mathbb{R}^n$ a commutative product

$$e_i e_j = \sum_k p_{ij,k} e_k \qquad (1 \leqslant i, j \leqslant n), \tag{2}$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. In such a way we obtain a commutative (nonassociative) baric algebra (A, s), where $s : A \to \mathbb{R}$ is a nonzero homomorphism over $\mathbb{R}$. Moreover, the pair (A, s) is a *simplicial stochastic algebra* (see [20, p. 150] for a general definition; we say "stochastic algebra" for short). This means that the product of any elements $x, y \in \Delta^{n-1}$ belongs to Δ^{n-1}. There is a remarkable connection between the stationary properties of an evolutionary operator and some property of the corresponding algebra A: the operator V is stationary if and only if the identity

$$(x^2)^2 = s(x)^2 x^2 \tag{3}$$

holds. This means that (A, s) is a Bernstein algebra (see [13] and [15], where identity (3) appeared for the first time; the term "Bernstein algebra" was introduced in [18]). One of most important conclusions that can be drawn from the preceding relation is that the Bernstein problem is equivalent to finding, for every Bernstein algebra, all stochastic bases with their corresponding multiplicative constants.

Let us remark that Bernstein algebras are not necessarily algebras with a *stochastic realization*; that is, there exist Bernstein algebras such that for every basis the simplex Δ, spanned by this basis, is not invariant with respect to the multiplication, i.e., $\Delta \cdot \Delta \not\subset \Delta$ (see [8]). A linearly independent set ϕ in A is said to be *stochastic* if $\Delta(\phi) \cdot \Delta(\phi) \subset \Delta(\phi)$, where $\Delta(\phi)$ is the simplex spanned by the set ϕ. Therefore, a Bernstein algebra is an algebra with a stochastic realization if and only if it has a stochastic basis. Now let $\Phi = \{e_1, \ldots, e_n\}$ be a stochastic basis of a Bernstein algebra (A, ω), and $\lambda_{ij,k}$ the corresponding multiplicative constants, that is, $e_i e_j = \sum_k \lambda_{ij,k} e_k$; since the basis is stochastic, $\lambda_{ij,k} \geqslant 0$ and $\sum_k \lambda_{ij,k} = 1$; since the algebra is commutative, $\lambda_{ij,k} = \lambda_{ji,k}$. Thus, we may consider $\lambda_{ij,k}$ as the hereditary coefficients of a stationary evolutionary operator, which we denote by V_Φ. In the following we will consider the elements of a stochastic basis indiscriminately as vectors of the algebra or as types of the s.e.o.

PROPOSITION 1.1. *If $\Phi = \{e_i\}$ is a basis of a Bernstein algebra (A, ω) and $\omega(e_i) = 1$ for all i, then the basis is stochastic if and only if the coordinates of $e_{j_1} e_{j_2}$ with respect to Φ are greater or equal to* 0 *for all j_1 and j_2.*

If $x \in A$ and $\omega(x) = 1$, then x^2 is an idempotent element. Thus, a Bernstein algebra has at least one nonzero idempotent element. Every nonzero idempotent element e, yields the Peirce decomposition $A = \mathbb{R}e \oplus U_e \oplus V_e$, where

$$\ker \omega = U_e \oplus V_e, \qquad U_e = \left\{x \in A \middle| ex = \tfrac{1}{2}x\right\}, \qquad V_e = \left\{x \in A \middle| ex = 0\right\}.$$

It was proved in [14] that $\dim U_e$ (and then $\dim V_e$) does not depend on the choice of the idempotent element, so one can define type $A = (m, \delta)$, where $m - 1 = \dim U_e$ and $\delta = \dim V_e$ (so $n = \dim A = m + \delta$). Products between elements of U_e and V_e satisfy the following relations:

$$U_e^2 \subset V_e, \qquad U_e V_e \subset U_e, \qquad V_e^2 \subset U_e, \tag{4}$$

$$(u^2)^2 = u^2(uv) = u^3 = u(uv) = u(v^2) = (uv)^2 = u^2v^2 = 0 \tag{5}$$

for all $u \in U_e$ and $v \in V_e$. On the other hand, the set of idempotent elements is $I(A) = \{e + u + u^2 \mid u \in U_e\}$, and if $\bar{e} = e + \bar{u} + \bar{u}^2$ is another idempotent, then

$$U_{\bar{e}} = \left\{u + 2u\bar{u} \mid u \in U_e\right\}, \qquad V_{\bar{e}} = \left\{v - 2(\bar{u} + \bar{u}^2)v \mid v \in V_e\right\}. \tag{6}$$

Moreover, dim U_e^2 and $\dim(U_eV_e + V_e^2)$ do not depend on the choice of the idempotent element. So we will use the following definitions introduced by Lyubich (see [14, 17, 20]): a Bernstein algebra in which $U_eV_e + V_e^2 = (0)$ is called *regular*, and a Bernstein algebra in which $U_e^2 = (0)$ is said to be *exceptional*. Every Bernstein algebra of type (m, δ) with $m - 1$ or δ less or equal to 1 is regular or exceptional. Also, an algebra A where $A^2 = A$ is called *nuclear*.

We will denote by U_0 the subset $U_e \cap \operatorname{ann} U_e$. This subset is independent of the idempotent considered, and it is an ideal of A. On the other hand, $\bar{A} = A/U_0$ is a Jordan-Bernstein algebra (see [5] for general information about U_0 and Jordan-Bernstein algebras).

The following basic concept of *invariant* linear form was introduced by Lyubich ([14]; see [20] for more information). This is a linear form f on a Bernstein algebra which satisfies the identity $f(x^2) = \omega(x)f(x)$ for all $x \in A$. The trivial invariant linear forms are ω and zero. If J_A is the set of invariant linear forms of A, then its orthogonal is $J_A^{\perp} = (U_eV_e + V_e^2) \oplus V_e$. For every stochastic basis Φ the space J_A can be provided with the norm $\|f\| = \max\{|f(x)| \mid x \in \Delta(\Phi)\}$. Obviously, $\|f\| = \max\{|f(e_i)| \mid e_i \in \Phi\}$, since $\Delta(\Phi)$ is the convex hull of Φ. The set $\Gamma_f = \{x \in \Delta(\Phi) \mid f(x) = \max\{f(y) : y \in \Delta(\Phi)\}$ is an invariant face of $\Delta(\Phi)$. We denote by $\Gamma_f^{(1)}$ and $\Gamma_f^{(2)}$ the invariant faces Γ_f and Γ_{-f} respectively.

A s.e.o. is called *regular*, *exceptional*, or of *type* (m, δ) according as the corresponding stochastic Bernstein algebra is regular, exceptional, or of type (m, δ). A type e_i of a s.e.o., is called *vanishing* if $x_i' = 0$, and a s.e.o. is *degenerate* if it has vanishing types. Also we call a stochastic basis *degenerate* if the corresponding s.e.o. is degenerate.

Now, we recall some definitions and results due to Lyubich that are very useful in the solution of this problem (see [20, p. 228–235] for more information). Let V be a Bernstein evolutionary operator. Consider an arbitrary face Γ of the basis simplex Δ, and let $C_\Gamma = \operatorname{Int}\Gamma \cap \operatorname{Im} V$ (as usual, $\operatorname{Int}\Gamma$ is the interior of the face Γ with respect to its affine hull). The face Γ is called *essential* if $C_\Gamma \neq \varnothing$. An essential face Γ is called a *k-dimensional* essential face if C_Γ has dimension k as topological space (we say "k-essential" for short). Let $x = \sum_{i=1}^n \alpha_i e_i$, $\alpha_i \geqslant 0$. Then the *support* of x, denoted by $\operatorname{supp}(x)$, is the set of basis vectors (or their corresponding indices) for which the coordinate of x is positive, i.e., $i \in \operatorname{supp}(x)$ if and only if $\alpha_i > 0$. If $x \in \Delta$ and Γ is a face, then $x \in \operatorname{Int}\Gamma$ if and only if $\operatorname{supp}(x) = \{i \mid e_i \in \Gamma\}$. Every essential face Γ is invariant, and $V|\Gamma$ is an operator nondegenerate of type $(m_\Gamma, \delta_\Gamma)$, where $m_\Gamma - 1$ is its topological dimension.

We can prove the following generalization of Theorem 5.7.1 of [20] related to the case $k = 0$.

LEMMA 1.1 [11]. *Let V be a Bernstein evolutionary operator of type (m, δ). Then the number of k-essential faces is at least $m - k$, $k = 0, 1, \ldots, m-1$.*

The next lemma follows immediately from definitions.

LEMMA 1.2. *Let Γ be a 0-essential face. Then*

(i) *if Γ_1 is essential and $\Gamma \cap \Gamma_1 \neq \varnothing$, then $\Gamma \subset \Gamma_1$;*
(ii) *if Γ_1 is 0-essential and $\Gamma \cap \Gamma_1 \neq \varnothing$, then $\Gamma = \Gamma_1$;*
(iii) *if f is an invariant linear form, then $f|_\Gamma$ is a constant function.*

Following [14] and [20] a linear form f on an algebra A is called *disappearing* if $f \in (A^2)^\perp$. Obviously, in this case $\ker f$ is a subalgebra containing A^2. This subalgebra is Bernstein if A is so.

A linear form f is called *hyperbolic* with respect to a stochastic basis $\Phi = \{e_i\}$ if $f = \sum_{k=1}^n \alpha_k e_k^* \neq 0$ and either there exists exactly one $\alpha_k \neq 0$ or there exists $\alpha_k \neq 0$ such that sign $\alpha_j = -$sign α_k for all $\alpha_j \neq 0$, $j \neq k$. In this case let us call α_k the *leading coefficient*. The following lemma is contained in [20, Corollary 3.7.6 and formulas 3.7.21].

LEMMA 1.3. *If a linear form f is hyperbolic with respect to a stochastic basis $\Phi = \{e_i\}$, then there exists a stochastic basis $\tilde{\Phi}$ of $\ker f$. Namely, if α_k is the leading coefficient in f, then*

$$\tilde{\Phi} = \left\{e_j \mid \alpha_j = 0\right\} \cup \left\{\tilde{e}_j \mid \alpha_j \neq 0,\ j \neq k\right\}, \tag{7}$$

where

$$\tilde{e}_j = \frac{|\alpha_j| e_k + |\alpha_k| e_j}{|\alpha_j| + |\alpha_k|}. \tag{8}$$

The stochastic set $\tilde{\Phi}$ is called *external reduction* of Φ [20, Section 3.9]. An *external kinship chain* is a sequence of stochastic sets

$$\Phi_1, \Phi_2, \ldots, \Phi_r$$

such that in every pair (Φ_k, Φ_{k+1}) the set Φ_{k+1} is an external reduction of Φ_k. If such a kinship chain does exist for given Φ_1 and Φ_r, one can say that Φ_1 and Φ_r are *externally kin*.

LEMMA 1.4. *If Φ and Ψ are externally kin stochastic bases and Φ is nondegenerate, then Ψ is nondegenerate as well.*

Proof. It is sufficient to prove the lemma for external reduction of a stochastic basis. Let $\Phi = \{e_i\}$ be a nondegenerate stochastic basis of a Bernstein algebra A and $\tilde{\Phi}$ an external reduction of Φ by a linear form $f = \sum_{i=1}^{n} \alpha_i e_i^*$. We can assume, by a reordination of the stochastic basis, that

$$\alpha_1 = \cdots = \alpha_r = 0, \qquad \alpha_{r+1}, \dots, \alpha_{n-1} < 0, \qquad \alpha_n = 1.$$

Let $x = \sum_{i=1}^{n} x_i e_i$ be an element of $\ker f$. Then by (7) and (8)

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{j=r+1}^{n-1} (1 - \alpha_j) x_j \tilde{e}_j + \left(x_n + \sum_{j=r+1}^{n-1} \alpha_j x_j \right) e_n$$

with $x_n + \sum_{j=r+1}^{n-1} \alpha_j x_j = f(x) = 0$, and hence

$$x = \sum_{i=1}^{r} x_i \tilde{e}_i + \sum_{j=r+1}^{n-1} (1 - \alpha_j) x_j \tilde{e}_j,$$

which immediately implies the lemma. ∎

For a finite sequence $\Phi = \{x_1, x_2, \dots, x_r\}$ of elements of an $\mathbb{R}$-algebra, we denote by $\langle x_1, x_2, \dots, x_r \rangle$ the vector subspace spanned by Φ, and by $[x_1, x_2, \dots, x_r]$ the convex hull of Φ.

2. STOCHASTIC BASES FOR THE TYPE $(3, n-3)$

Bernstein solved the Bernstein problem for $n = 3$. (The case $n = 2$ is trivial.) For all $n > 3$ the Bernstein problem in the regular case and the exceptional case was completely solved by Yu. Lyubich in a series of papers (see the book [20] and the references there). In [7] we described explicitly all s.e.o.'s of type (3, 2) in the nonregular and nonexceptional case and thereby completed the solution of the problem for $n = 5$. Also, in [9], for all n, we described all s.e.o.'s of type $(n-2, 2)$ in the nonregular nonexceptional nonnuclear case. Finally, in [10] we solved completely the Bernstein problem for $n = 6$, talking into account the above-mentioned result of Lyubich and the following one recently obtained by Grishkov.

LEMMA 2.1 [12]. *Every nuclear s.e.o. of type* (4, 2) *is regular.*

Thus, the following Lyubich conjecture is true for $n \leqslant 6$:

CONJECTURE 2.1. *Every nuclear s.e.o. is regular.*

In this paper, we will describe, for all n, the stochastic bases of a Bernstein algebra of type $(3, n-3)$ and with $\dim U_0 = 1$. It has been proved in [19] (see [6] for more information) that if A is a nonregular nonexceptional Bernstein algebra of type $(3, n-3)$, then A^2 is isomorphic to the algebra $A_{1,1}$ (see Corollary 2.3 of [8]), i.e., A^2 has a basis $\{e, u_1, u_2, v_1\}$ with products nonzero $e^2 = e$, $eu_1 = u_1/2$, $eu_2 = u_2/2$, and $u_1^2 = v_1$. Consequently, with this work, the Bernstein problem for the type $(3, n-3)$ is completely solved.

From now on A will be a nonexceptional Bernstein algebra with dimension of U_0 equal to 1 (so $n = \dim A \geqslant 4$). A basis $\Psi = \{e, u_1, u_2, v_1, v_2, \dots, v_{n-3}\}$ of A is called *standard* if e is an idempotent element, u_1, u_2 a basis of U_e, $v_1, \dots, v_{n-3}$ a basis of V_e, $u_2 \in U_0$, and $u_1^2 = v_1$. (See Propositions 3 and 4 of [6] for more information about the standard bases.)

PROPOSITION 2.1 [6]. *Let* $\{e, u_1, u_2, v_1, \dots, v_{n-3}\}$ *be a standard basis of* A. *Then*:

(i) *The basis* $\{e, u'_1, u'_2, v'_1, \dots, v'_{n-3}\}$ *is standard if and only if there exist* $\alpha, \beta \in \mathbb{R}^*$ *and* $\xi \in \mathbb{R}$ *such that* $u'_1 = \alpha(u_1 + \xi u_2)$, $u'_2 = \beta u_2$, $v'_1 = \alpha^2 v_1$.

(ii) *If* $\bar{e} = e + \bar{u} + \bar{u}^2$ *is another idempotent of* A, $\bar{u} = pu_1 + qu_2$, *then* $\{\bar{e}, \bar{u}_1, \bar{u}_2, \bar{v}_1, \dots, \bar{v}_{n-3}\}$ *is a standard basis, where*

$$\bar{u}_1 = u_1 + 2pv_1, \qquad \bar{u}_2 = u_2,$$

$$\bar{v}_1 = v_1, \qquad \bar{v}_i = v_i - 2(\lambda_{1i}p + \lambda_{2i}q + \mu_{1i}p^2)u_2$$

for $2 \leqslant i \leqslant n-3$.

Some properties of the algebra A are:

(1) The subspace U_e^2 does not depend on choice of the idempotent element and it has dimension 1. We will denote this subspace by V_0.

(2) $U_e^3 = U_e V_0 = V_0^2 = (0)$ and $U_e V_e + V_e^2 \subset U_0$.

(3) If $\Psi = \{e, u_1, u_2, v_1, \dots, v_{n-3}\}$ is a standard basis, then the set of idempotent elements of A is $I_A = \{e + \alpha u_1 + \beta u_2 + \alpha^2 v_1 \mid \alpha, \beta \in \mathbb{R}\}$, and $J_A = \langle e^*, u_1^* \rangle$ with respect to its dual basis.

(4) Let Φ be a stochastic basis of A. Then the set $\{\langle \Gamma_f^{(i)} \rangle \mid i = 1, 2\}$ does not depend on choice of a nontrivial invariant linear form f on A, and if (m_i, δ_i) is the type of the subalgebra $A_i := \langle \Gamma_f^{(i)} \rangle$, then $m_i \leqslant 2$. On the other hand, the set $\Phi_f^{(i)} = \{e_i \in \Phi \mid e_i \in \Gamma_f^{(i)}\}$ is a stochastic basis of A_i.

The following lemma is a reformulation of a theorem obtained by Lyubich (see [16]).

LEMMA 2.2. *Let a Bernstein algebra be exceptional and of type* $(2, n - 2)$. *Then its stochastic bases are of the following forms*

$$e_1 = e + \sum_{k=1}^{r-1} \alpha_k v_k,$$

$$e_i = e + v_{i-1} \qquad (i = 2, \ldots, r),$$

$$e_{r+1} = e + \mu_{r+1} u + \sum_{k=r}^{s-2} \alpha_k v_k,$$

$$e_j = e + \mu_j u + v_{j-2} \qquad (j = r + 2, \ldots, s),$$

$$e_l = e + \mu_l u + v_{l-2} \qquad (l = s + 1, \ldots, n).$$

The element e *is an idempotent,* $u \in U_0$, $v_1, \ldots, v_{n-2}$ *is a basis of* V_e, *and* $\alpha_k < 0$.

In addition, the faces $[e_1, \ldots, e_r]$ *and* $[e_{r+1}, \ldots, e_s]$ *are* 0*-essential, and the vectors* e_l *are vanishing.*

Proof. Let $\Phi = \{e_i\}$ be a stochastic basis of an exceptional Bernstein algebra A of type $(2, n - 2)$. By Lemma 1.1, the simplex of this basis contains two 0-essential faces. Thus, by Lemma 1.2 and a suitable arrangement of the stochastic basis we can suppose that

$$\Gamma_1 = [e_1, \ldots, e_r] \quad \text{and} \quad \Gamma_2 = [e_{r+1}, \ldots, e_s]$$

are the 0-essential faces. We denote by e the element of Γ_1^2. Then e is an idempotent element, and there exist linearly independent vectors $v_1, \ldots, v_{r-1}$ in V_e such that $e_1 = e + \sum_{k=1}^{r-1} \alpha_k v_k$ and $e_i = e + v_{i-1}$ for $i = 2, \ldots, r$. Now, as

$$\dim(\langle \Gamma_1 \cup \Gamma_2 \cup U_0 \rangle / \langle \Gamma_1 \cup U_0 \rangle) = \dim\langle \Gamma_2 \rangle - 1,$$

it follows that there exist vectors $v_r, \ldots, v_{s-2}$ in V_e and u in U_0 such that $v_1, v_2, \ldots, v_{s-2}$ are linearly independent, $e_{r+1} = e + \mu_{r+1}u + \sum_{k=r}^{s-2}\alpha_k v_k$, and $e_j = e + \mu_j u + v_{j-2}$ for $j = r+2, \ldots, s$.

Finally, by the relation $A^2 \subset \langle \Gamma_1 \cup \Gamma_2 \rangle$ we obtain that the types e_l, for $s+1 \leqslant l \leqslant n$, are vanishing. This completes the proof of the lemma. ■

COROLLARY 2.1. *Let B be a Bernstein subalgebra of A of type $(2, n'-2)$, $n' = \dim B$, such that U_0 and V_0 are in B. Then its stochastic bases, after a suitable reordination, are of the following form:*

$$e_1 = e + \sum_{k=2}^{r} \alpha_k v_k,$$

$$e_i = e + v_i \qquad (i = 2, \ldots, r),$$

$$e_{r+1} = e + \mu_{r+1}u_2 + \sum_{k=1}^{n'-2} \beta_k v_k,$$

$$e_{r+2} = e + \mu_{r+2}u_2 + \sum_{k=1}^{n'-2} \gamma_k v_k,$$

$$e_j = e + \mu_j u_2 + v_{j-2} \qquad (j = r+3, \ldots, n').$$

The element e is an idempotent, $u_2 \in U_0$, $v_1 \in V_0$, and $v_1, \ldots, v_{n'-2}$ are linearly independent vectors in V_e. The face $[e_1, \ldots, e_r]$ is 0-essential.

THEOREM 2.1. *Let $\Phi = \{e_1, e_2, \ldots, e_n\}$ be a stochastic basis of A. Then there exist Φ_1, Φ_2, Φ_3 three subsets of Φ, and a standard basis $\Psi = \{e, u_1, u_2, v_1, \ldots, v_{n-3}\}$ such that the following statements hold:*

(i) *$\Phi_i \cap \Phi_j = \varnothing$ if $i \neq j$;*

(ii) *Δ_1 and Δ_2 are 0-essential faces and $\Delta_3^2 \subset \Delta_3$ ($\Delta_i := \Delta(\Phi_i)$);*

(iii) *$\Delta_1 \cap A^2 = \{e\}$, $\Delta_2 \cap A^2 = \{e + u_2\}$, and $\Delta_3 \cap A^2 = [e + u_1 + \varepsilon_1 v_1, e + u_1 + \varepsilon_2 v_1]$, where $\varepsilon_1 \leqslant 0$ and $1 \leqslant \varepsilon_2$;*

(iv) *if f is the invariant linear form that verifies $f(e) = 0$ and $f(u_1) = 1$, then $f(\Phi) \geqslant 0$ and $\|f\| = 1$;*

(v) *every element of Φ either is a vanishing type or belongs to $\Phi_1 \cup \Phi_2 \cup \Phi_3$.*

Proof. We shall use induction on $n = \dim A$. The first step of induction is for $n = 4$. In Theorem 3.1 of [8] (see also Theorem 3.2 of [7]), we describe explicitly all stochastic bases of A when $\dim A = 4$. Also, in [7] and [10] we describe explicitly all stochastic bases of A, when $\dim A = 5$ and when $\dim A = 6$ and A is nonregular respectively. Consequently, the theorem has been proved in all the cases for $n \leqslant 5$.

Let $n > 5$. Assume that the theorem is valid for Bernstein algebras of type $(3, n' - 3)$ with $\dim U_0 = 1$ and $n' < n$. Let $\Phi = \{e_i\}$ be a stochastic basis of A. If the stochastic basis has vanishing types, then the nonvanishing types form a stochastic basis of a subalgebra of A that contains A^2. Therefore, by the induction assumption we obtain the result. In the rest of the proof we assume that the stochastic basis has no vanishing types. Let f be a nontrivial invariant linear form of A (we can assume that $\min f(\Phi) = 0$ and $\|f\| = 1$) and A_i the associated Bernstein subalgebras with respect to the invariant form f. We can select the invariant linear form f such that $m_1 \geqslant m_2$. Also, we can assume, by a reordering of the stochastic basis, that $A_1 = \langle e_1, e_2, \ldots, e_{n_1} \rangle$, $A_2 = \langle e_{n_1+1}, e_{n_1+2}, \ldots, e_{n_2} \rangle$. We will prove that $m_1 = 2$:

(1) If $(m_1, m_2) = (1, 1)$ and $u_2, v_1 \notin A_1 + A_2$, then there exists a standard basis and a suitable arrangement of the stochastic basis such that the coordinates of Φ with respect to the standard basis are as shown in Table 1, where $e + u_1 + v_1 \in A_2$ and $0 < \lambda_l < 1$ for all l. By the induction assump-

TABLE 1

1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	1	$\cdots$	1
0	0	$\cdots$	0	1	1	$\cdots$	1	λ_{n_2+1}	$*$	$*$	$\cdots$	λ_n
0	0	$\cdots$	0	μ_{n_1+1}	$*$	$\cdots$	$*$	$*$	$*$	$*$	$\cdots$	μ_n
0	0	$\cdots$	0	1	1	$\cdots$	1	$*$	$*$	$*$	$\cdots$	$*$
α_2	1			0				β_2	γ_2			
$\vdots$		$\ddots$		$\vdots$		0		$\vdots$	$\vdots$		0	
α_{n_1}			1	0				β_{n_1}	γ_{n_1}			
				α_{n_1+1}	1			β_{n_1+1}	γ_{n_1+2}			
				$\vdots$		$\ddots$		$\vdots$	$\vdots$		0	
				α_{n_2-1}			1	β_{n_2-1}	γ_{n_2-1}			
								β_{n_2+1}	γ_{n_2+1}	1		
								$\vdots$	$\vdots$		$\ddots$	
								β_{n-3}	γ_{n-3}			1

tion and Lemma 1.3 we have that $n = n_2 + 2$ and α_i, β_i, and γ_i are different from zero for $i = 2, \dots, n_2 - 1$. Therefore, $\Gamma = [e_1, \dots, e_{n_1}]$ and $\Gamma' = [e_{n_1+1}, \dots, e_{n-2}]$ are 0-essential faces. This means that $\Gamma^2 = \{e\} \subset \operatorname{Int}\Gamma$ and $(\Gamma')^2 = \{e + u_1 + v_1\} \subset \operatorname{Int}\Gamma'$. Now by Lemmas 1.1 and 1.2, the face $[e_{n-1}, e_n]$ is 0-essential and also the faces $[e_1, \dots, e_{n_1}, \dots, e_{n-1}, e_n]$ and $[e_{n_1+1}, \dots, e_{n-2}, e_{n-1}, e_n]$ are 1-essential. This implies that $e_1^2 e_n^2 \in \Gamma \cup [e_{n-1}, e_n]$ and $e_{n_1+1}^2 e_n^2 \in \Gamma' \cup [e_{n-1}, e_n]$, so $n_1 = 1$ and $n_2 = n_1 + 1$. Therefore $\Phi = \{e, e + u_1 + v_1, e + \lambda_3 u_1 + \mu_3 u_2 + \beta_2 v_1, e + \lambda_3 u_1 + \mu_4 u_2 + \gamma_2 v_1\}$, but by Theorem 3.1 of [8] this basis is not stochastic.

(2) Let $(m_1, m_2) = (1, 1)$ and either $v_1 \in A_1 + A_2$ or $u_2 \in A_1 + A_2$. If $\dim[(A_1 + A_2) \cap A^2] = 4$, then $A_1 + A_2 = A$, since $A^2 \subset A_1 + A_2$ and Φ has no vanishing types. By Lemma 1.1 the simplex Δ has three 0-essential faces. It is clear by Lemma 1.2 that every 0-essential face of Δ is contained in A_1 or A_2. Thus, one of the subalgebras A_1, A_2 contains two 0-essential faces; but this is impossible, since the type of A_i is $(1, \dim A_i - 1)$ and therefore $\dim A_i^2 = 1$. If $\dim[(A_1 + A_2) \cap A^2] = 3$, then by Lemma 1.3 there exists e' such that Φ and $\Phi_1 = \{e_1, \dots, e_{n_2}, e'\}$ are externally kin. By the induction assumption $n = n_2 + 1$ and $e' = e_n$. Now $[e_n]$ must be a 0-essential face by Lemma 1.1. This means that e_n is an idempotent element. On the other hand, the subspace V_0 has dimension 1, so either $V_0 \not\subset A_1$ or $V_0 \not\subset A_2$. We can assume that $V_0 \not\subset A_1$. We have

$$e_n \in A^2, \qquad \dim\left[A/\left(A_1 + A^2\right)\right] = \dim A_2 - 2;$$

therefore by Lemma 1.3 there exists $\Phi_2 = \{e_1, \dots, e_{n_1}, e_1', e_2', e_n\}$ such that Φ and Φ_2 are externally kin. But induction assumption $\Phi = \Phi_2$ and $A_2 = \langle e_{n-2}, e_{n-1} \rangle$. Therefore, there exists a standard basis such that the coordinates of the stochastic basis with respect to this standard basis are of the form shown in Table 2, where $0 < \lambda < 1$, and by Lemma 1.3 and the induction assumption, the scalars $\alpha_i, \beta_i, \gamma_i$ must be different from zero for $2 \leqslant i \leqslant n$

TABLE 2

1	1	$\cdots$	1	1	1	1
0	0	$\cdots$	0	1	1	λ
0	0	$\cdots$	0	μ_{n-2}	μ_{n-1}	μ_n
0	0	$\cdots$	0	β_1	γ_1	λ^2
α_2	1			β_2	γ_2	0
$\vdots$		$\ddots$		$\vdots$	$\vdots$	$\vdots$
α_{n-3}			1	β_{n-3}	γ_{n-3}	0

TABLE 3

1	1	$\cdots$	1	1	1	1
0	0	$\cdots$	0	0	0	1
0	0	$\cdots$	0	μ_{n-2}	μ_{n-1}	0
0	0	$\cdots$	0	β_1	γ_1	1
α_2	1			β_2	γ_2	0
$\vdots$		$\ddots$		$\vdots$	$\vdots$	$\vdots$
α_{n-3}			1	β_{n-3}	γ_{n-3}	0

$-$ 3. Then $\Gamma = [e_1, \ldots, e_{n-3}]$, $\Gamma' = [e_{n-2}, e_{n-1}]$, and $\Gamma'' = [e_n]$ are 0-essential faces. We have that $ee_n \notin \Gamma \cup \Gamma''$, so $\Gamma \cup \Gamma'$ and $\Gamma' \cup \Gamma''$ must be 1-essential faces. Then, $e_{n-1}^2 e_n = (e_{n-1}^2 + e_n)/2$; but this is impossible. Therefore, we have proved that $m_1 = 2$.

In the rest of the proof we will assume that $\Gamma_1 = [e_1, \ldots, e_r]$, $\Gamma_2 = [e_{r+1}, \ldots, e_t]$, and $\Gamma_3 = [e_{n_1+1}, \ldots, e_s]$ are 0-essential faces and $V_0 \cap \Gamma_1 = \varnothing$. We will consider a standard basis such that

$$e \in \Gamma_1, \qquad e + u_2 \in \Gamma_2, \qquad e + u_1 + v_1 \in \Gamma_3.$$

The dimension of the subspace $A_1 \cap A^2$ is equal to 2 or 3, since $m_1 = 2$. If $\dim(A_1 \cap A^2) = 3$, then $A^2 \subset A_1 + A_2$. This implies that $A = A_1 + A_2$. Also, we have that $\dim[A/(A_1 + A^2)] = \dim A_2 - 1$. Therefore, by Lemma 1.3 we obtain that $n = n_1 + 1$. This means that $A_2 = \langle e_n \rangle$ and $\Gamma_3 = [e_n]$. The element e_n is an idempotent element and $e_n \in A^2$. Now Lemmas 2.2 and 1.3 imply that $n_1 = r + 2$, and therefore there exists a standard basis such that the coordinates of the stochastic basis Φ with respect to the standard basis are of the form shown in Table 3, where the scalars α_i, β_i, γ_i are different from zero. Thus, $\Gamma_2 = [e_{n-2}, e_n]$, and by Lemma 1.1 one of the faces $\Gamma_1 \cup \Gamma_3$, $\Gamma_2 \cup \Gamma_3$ is 1-essential. But this is impossible, since $ee_n \notin \Gamma_1 \cup \Gamma_3$ and $e_{n-1}^2 e_n \notin \Gamma_2 \cup \Gamma_3$. Therefore, we have obtained a contradiction that comes from the supposed condition $\dim(A_1 \cap A^2) = 3$. This implies that $\dim(A_1 \cap A^2) = 2$.

If $V_0 \not\subset A_1 + A_2$, then $\dim[A/(A_1 + A_2 + A^2)] = n - \dim(A_1 + A_2) + 1$. Therefore, by Lemma 1.3 and the induction assumption, $n = n_2 + 1$.

TABLE 4

1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	$\cdots$	1	1
0	0	$\cdots$	0	0	0	$\cdots$	0	1	1	$\cdots$	1	λ_n
0	0	$\cdots$	0	μ_{r+1}	*	$\cdots$	*	*	*	$\cdots$	*	μ_n
0	0	$\cdots$	0	0	0	$\cdots$	0	1	1	$\cdots$	1	*
α_2	1			0				0				β_2
$\vdots$		$\ddots$		$\vdots$		0		$\vdots$		0		$\vdots$
α_r			1	0				0				β_r
				α_{r+1}	1			0				β_{r+1}
				$\vdots$		$\ddots$		$\vdots$		0		$\vdots$
				α_{n_1-1}			1	0				β_{n_1-1}
								α_{n_1}	1			β_{n_1}
								$\vdots$		$\ddots$		$\vdots$
								α_{n-3}			1	β_{n-3}

Thus, the coordinates of the stochastic basis with respect to a suitable standard basis are as shown in Table 4. Hence, there exists

$$\tilde{\Phi} = \{e, e + u_2, e + u_1 + v_1, e + \lambda u_1 + \mu u_2 + \beta v_1\},$$

$0 < \lambda < 1$, such that Φ and $\tilde{\Phi}$ are externally kin. But, by Theorem 3.1 of [8], this is impossible.

At this point, we have proved that $\dim(A_1 \cap A^2) = 2$ and $V_0 \subset (A_1 + A_2) - A_1$; besides, $A^2 \subset A_1 + A_2$ and $n = n_2$, since the stochastic basis under consideration has no vanishing types. Hence, the coordinates of the stochastic basis with respect to a suitable standard basis are of the form shown in Table 5. The scalars α_i are different from zero for $i = 2, \ldots, n_1 - 1$, since if $\alpha_k = 0$, then there exist two vectors in $[e_{k+1}, e_{n_1+1}, e_{n_1+2}]$ such that these vectors jointly with the vectors e_j, $j \neq k + 1, n_1 + 1, n_1 + 2$, form an external reduction of Φ; but by Lemma 1.4 and the induction assumption, this is impossible. Therefore $\Gamma_2 = [e_{r+1}, \ldots, e_{n_1}]$. On the other hand, there exists

$$\tilde{\Phi} = \{e_1, e_2, \ldots, e_{n_1}, \tilde{e}_{n_1+1}, \tilde{e}_{n_1+2}\},$$

TABLE 5

1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	1	$\cdots$	1
0	0	$\cdots$	0	0	0	$\cdots$	0	1	1	1	$\cdots$	1
0	0	$\cdots$	0	μ_{r+1}	*	$\cdots$	*	*	*	*	$\cdots$	μ_n
0	0	$\cdots$	0	0	0	$\cdots$	0	β_1	γ_1	0	$\cdots$	0
α_2	1			0				β_2	γ_2			
$\vdots$		$\ddots$		$\vdots$		0		$\vdots$	$\vdots$		0	
α_r			1	0				β_r	γ_r			
				α_{r+1}	1			β_{r+1}	γ_{r+1}			
				$\vdots$		$\ddots$		$\vdots$	$\vdots$		0	
				α_{n_1-1}			1	β_{n_1-1}	γ_{n_1-1}			
								β_{n_1}	γ_{n_1}	1		
								$\vdots$	$\vdots$		$\ddots$	
								β_{n-3}	γ_{n-3}			1

with $\tilde{e}_{n_1+i} = e + u_1 + \mu'_i u_2 + \sum_{k=1}^{n_1-1} \beta_{i,k} v_k$, $i = 1, 2$, such that Φ and $\tilde{\Phi}$ are externally kin.

If there exists k with $2 \leqslant k \leqslant n_1 - 1$ such that $\beta_{1,k} = 0$ or $\beta_{2,k} = 0$, then by Lemma 1.4 and the induction assumption $\beta_{1,k} = \beta_{2,k} = 0$. Hence, $[\tilde{e}_{n_1+1}, \tilde{e}_{n_1+2}]$ is a 0-essential face of $\Delta(\tilde{\Phi})$ or $\tilde{e}_{n_1+1}$ and $\tilde{e}_{n_1+2}$ are in A^2. If $[\tilde{e}_{n_1+1}, \tilde{e}_{n_1+2}]$ is a 0-essential face, then by Lemma 1.1 $\Gamma_1 \cup [\tilde{e}_{n_1+1}, \tilde{e}_{n_1+2}]$ or $\Gamma_2 \cup [\tilde{e}_{n_1+1}, \tilde{e}_{n_1+2}]$ is 1-essential face. We can assume, by a reordering of the stochastic basis and the standard basis, that the first statement is true. Since $\Gamma_1 \cup [\tilde{e}_{n_1+1}, \tilde{e}_{n_1+2}]$ is essential, the products $e\tilde{e}_{n_1+1}$ and $e\tilde{e}_{n_1+2}$ are in $\Gamma_1 \cup [\tilde{e}_{n_1+1}, \tilde{e}_{n_1+2}]$. This implies that $\mu'_1 = \mu'_2 = 0$ and $\beta_{1,k} = \beta_{2,k} = 0$ for $k = r + 1, \ldots, n_1 - 1$. Finally, the products $(e + u_2)\tilde{e}_{n_1+1}$ and $(e + u_2)\tilde{e}_{n_1+2}$ imply that $\beta_{1,k} = \beta_{2,k} = 0$ for $k = 2, \ldots, r$. Therefore, in all the cases we obtain that $\tilde{e}_{n_1+1}$ and $\tilde{e}_{n_1+2}$ are in A^2 and $\mu'_1 = \mu'_2 = 0$. This implies that

$$\mu_{n_1+1} = \cdots = \mu_n = 0 \qquad \beta_2 = \cdots = \beta_{n_1-1} = 0,$$

$$\gamma_2 = \cdots = \gamma_{n_1-1} = 0,$$

which proves the theorem. ∎

As a result, we also obtain

THEOREM 2.2. *Let a Bernstein algebra be of type* $(3, n-3)$ *and with* $\dim U_0 = 1$. *Then its nondegenerate stochastic bases have the following form:*

$$e_1 = e + \sum_{k=2}^{r} \alpha_k v_k,$$

$$e_i = e + v_i \qquad (i = 2, \ldots, r),$$

$$e_{r+1} = e + \mu_2 u_2 + \sum_{k=r+1}^{s-1} \alpha_k v_k,$$

$$e_j = e + \mu_j u_2 + v_{j-1} \qquad (j = r+2, \ldots, s),$$

$$e_{s+1} = e + u_1 + \sum_{k=s}^{n-3} \beta_k v_k,$$

$$e_{s+2} = e + u_1 + \sum_{k=s}^{n-3} \gamma_k v_k,$$

$$e_l = e + u_1 + v_{l-3} \qquad (l = s+3, \ldots, n).$$

The element e is an idempotent; $u_2 \in U_0$, $v_1 \in V_0$; u_1, u_2 *is a basis of* U_e; $v_1, v_2, \ldots, v_{n-3}$ *is a basis of* V_e; *and* $\alpha_k < 0$ *for all k.*

The faces $\Gamma_1 = [e_1, \ldots, e_r]$ *and* $\Gamma_2 = [e_{r+1}, \ldots, e_s]$ *are* 0-*essential, and* $\Gamma_3 = [e_{s+1}, \ldots, e_n]$ *is invariant. In addition* $\Gamma_1^2 = \{e\}$, $\Gamma_2^2 = \{e + u_2\}$, *and there exist two scalars* $\epsilon_1 \leqslant 0$ *and* $\epsilon_2 \geqslant 1$ *such that* $[e_{s+1}, \ldots, e_n] \cap A^2 = [e + u_1 + \epsilon_1 v_1, e + u_1 + \epsilon_2 v_1]$.

We conclude that all these evolutionary operators are not normal in the sense of [20, p. 167]. This is new support for the Lyubich conjecture [20, p. 232]. According to the Lyubich conjecture, if V is a *normal* s.e.o., then the corresponding Bernstein algebra is regular. A nondegenerate s.e.o. V is normal if there is no a pair j_1, j_2 such that x'_{j_1} and x'_{j_2} are proportional and there is no a pair j_1, j_2 such that all x'_j's depend only on $x'_{j_1} + x'_{j_2}$, x'_i $(i \neq j_1, j_2)$.

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Received 22 April 1996; final manuscript accepted 2 December 1996