Complex Analysis Math 220C—Spring 2008
Final Examination—Part I—June 2, 2008—due June 9, 2008

1. Let $\{z_n\}$ be a sequence of complex numbers satisfying $\operatorname{Re} z_n \geq 0$. Suppose that $\sum z_n$ and $\sum z_n^2$ both converge. Show that $\sum |z_n|^2$ converges.

2. Suppose that f is entire and that $f(z)$ is real if and only if z is real. Show that f can have at most one zero. (Put a circle C around a couple of supposed zeros of f and apply the argument principle. Then break up the image curve $f \circ C$ into three pieces with index 0,0,1 respectively with respect to 0.)

3. Find the number of zeros of (two of these is enough)

   $f_1(z) = 3e^z - z$ in $|z| \leq 1$
   $f_2(z) = e^z/3 - z$ in $|z| \leq 1$
   $f_3(z) = z^4 - 5z + 1$ in $1 \leq |z| \leq 2$
   $f_4(z) = z^6 - 5z^4 + 3z^2 - 1$ in $|z| \leq 1$

4. Find a conformal map between the domains S and T if (three of these is enough)

   $S = \{z = x + iy : -2 < x < 1\}$ and $T = \{|z| < 1\}$
   $S = T =$ open upper half plane
   $S = \{z = re^{i\theta} : r > 0, 0 < \theta < \pi/4\}$ and $T = \{x + iy : 0 < y < 1\}$
   $S = \{|z| < 1\} - [0, 1)$ and $T = \{|z| < 1\}$
   $S =$ the region between $|z| = 1$ and $|z - 1/2| = 1/2$ and T is a half plane
   $S =$ the inside of the right-hand branch of the hyperbola $x^2 - y^2 = 1$ and T is the open unit disc. (Map the focus to 0 and the vertex to -1)

5. Find the Laurent expansion for

   $1/(z^4 + z^2)$ about $z = 0$
   $\exp(1/z^2)/(z - 1)$ about $z = 0$
   $1/(z^2 - 4)$ about $z = 2$

6. Let f_n be analytic on an open set D and suppose f_n converges uniformly on compact subsets of D. Let $S = \{z \in D : f_n(z) = 0 \text{ for some } n \geq 1\}$ and suppose that f is not identically zero. Show that f vanishes exactly at the limit points of S.

7. Prove that if u is continuous and bounded on the closed upper half plane, harmonic on the open upper half plane, and vanishes on the real axis, then it is a constant.

8. For z in the upper half plane, let $u(z)$ be the angle under which the interval $[0, 1]$ is seen from the point z. Show that u is a harmonic function by finding an analytic function f such that $u = \operatorname{Re} f$. (Consider first the angle under which the real axis from 0 to ∞ is seen from z.)