

DERIVATIONS IN GENETIC ALGEBRAS

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INTRODUCTION

In [7] the authors classify the derivations of the n -ploidy genetic algebra. Upon reviewing this paper for Mathematical Reviews I discovered that the computations simplify greatly when a canonical basis is used. I developed the theory using a canonical basis and extended the results to other genetic algebras. Somewhat later I was given [1] and [2] (where Costa treats ploidy with multiple alleles) to review and discovered that one of the authors used canonical bases for the algebras studied in [7] as well as for some more general algebras. I was surprised that [1] and [2] are not mentioned in [7] although they appeared earlier.

In Section 1 of this paper we prove several general theorems which are useful in computation of derivations of genetic algebras. In the remaining sections we apply these methods to several genetic algebras arising in the literature. In Sections 2-4 we recover the results of Costa cited above and in Sections 5-8 we consider algebras involving mutations, sex linkage and crossing over.

For background in genetic algebras including definitions of standard terms used in the theory I suggest [8] as a good reference. All algebras considered are assumed to be commutative.

1. GENERAL THEOREMS

Since many of the computations are similar for various different genetic algebras it seems worthwhile for the sake of efficiency to state some rather general results especially since they are very easy.

Theorem 1. Let A be a basic algebra with a non-zero idempotent and all train roots distinct from 1. Then any derivation d maps every element into an element of weight 0.

Proof. We choose a basis c_0, c_1, \dots, c_n where $c_0^2 = c_0$ and $c_0 c_i = \sum_{j=0}^n x_{0ij} c_j$ where $x_{0ij} = 0$ for $j < i$. (Since idempotents have weight 1 this can always be done.) From $d(c_0) = d(c_0^2) = 2c_0 d(c_0)$ we obtain that $w(d(c_0)) = 2w(d(c_0))$ so $w(d(c_0)) = 0$.

It is convenient to use complete backward induction on i where $1 \leq i \leq n$, i.e., suppose that $d(d_j)$ has weight 0 for all $j > i$. From $d(c_0 c_i) = c_0 d(c_i) + c_i d(c_0)$ we obtain $\sum_{k=0}^n x_{0ik} d(c_k) = c_0 d(c_i) + d_i d(c_0)$. We may write this as $x_{0i} d(c_i) + \sum_{k>i} d(c_k) = c_0 d(c_i) + c_i d(c_0)$. By taking weights of both sides we obtain $x_{0i} w(d(c_i)) = w(d(c_i))$. Since $x_{0i} \neq 1$ by hypothesis, $w(d(c_i)) = 0$.

Remark 1. Note from the proof of Theorem 1 that if no train root is $\frac{1}{2}$ then $d(c_0)$ is necessarily 0.

Remark 2. Note that we do not assume that A is a genetic algebra although we are interested only in genetic algebras at this point. Secondly, in [8, p. 71] it is shown that idempotents always exist in genetic algebras with genetic realization and it is proved on page 73 that all the train roots of such algebras are bounded above in absolute value by $\frac{1}{2}$. Thus the hypotheses are automatically satisfied in all such genetic algebras. In [8, p. 67] there is an example of a genetic algebra with no idempotents. In [3, p. 43] we obtain an example of a genetic algebra with train roots 1 by taking $r + s = -1$ in the mutation algebra. However r and s are genetically meaningful only when they are non-negative. Furthermore it is shown in [1] that if A has a unique weight function then the hypotheses on idempotents and train roots are not

needed. This applies in particular to genetic algebras since they necessarily have unique weight functions.

Remark 3. The conclusion of Theorem 1 can be strengthened if the c_i are assumed to be a genetic algebra basis by considering the expansion of $d(c_i)$ in terms of the c_j rather than just its weight. Then the least k such that $d(c_k)$ has a non-zero coefficient is bounded below by the least k such that $x_{0kj} \neq 0$ for some $j \geq i$. (Otherwise we obtain a contradiction by equating the coefficients of the earliest c_j whose coefficient in the expansion of $d(c_i)$ is different from 0 in the equation occurring at the end of the proof.) This result is slightly awkward to state in the above form. Fortunately, in many genetic algebras the train roots $\lambda = x_{0ii}$ take on a special form, e.g., if all train roots are distinct as in the gametic polyploidy algebra, this says that $d(c_i)$ has the form $\sum_{j \geq i} x_j c_j$. Even if the train roots are not distinct there is often a pattern among them as well as a multiplication table which is more tractable than the one which occurs in the general genetic algebra. For example, $c_i c_j$ has the simple form $x_k c_k$ surprisingly often!

Many genetic algebras have a natural structure as graded algebras. In this connection the referee of the original draft of this paper pointed out that much of the work can be simplified by means of a general theorem on derivations of graded algebras. Let $\sum_{n \geq 0} C_n$ be a graded algebra, i.e., $C_i C_j \subseteq C_{i+j}$ for all integers i, j . (Actually in all cases of interest here $C_n = 0$ for all n sufficiently large.) Then it is easy to see that the algebra of derivations of C also has the structure of a graded algebra: in fact, $Der C = \sum_{n \in \mathbb{Z}} (Der C)_n$ where $(Der C)_n = \{d \in Der C \mid d(C_i) \subseteq C_{i+n} \text{ for all } i\}$.

Theorem 2. Let $C = \sum_{i \geq 0} C_i$ be a graded algebra satisfying the following conditions.

- I. C_0 is one-dimensional and contains a non-zero idempotent c_0 .
- II. $C_0 + C_1$ generates C .

III. Let T denote left multiplication by c_0 . If Λ_j denotes the set of eigenvalues of T restricted to C_j then $(\frac{1}{2} \cup \Lambda_1) \cap \Lambda_j = \emptyset$ for $j \neq 1$.

Then $DerC = D_0 + D_1$ where $D_0 = (DerC)_0$, $D_1 = (DerC)_1$, and we have $D_0(c_0) = 0$, D_0 preserves the generalized eigenspaces of T acting on C_1 and $D_1(c_0)$ is contained in the $\frac{1}{2}$ -eigenspace for T acting on C_1 . Furthermore the map

$$\phi : DerC \rightarrow (EndC_1) \oplus C_1$$

defined by

$$\phi : d \mapsto d_0 |_{C_1} \oplus d_1(c_0)$$

is a Lie algebra injection into the holomorph of C_1 .

Note. The holomorph of a vector space V is the Lie algebra defined as follows. The underlying space is the direct sum of the full matrix ring $EndV$ over V and V . We first define an associative product as follows: If $m_1, m_2 \in EndV$ then $m_1 m_2$ is the matrix product. If $m \in EndV$ and $x \in V$ then mx is the result of m operating on x . All other products are zero. It is easy to see that this algebra is associative. The holomorph is the standard Lie algebra obtained by taking Lie products.

Proof. Let $0 \neq d \in (DerC)_n$. First suppose $d(c_0) \neq 0$. From $c_0^2 = c_0$ we obtain $d(c_0) = d(c_0^2) = 2c_0 d(c_0)$. Hence $d(c_0)$ is in the $\frac{1}{2}$ -eigenspace of T . By condition III this implies that $n = 1$.

Now suppose $d(c_0) = 0$. Then $d(c_0 x) = c_0 d(x) + d(c_0)x = c_0(dx)$. Therefore d commutes with T . If $(T - \lambda)^n x = 0$ then $(T - \lambda)^n dx = d(T - \lambda)^n x = 0$. Hence d preserves the generalized eigenspaces of T . Since $d \neq 0$ and $d(c_0) = 0$ it follows from II that $d(C_1) \neq 0$. It then follows from III that $n = 0$. We have shown that $(DerC) = (DerC)_0 \oplus (DerC)_1 = D_0 \oplus D_1$.

To show that ϕ is injective we consider D_0 and D_1 separately. For $d \in D_0$ we have shown that $d(c_0) = 0$. By II d is determined uniquely by

its restriction to C_1 hence ϕ is injective on D_0 . For $d_1, d_2 \in D_1$ suppose $d_1(c_0) = d_2(c_0)$. Then $d_1 - d_2 \in D_1$ and $(d_1 - d_2)(c_0) = 0$. If $d_1 - d_2 \neq 0$ then $d_1 - d_2 \neq D_0$ by the above which is a contradiction. Therefore $d_1 = d_2$.

It is almost immediate that ϕ is a Lie algebra mapping. On D_0 this is essentially nothing but the definition. For $d_1, d_2 \in D_1$ the Lie product is contained in $(DerC)_2$ which is 0. Now suppose $d_0 \in D_0$ and $d_1 \in D_1$. Then the Lie product is in C_1 . Note that $d_0(c_0) = 0$. Therefore $d_1 d_0(c_0) = 0$. Also $d_0 d_1(c_0)$ is by definition d_0 operating on $d_1(c_0)$. But this is precisely the product of the restriction of d_0 to C_1 and $d_1(c_0)$ in the associative algebra used in defining the holomorph. This completes the proof.

Note that the above theorem involves necessary conditions only. However, this is very useful since often the hardest part of a computational proof in the classification of derivations concerns necessary conditions. Actually every graded algebra has at least one non-zero derivation. Namely if $C = \sum_{n \geq 0} C_n$ then the "degree derivation" which is multiplication by n on C_n for all n is easily seen to be a derivation.

Recall that if C is any commutative algebra an element $a \in C$ is called a strong nilpotent if $ax = 0$ for all $x \in C$. Let $R(C)$ denote the set of all strong nilpotents in C . Let C^2 denote $span\{c_1 c_2 | c_1, c_2 \in C\}$. The following easy result will be useful.

Theorem 3. (a) If $d \in DerC$ then $d(R(C)) \subseteq R(C)$.

(b) If $d \in EndC$, $d(C^2) = 0$ and $d(C) \subseteq R(C)$ then $d \in DerC$.

(c) If $d \in EndC$ and $d(R(C)) \subseteq R(C)$ then $d(xy) = (dx)y + x(dy)$ whenever $x \in R(C)$.

We will call derivations satisfying (b) of this theorem "trivial derivations".

2. n-PLIODY GAMETIC ALGEBRAS

As shown in [3] the n-pliedy gametic algebra $G(n)$ has basis $\{c_i | 0 \leq i \leq n\}$ and the multiplication table has the form

$$c_i c_j = \binom{2n}{i+j}^{-1} \binom{n}{i+j} c_{i+j}$$

where the righthand side is understood to be 0 if $i + j > n$.

This algebra has an obvious grading: $G(n)_i = \text{span}\{c_i\}$. The hypotheses of Theorem 2 are satisfied since all train roots beyond the first one are less than $\frac{1}{2}$. Hence there are at most 2 independent derivations. By the remark after the proof of Theorem 2 we know that the linear map which sends c_i into ic_i for all i is a derivation. Thus to prove the derivation algebra is two-dimensional it is enough to check if there is a non-trivial derivation mapping c_0 into C_1 .

Theorem 4. (Costa [1,2]) $\text{Der } G(n) = (\text{Der } G(n))_0 + (\text{Der } G(n))_1$ where $(\text{Der } G(n))_0$ is spanned by the degree derivation and $(\text{Der } G(n))_1$ is spanned by d satisfying $d(c_i) = \binom{n-i}{n} c_{i+1}$ for $0 \leq i \leq n-1, d(c_n) = 0$.

Proof. Let d be a linear map satisfying $dc_i = \mu_i c_{i+1}$ for $0 \leq i \leq n-1, dc_n = 0$.

Write

$$\lambda_i = \binom{2n}{i}^{-1} \binom{n}{i}$$

so that $c_i c_j = \lambda_{i+j} c_{i+j}$ ($i, j \geq 0, i + j \leq n$). Then d is a derivation of $G(n)$ if and only if

$$(\mu_i + \mu_j) \lambda_{i+j+1} = \mu_{i+j} \lambda_{i+j} \tag{2.1}$$

for $i, j \geq 0, i + j \leq n-1$. Since $\lambda_{i+1} = \binom{n-i}{2n-i} \lambda_i$ and $\lambda_0 \neq 0$ we have $\lambda_i \neq 0$ for all $i, 0 \leq i \leq n$, and (2.1) is equivalent to

$$(\mu_i + \mu_j) \binom{n-i-j}{2n-i-j} = \mu_{i+j} \tag{2.2}$$

for $i, j \geq 0, i + j \leq n-1$. Since $\mu_i = \frac{n-i}{n}, 0 \leq i \leq n-1$, is a solution of (2.2) the lemma is proved. (Note that the formula for d may be derived by taking $\mu_0 = 1$, setting $j = 0$ in (2.2) and solving for μ_i .)

3. n-PLUIDY ZYGO TIC ALGEBRAS

The n-pluidy zygotic algebra $Z(n)$ has basis $\{c_i | 0 \leq i \leq 2n\}$ with multiplication table:

$$c_i c_j = \binom{n}{i} \binom{2n}{i}^{-1} \binom{n}{j} \binom{2n}{j}^{-1} c_{i+j}$$

where the right hand side is understood to be 0 if $i > n$ or $j > n$. (Note that this is different from the algebra studied in [2] since we use the identification of basis elements as in [3].)

This algebra is graded by letting $Z(n)_i = \text{span}\{c_i\}$. As before all we need do to prove $\text{Der } Z(n)$ is two-dimensional is find a non-trivial derivation mapping $Z(n)_0$ into $Z(n)_1$.

Theorem 5. (Costa [1,2]) $\text{Der } Z(n) = (\text{Der}(Z(n))_0 + (\text{Der}(Z(n))_1)$ where $(\text{Der}(Z(n))_0$ is spanned by the degree derivation and $(\text{Der}(Z(n))_1$ is spanned by the linear map d which satisfies $d(c_i) = \binom{2n-i}{2n} c_{i+1}$ for $0 \leq i < 2n, d(c_{2n}) = 0$.

Proof. Let d be a linear map with $dc_i = \mu_i c_{i+1}$ for $0 \leq i \leq 2n-1, dc_{2n} = 0$.

Write

$$\lambda_{i,j} = \binom{n}{i} \binom{2n}{i}^{-1} \binom{n}{j} \binom{2n}{j}^{-1}$$

so that $c_i c_j = \lambda_{i,j} c_{i+j}$ for $0 \leq i, j \leq n$. Then d is a derivation of $Z(n)$ if and only if

$$\mu_i \lambda_{i+1,j} + \mu_j \lambda_{i,j+1} = \mu_{i+j} \lambda_{i,j} \tag{3.1}$$

for $0 \leq i, j \leq n$. Since $\lambda_{i+1,j} = \binom{n-i}{2n-i} \lambda_{i,j}$, $\lambda_{i,j+1} = \binom{n-j}{2n-j} \lambda_{i,j}$ and $\lambda_{0,0} \neq 0$ we have $\lambda_{i,j} \neq 0$ for $0 \leq i, j \leq n$ and (3.1) is equivalent to

$$\mu_i \binom{n-i}{2n-i} + \mu_j \binom{n-j}{2n-j} = \mu_{i+j} \tag{3.2}$$

for $0 \leq i, j \leq n$. Since $\mu_i = \binom{2n-i}{2n}$, $0 \leq i \leq 2n-1$, is a solution of (3.2) the lemma is proved.

It is interesting that the gametic and zygotic cases are so closely parallel even though the multiplication tables are different. On the other hand such

similarity can be expected since the gametic algebra is a quotient of the zygotic algebra by an ideal consisting of strongly nilpotent elements. This illustrates the fact that genetic algebras arising naturally tend to have nice properties.

4. MULTIPLE ALLELES

Let us begin with the elementary case where $n = 1$. We have m alleles D_1, D_2, \dots, D_m and so the algebra, denoted $G(m, 1)$ has basis $\{D_1, \dots, D_m\}$ with multiplication table $D_i D_j = \frac{1}{2}(D_i + D_j)$. If we write $a_i = D_1 - D_i$ for $i \geq 2$ we obtain the basis D_1, a_2, \dots, a_m for $G(m, 1)$ with products $D_1^2 = D_1, D_1 a_i = \frac{1}{2} a_i$; and $a_i a_j = 0$. Note that all train roots are $\frac{1}{2}$.

$G(m, 1)$ is a graded algebra: $G(m, 1) = G(m, 1)_0 + G(m, 1)_1$ where $G(m, 1)_0$ is spanned by $\{D_1\}$ and $G(m, 1)_1$ is spanned by $\{a_2, \dots, a_m\}$. Let d be any linear map of $G(m, 1)$ to $G(m, 1)_1$. Then $d(D_1^2) = 2D_1 d(D_1), d(D_1 a_i) = D_1 d(a_i) + a_i d(D_1)$ (since $a_i d(D_1) = 0$) and $d(a_i a_j) = (d a_i) a_j + a_i (d a_j)$ (since all terms are zero). Hence $Der G(m, 1)$ is isomorphic to the holomorph of $G(m, 1)_1$.

We now combine multiple alleles and ployploidy.

We use the description of the m -allelic n -ploidy gametic algebra $G(m, n)$ from [4, Theorem 5.2] (where the work is actually simpler than that used in [3] for the special case of two alleles). It is shown there that if the m alleles are D_1, \dots, D_m (with $D_i D_j = \frac{D_i + D_j}{2}$) and if $a_i = D_1 - D_i$ for $i = 2, \dots, m$ then $G(m, n)$ has basis

$$\{D^{n-i} b \mid 0 \leq i \leq n, b \text{ a monomial of degree } i \text{ in } a_2, \dots, a_m\}$$

and multiplication

$$(D^{n-i} b)(D^{n-j} c) = \lambda_{i+j} D^{n-i-j} (bc)$$

(where λ_{i+j} is as in Section 2) if $i + j \leq n$ and $= 0$ otherwise.

Now let C denote the field of complex numbers and \mathcal{P} denote the polynomial algebra $C[a_2, \dots, a_m]$ graded, as usual, by degree: \mathcal{P}_i is the space of

homogeneous polynomials of degree i . Then $G(n) \otimes \mathcal{P}$ is an algebra (with multiplication determined by $(c \otimes u)(c' \otimes u') = (cc') \otimes (uu')$ for $c, c' \in G(n), u, u' \in \mathcal{P}$. Clearly $\sum_{i=0}^n G(n)_i \otimes \mathcal{P}_i$ is a subalgebra and there is an isomorphism $\tau : G(m, n) \rightarrow \sum_{i=0}^n G(n)_i \otimes \mathcal{P}_i$ satisfying

$$\tau : D^{n-i} b \mapsto c_i \otimes b$$

for $0 \leq i \leq n, b$ a monomial of degree i . We now identify $G(m, n)$ with $\sum_{i=0}^n G(n)_i \otimes \mathcal{P}_i$.

Then $G(m, n)$ is graded, with $G(m, n)_i = G(n)_i \otimes \mathcal{P}_i$ and $G(m, n)$ satisfies the hypotheses of Theorem 2. Furthermore, if d denotes the derivation of degree 1 of $G(n)$ given by Theorem 4 and if $a \in \mathcal{P}_1$ then the endomorphism of $G(n) \otimes \mathcal{P}$ given by

$$c \otimes u \mapsto d(c) \otimes au$$

is a derivation of $G(n) \otimes \mathcal{P}$ which preserves $\sum G(n)_i \otimes \mathcal{P}_i$ and hence restricts to a derivation of $G(m, n)$. Also, if e is any derivation of \mathcal{P} of degree 0 then the endomorphism $1 \otimes e$ of $G(n) \otimes \mathcal{P}$ is a derivation of $G(n) \otimes \mathcal{P}$ which preserves $\sum G(n)_i \otimes \mathcal{P}_i$ and hence restricts to a derivation of $G(m, n)$. Thus the map ϕ of Theorem 2 is onto and we have proved the following theorem. Theorem 6. (Costa [1,2]) $Der G(m, n) = (Der G(m, n))_0 + (Der G(m, n))_1$ is isomorphic to the holomorph of $G(m, n)_1$.

5. POLYPLOIDY WITH MUTATIONS

At first, this algebra appears to be somewhat complicated. However [5, page 294] gives the multiplication table in a nice form. Specifically the product of two basis elements is obtained by taking the product in the ployploidy algebra without mutations and then applying the mutation map S defined as follows: Write c_i in the form $D^{n-i}(D - R)^i$. Then $S[D^{n-i}(D - R)^i] = [D - R(D - R)]^{n-i}(1 - R - S)^i(D - R)^i$. Although $\{c_i\}$ is a good enough basis to exhibit that the algebra is generic, a simpler basis is preferable for our pur-

pose. In the special case of the gametic algebra for $n = 2$, $E = \frac{D+rR}{s+r}$ is easily seen to be an idempotent. (This may also be written as $D - \frac{r}{r+s}(D - R)$.) Since every element of weight one in the gametic algebra without mutations for $n = 1$ is an idempotent, the statement that this is so for an element E in the mutation algebra is equivalent to the statement that E is fixed under the mutation map S . We now obtain in the notation of [5] that $S[E^{n-i}(D - R)^i] = (1 - r - s)^i E^{n-i}(D - R)^i$. If we now let $c_i = E^{n-i}(D - r)^i$ we obtain the formula $c_i c_j = (1 - r - s)^{i+j} \lambda_{i+j} c_{i+j}$ (where λ_i is as in Section 2) for the gametic algebra and a similar result for the zygotic algebra.

Although this result was discovered on the way in the attempt to classify derivations we feel that this is interesting for its own sake (probably more so than the whole issue of derivations) since it shows that in spite of the *a priori* reasonable belief that the inclusion of mutations would give rise to a complicated algebra, a suitable choice of basis leads to a surprisingly simple multiplication table. For example, we still have a graded algebra.

Furthermore, we have the ironic fact that the inclusion of mutations actually simplifies the classification of derivations! Let us see why. Since r and s are both non-negative and are not both zero (otherwise we are back in the case of no mutations) the extra factor $(1 - r - s)^i$ occurring in the train roots guarantees that no train root is $\frac{1}{2}$. By Theorem 2 $d(c_0) = 0$.

If $r + s \neq 1$ we obtain that the derivation algebra is the one-dimensional space generated by the degree map. If $r + s = 1$ Theorem 2 no longer applies. However, Theorem 3 tells us that every linear transformation on the $n - 1$ dimensional space spanned by $\{c_i | i \neq 0\}$ gives rise to a derivation. This together with the fact that $d(c_0) = 0$ completes the picture.

6. MULTIPLE ALLELES, POLYPLOIDY AND MUTATIONS

We will denote by $G(m, n, S)$ the m -allelic, n -ploidic gametic algebra with mutation map S . Its underlying vector space is $G(m, n)$

($= \sum_{i=0}^n G(m, n)_i \otimes P_i$) and multiplication is given by

$$(c_i \otimes f)(c_j \otimes g) = \lambda_{i+j} c_{i+j} \otimes S(fg) \tag{6.1}$$

(where $\lambda_i = \binom{n}{i} \binom{2n}{i}^{-1}$ as in Section 2.) Note that the mutation map S on the span of $\{a_2, \dots, a_m\}$ extends to a degree preserving homomorphism (again denoted S) of \mathcal{P} and that if μ is any eigenvalue of S then $|\mu| \leq 1$.

Note that while $G(m, n, S)$ is graded it does not necessarily satisfy the hypotheses of Theorem 2.

We now give examples of some derivations of $G(m, n, S)$.

1) Derivations of $G(m, n)$ commuting with $1 \otimes S$.

If $D \in \text{Der}G(m, n)$ and $D(1 \otimes S) = (1 \otimes S)D$ then $D \in \text{Der}G(m, n, S)$.

In view of the proof of Theorem 6 this shows that for any $f \in \mathcal{P}_1$ satisfying $Sf = f$ the map

$$d_f : c_i \otimes g \mapsto \binom{n-i}{i} c_{i+1} \otimes fg$$

is a derivation belonging to $(\text{Der}G(m, n, S))_1$ and that for any $d \in (\text{Der}\mathcal{P})_0$ satisfying $dS = Sd$ the map

$$1 \otimes d : c_i \otimes g \mapsto c_i \otimes dg$$

is a derivation belonging to $\text{Der}(G(m, n, S))_0$.

2) Trivial derivations.

Note that $(G(m, n, S))^2 = \text{span}\{c_i \otimes SP_i\}$ and $R(G(m, n, S)) = \text{span}\{c_i \otimes (\ker S)_i\}$. Hence if E is any linear map of degree l of \mathcal{P} satisfying $ES = SE = 0$ then the map

$$d_E : c_i \otimes f \mapsto c_{i+l} \otimes Ef$$

for $0 \leq i + l \leq n$ and

$$d_E : c_i \otimes f \mapsto 0$$

otherwise is a derivation belonging to $(\text{Der}G(m, n, S))_l$. (In fact, this is a

trivial derivation in the sense of the definition following Theorem 3.) Note that if S is an isomorphism the only trivial derivation is 0.

7. SEX LINKAGE

We use the notation in [6] and consider the simplest case only. We express the basis of the sex linked algebra in the form $\{D, a, DD, Da, and aD\}$ with multiplication table

$$(D)(DD) = \frac{1}{2}(D + DD), (a)(DD) = \frac{1}{2}Da$$

$$(D)(Da) = \frac{1}{4}Da + \frac{1}{4}a, (a)(Da) = \frac{1}{4}aa$$

with all other products 0. This is not a genetic algebra. However, the 4 dimensional subalgebra spanned by $e = D + DD, b = a + 2Da, c = a - \frac{b}{2}, ec = -\frac{a}{4}, b^2 = f, c^2 = -\frac{f}{2}, bc = -\frac{f}{4}, fa = fb = fc = fe = f^2 = 0$.

This is a graded algebra $C = C_0 + C_1 + C_2$ where C_0 is spanned by e, C_1 by b and c and C_2 by f . We will show that $DerC$ is two-dimensional. Again Theorem 2 applies. We thus know that a derivation in $(DerC)_0$ must satisfy $d(e) = 0, d(b) = \beta b, d(c) = \gamma c$ for some scalars β and γ . Now $d(b^2) = 2bd(b) = 2\beta b^2, d(c^2) = 2cd(c) = 2\gamma c^2$. Since b^2 and c^2 are both scalar multiples of f it follows that $d(f) = 2\beta f = 2\gamma f$, i.e., $\beta = \gamma$. Thus $(DerC)_0$ is at most one-dimensional. Since the degree map is a derivation this is enough to show that $(DerC)_0$ is the one-dimensional space generated by the degree map.

Since we know by Theorem 2 that $(DerC)_1$ is at most one-dimensional it is enough to check that there exists a derivation $d \in (DerC)_1$ satisfying $d(e) = b$. In fact, let d be the linear map satisfying $d(e) = b, d(b) = 2f, d(c) = f$, and $d(f) = 0$. By Theorem 3(b) (applied to the subalgebra spanned by $\{b, c, f\}$) we see that $d(xy) = d(x)y + xd(y)$ whenever $x, y \in \{b, c, f\}$. Thus it is sufficient to check the formula for $x = e, y \in \{e, b, c, f\}$. This is immediate by direct computation.

The situation which makes this case slightly different from those considered above is the fact that different products of basis elements of level 1 give rise to the same basis element of level 2. At any rate, we have seen basic similarities in the nature of derivations in spite of this difference.

8. CROSSING OVER

Crossing over has been extensively studied although I personally have concentrated on polyploidy. The situation is analogous to that of polyploidy although there are important differences. I will assume two alleles at each locus for convenience of notation. Mutations will not be considered since the generalization is basically similar to the polyploidy case.

The natural basis consists of all ordered n -tuples where the i^{th} term (called the i^{th} locus genetically) is either D_i or R_i . Let τ be a non-negative real-valued function defined on the subsets σ of $\{i | 1 \leq i \leq n\}$ satisfying $r_\sigma = r_{\sigma'}$ (where σ' denotes the complement of σ) and $\sum r^\sigma = 1$. The values r_σ may be thought of as cross-over probabilities, i.e., r_σ is the probability that the offspring gamete will take its genes from the first parent gamete precisely for those loci in σ . We have chosen a minimal set of requirements on the r_σ to make this meaningful. (Experimental work in cytology gives rise empirically to various other restrictions that do not concern us here.) Two extremes that naturally arise are: (a) $r_\emptyset = \frac{1}{2}$, and (b) $r_\sigma = 2^{-n}$ for all σ . The first is the case of complete linkage, i.e., no crossing over and the second is the case of no linkage whatsoever. The function τ parameterizes the family of genetic algebras we will consider. Denote the algebra corresponding to τ by $C(\tau)$. Given two n -tuples of D 's and R 's the product has the form $\sum r_\sigma D_\sigma$ where D_σ is the n -tuple obtained by choosing the term in the first factor for all loci in σ and the term in the second factor otherwise.

The "multilinear strategy" works here just as well as in the polyploidy case discussed in [5], i.e., it makes sense to have each term in the n -tuple of the

form $aD_i + bR_i$ and use the same formula formally for the product providing for every term omitted we multiply by the weight of that factor. This allows us to choose a basis consisting of n -tuples where the i 'th term is either D_i or $a_i = D_i - R_i$. This basis not only exhibits the algebra as a generic algebra but also has the convenient property referred to earlier that the product of two basis elements involves only one basis element. Specifically let c_σ be the basis element obtained by choosing the term a_i at locus i if and only if $i \in \sigma$. Then

$$c_\sigma c_\tau = R(\sigma, \tau) c_{\sigma \cup \tau}$$

where

$$R(\sigma, \tau) = \sum_{\sigma \subseteq \delta \subseteq \tau'} r_\delta.$$

(Note that $R(\sigma, \tau) = 0$ if $\sigma \cap \tau \neq \emptyset$.)

We will now show that $C(r)$ can be given the structure of a graded algebra satisfying the hypotheses of Theorem 2. The grading will be defined in terms of certain subsets (called indecomposable subsets) given by the following definition.

Definition: A subset $\emptyset \neq \sigma \subseteq \{1, \dots, n\}$ is said to be indecomposable if $r_\sigma = 0$ unless $\sigma \subseteq \delta$ or $\delta \subseteq \sigma'$.

For example (in the extreme cases for r mentioned above) if $r_\emptyset = \frac{1}{2}$ then every nonempty subset of $\{1, \dots, n\}$ is indecomposable and if $r_\sigma = 2^{-n}$ for all δ , then $\{1\}, \dots, \{n\}$ are the only indecomposable subsets.

Lemma 1. Let $\emptyset \neq \sigma \subseteq \{1, \dots, n\}$. Then the following are equivalent:

- (a) σ is indecomposable;
- (b) $c_\emptyset c_\sigma = (\frac{1}{2})c_\sigma$;
- (c) $0 = \sum_{\sigma \cap \delta \neq \emptyset, \sigma \cap \delta' \neq \emptyset} r_\delta$.

Proof. It is immediate that (a) implies (c) and the reverse implication holds since each $r_\delta \geq 0$. Now

$$R(\emptyset, \sigma) = \sum_{\delta \subseteq \sigma'} r_\delta = R(\sigma, \emptyset) = \sum_{\sigma \subseteq \delta} r_\delta.$$

Since $\sigma \neq \emptyset$ we have $2R(\emptyset, \sigma) = \sum_{\sigma \subseteq \delta} r_\delta = \sum_{\sigma \subseteq \sigma'} r_\delta$. Therefore (b) is equivalent to $1 = \sum_{\sigma \subseteq \delta} r_\delta$ or $\delta \subseteq \sigma'$. Since $\sum r_\delta = 1$ this is equivalent to (c).

Define a relation \sim on $\{1, \dots, n\}$ by $i \sim j$ if and only if $\{i, j\}$ is indecomposable. (Genetically this means that locus i and locus j are linked.) Then \sim is easily seen to be an equivalence relation on $\{1, \dots, n\}$. (Use Lemma 1(c) to check transitivity.) Furthermore, from Lemma 1(c) we see that a nonempty set σ is indecomposable if and only if $i \sim j$ for all $i, j \in \sigma$. Write $\bar{\sigma} = \{i | i \sim j \text{ for some } j \in \sigma\}$. Let $\lambda_1, \dots, \lambda_k$ denote the \sim equivalence classes in $\{1, \dots, n\}$ and for $\sigma \subseteq \{1, \dots, n\}$ let $\sigma_i = \sigma \cap \lambda_i$. Then $\bar{\sigma} = \bar{\sigma}_1 \cup \dots \cup \bar{\sigma}_k$.

Lemma 2: Let $\sigma, \tau \subseteq \{1, \dots, n\}$. Then:

- (a) $R(\sigma, \tau) = R(\bar{\sigma}, \bar{\tau})$;
- (b) If σ and τ are indecomposable then $R(\sigma, \tau) = 0$ if and only if $\bar{\sigma} \neq \bar{\tau}$.

Proof. For (a) note that if $\sigma \subseteq \delta, \bar{\sigma} \not\subseteq \delta$ there are some i, j such that $i \in \delta, j \notin \delta$ and $i \sim j$. Then by Lemma 1(c) $r_\delta = 0$ since $\{i, j\}$ is indecomposable. Similarly $\delta \subseteq \tau', \delta \not\subseteq \tau'$ implies $r_\delta = 0$. Thus $R(\sigma, \tau) = \sum_{\sigma \subseteq \delta \subseteq \tau'} r_\delta = \sum_{\bar{\sigma} \subseteq \delta \subseteq \bar{\tau}'} r_\delta = R(\bar{\sigma}, \bar{\tau})$, proving (a).

For (b) observe that if σ and τ are indecomposable (hence $\neq \emptyset$) and $\bar{\sigma} \neq \bar{\tau}$ then (as $\bar{\sigma} \cap \bar{\tau} \neq \emptyset$) $0 = R(\bar{\sigma}, \bar{\tau}) = R(\sigma, \tau)$. If $\bar{\sigma} \neq \bar{\tau}$ then by Lemma 1(c) $r_\delta \neq 0$ for some δ with $\bar{\sigma} \cup \bar{\tau} \not\subseteq \delta, \delta \not\subseteq (\bar{\sigma} \cup \bar{\tau})'$. Now $\bar{\sigma}, \bar{\tau}$ are indecomposable, so $r_\delta \neq 0$ implies $\bar{\sigma} \subseteq \delta$ or $\delta \subseteq \bar{\sigma}'$, and also $\bar{\tau} \subseteq \delta$ or $\delta \subseteq \bar{\tau}'$. This implies $\bar{\sigma} \subseteq \delta \subseteq \bar{\tau}'$ or $\bar{\tau} \subseteq \delta \subseteq \bar{\sigma}'$ so $R(\bar{\sigma}, \bar{\tau}) \neq 0$.

We can now construct some derivations.

Definition: Let $\sigma \subseteq \{1, \dots, n\}$ be indecomposable. Define a linear map $d_\sigma : C(r) \rightarrow C(r)$ by

$$d_\sigma(c_\tau) = 0 \text{ if } \sigma \cap \bar{\tau} \neq \emptyset$$

and

$$d_\sigma(c_\tau) = c_{\sigma \cup \tau} \text{ if } \sigma \cap \bar{\tau} = \emptyset.$$

Lemma 3: Let $\sigma \subseteq \{1, \dots, n\}$ be indecomposable. Then:

- (a) $R(\alpha, \beta) = R(\alpha \cup \sigma, \beta) + R(\alpha, \beta \cup \sigma)$ for all $\alpha, \beta \subseteq \{1, \dots, n\}$;
- (b) $d_\sigma \in \text{Der}C(\tau)$.

Proof. For (a) note that $R(\alpha \cup \sigma, \beta) + R(\alpha, \beta \cup \sigma) = \sum_{\alpha \cup \sigma \subseteq \delta \subseteq \beta \cup \tau} R(\alpha, \beta) + \sum_{\alpha \subseteq \delta \subseteq \beta \cup \tau} R(\alpha, \beta)$. Since σ is indecomposable this is equal to $\sum_{\alpha \subseteq \delta \subseteq \beta \cup \tau} R(\alpha, \beta)$, proving (a). For (b) we must verify

$$d_\sigma(c_\alpha c_\beta) = (d_\sigma c_\alpha) c_\beta + c_\alpha (d_\sigma c_\beta) \tag{8.1}$$

for all α, β . If $\sigma \cap (\alpha \cup \beta) \neq \emptyset$ then every term in (8.1) is zero (by the definition of d_σ , the formula for multiplication and Lemma 2(a)) so (8.1) holds. If $\sigma \cap (\alpha \cup \beta) = \emptyset$ then (8.1) follows from (a).

Definition: Let σ, σ' be indecomposable with $\bar{\sigma} = \bar{\sigma}' = \lambda$. (Note that $\sigma = \sigma'$ is allowed.) Define a linear mapping

$$d_{\sigma, \sigma'} : C(\tau) \rightarrow C(\tau)$$

by

$$d_{\sigma, \sigma'}(c_\tau) = 0 \quad \text{if } \tau_i \neq \sigma$$

and

$$d_{\sigma, \sigma'}(c_\tau) = c_{(\tau - \sigma) \cup \sigma'} \quad \text{if } \tau_i = \sigma.$$

Lemma 4: Let σ, σ' be indecomposable, $\bar{\sigma} = \bar{\sigma}' = \lambda$. Then $d_{\sigma, \sigma'} \in \text{Der}C(\tau)$.

Proof. We must verify

$$d_{\sigma, \sigma'}(c_\mu c_\tau) = (d_{\sigma, \sigma'} c_\mu) c_\tau + c_\mu (d_{\sigma, \sigma'} c_\tau) \tag{8.2}$$

for all μ, τ . If $\mu_i \neq \emptyset, \tau_i \neq \emptyset$ then every term in (8.2) equals zero (using Lemma 2(a) and $\bar{\mu} \cap \bar{\tau} \neq \emptyset, \bar{\mu} \cap \sigma' \neq \emptyset, \bar{\tau} \cap \sigma' \neq \emptyset$). If $\mu_i = \emptyset$ and $\tau_i \neq \sigma$ then every term in (8.2) is zero by the definition of $d_{\sigma, \sigma'}$. If $\mu_i = \emptyset, \tau_i = \sigma$ then $d_{\sigma, \sigma'}(c_\mu c_\tau) = R(\mu, \tau) c_{(\mu \cup \tau - \sigma) \cup \sigma'}$, $(d_{\sigma, \sigma'} c_\mu) c_\tau = 0$ and $c_\mu (d_{\sigma, \sigma'} c_\tau) = R(\mu, (\tau - \sigma) \cup \sigma') c_{(\mu \cup \tau - \sigma) \cup \sigma'}$. As $\bar{\tau} = \overline{(\tau - \sigma) \cup \sigma'}$, (8.2) holds. The remaining cases ($\tau_i = \emptyset$) are similar, so the lemma is proved.

Now set $l(\mu)$ equal to the number of i such that $\mu_i \neq \emptyset$ and $C(\tau)_j = \text{span}\{c_\mu | l(\mu) = j\}$. Then $C(\tau) = \sum_{j=0}^k C(\tau)_j$ is a graded algebra which satisfies conditions I-III of Theorem 2. (Condition I is clear. For condition II suppose that $l(\mu) > 1$. Since μ is not indecomposable there exists δ such that $\mu \cap \delta \neq \emptyset, \mu \cap \delta' \neq \emptyset$ and $\tau_\delta \neq 0$. Then $R(\mu \cap \delta, \mu \cap \delta') \geq \tau_\delta > 0$. Since $l(\mu \cap \delta), l(\mu \cap \delta') < l(\mu)$, this gives II. Finally, Lemma 1(b) gives condition III.)

Thus Theorem 2 applies to $C(\tau)$ and so we obtain the following theorem.

Theorem 7: $\text{Der}C(\tau) = (\text{Der}C(\tau))_0 + (\text{Der}C(\tau))_1$. Furthermore,

$$(\text{Der}C(\tau))_1 = \text{span}\{d_\sigma | \sigma \text{ indecomposable}\}$$

and

$$(\text{Der}C(\tau))_0 = \text{span}\{d_{\sigma, \sigma'} | \sigma, \sigma' \text{ indecomposable, } \bar{\sigma} = \bar{\sigma}'\}.$$

Proof. In view of Theorem 2 all that remains to prove is that

$$(\text{Der}C(\tau))_0 = \text{span}\{d_{\sigma, \sigma'} | \sigma, \sigma' \text{ indecomposable, } \bar{\sigma} = \bar{\sigma}'\}.$$

Let $D \in (\text{Der}C(\tau))_0$. Then if τ is indecomposable we have $0 = D((c_\tau)^2) = 2c_\tau(D(c_\tau))$. Thus by Lemma 2(b) $D(c_\tau) \in \text{span}\{\tau' | \bar{\tau} = \bar{\tau}'\}$. Thus $D(c_\tau) = \sum_{\tau'} a_{\tau'} c_{\tau'}$ and so $E = D - \sum_{\tau'} a_{\tau'} d_{\tau, \tau'} \in (\text{Der}C(\tau))_0, E(C(\tau)_1) = (0)$. By Theorem 2 this implies $E = 0$, proving the theorem.

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DISJOINT IDEALS

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Abstract. This paper develops new techniques for studying modules over a ring from knowledge of homomorphic images of the modules. The concept of disjoint ideals is introduced in this regard. Applications are given to the construction of almost split sequences over Artinian algebras.

§1. INTRODUCTION

Given a ring R , a standard technique used in the study of R -modules is to study homomorphic images of the modules. Let $\text{Mod}(R)$ denote the category of (left) R -modules. If A is an ideal in R , the functor $R/A \otimes_R - : \text{Mod}(R) \rightarrow \text{Mod}(R/A)$ provides the categorical connection. The forgetful functor $F : \text{Mod}(R/A) \rightarrow \text{Mod}(R)$ is a right adjoint to $R/A \otimes_R -$. In this paper, given two ideals in R , we introduce a new ring S and adjoint functors $\Phi : \text{Mod}(R) \rightarrow \text{Mod}(S)$ and $\Psi : \text{Mod}(S) \rightarrow \text{Mod}(R)$. In the special case where the two ideals are equal to an ideal A then $\Psi\Phi$ is naturally equivalent to $F \circ (R/A \otimes_R -)$. Of more interest is the case when two ideals are far from equal. The notion of (left) *disjoint ideals* is defined in section 3 and studied for the remainder of the paper. We will see, for example, that when the two ideals are disjoint, $\Psi\Phi$ is naturally equivalent to the identity functor on $\text{Mod}(R)$.

The main objective of this paper is to show how the study of R -modules can be reduced to the study of modules and maps involving homomorphic images of the ring R . The constructions introduced in sections 1 and 2 are especially powerful in case the ideals are disjoint. Sections 3 and 6 define and classify disjoint ideals and sections 3 and 4 study their general properties. Section 5 applies the previous results in the Artinian case to the study of almost split sequences. This section demonstrates the power of the reduction techniques. Section 6 shows that disjoint ideals usually exist for Artinian rings and provides a canonical construction of a set of disjoint ideals in such rings.

We begin by introducing notation which will be used for the remainder of the paper. Let A and B be ideals in R . If M is a (left) R -module, we denote the canonical surjections $M \rightarrow M/AM$ and

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